ONE DIMENSIONAL DISCRETE SYSTEMS

We here consider discrete dynamical systems generated by iterates of a function. For a general definition of a discrete dynamical system we refer to

`http://en.wikipedia.org/wiki/Discrete-time_dynamical_system`

The n-th iterate x_n, $n = 0, 1, 2, ...$ of a function is defined by recursion $x_n = f(x_{n-1})$, $n > 1$, where x_0 is given. Thus the iterates are solutions of the recurrence equation $x_n = f(x_{n-1})$, where x_0 is an initial condition.

In practice x_n means the magnitude of the state variable at discrete time n. For example, in ecology it can be the size of a population at generation number n. In other applications it can mean the distance to an object at some time moment number n. It can also be thought of as the magnitude of currents in physics, the concentration in chemistry, the number of sick people in epidemics, the amount of money in economics, the humidity in climate and so on ..

The equation $x_n = f(x_{n-1})$ means that the state is completely defined by the previuos state. No other effects are accounted. So the model is valid only if the state can be determined from the last one and all other effects are neglible. For example, in populations it means that we always know the size of the population if we know the size of the population in the previous generation.

The sequence x_0, x_1, x_2, ... is thus giving the behaviour of the state variable for discrete time. The set $\{x_0, x_1, x_2, ...\} = \{f^n(x_0) | n \geq 0\}$ is called the orbit of the point. We define f^n as the function given by $f^n(x) = f(f(..f(x)..))$, where we have the letter f for n times in the right hand side and we suppose $f^0 = f$. Often we also use notations:

$$x \to f(x), \quad x_n \to x_{n+1}$$

and

$$x_0 \to x_1 \to x_2 \to \cdots$$

We suppose $x_n \in R^m$ and in our first part we consider the case when $m = 1$. Some definitions like fixed points, periodic orbits are also valid for $m > 1$. So we now consider one dimensional systems. Indeed we consider only systems on intervals and do not include the important and interesting dynamics on circles. Let us look at some examples.

Example. Let $f(x) = 2x$. Then the magnitude of the state variable x is doubled for each iterate. If, for example, $x_0 = 1$ the we get

$$1 \to 2 \to 4 \to 8 \to \cdots \to 2^n \to \cdots$$

If $x = -1$ we get

$$-1 \to -2 \to -4 \to -8 \to \cdots \to -2^n \to \cdots$$

If $x_0 = 0$ then $x_n = 0$, $n = 0, 1, 2, ...$ and we get

$$0 \to 0 \to 0 \to 0 \to \cdots$$

The general solution is $x_n = x_0 2^n$ and the behaviour of the solution can be classified as: $x_n \to \infty$ for $x_0 > 0$, $x_n \to -\infty$ for $x_0 < 0$ and $x_n = 0$ for

$x_0 = 0$. For positive x this model is useful in modelling some growth processes without limitations in growth.

Let us take some other examples.

Example. Let $f(x) = 2x + 3$. We then get

$$0 \to 3 \to -2 \to 7 \to -11 \to 25 \to \cdots$$

and

$$1 \to 1 \to 1 \to 1 \to 1 \cdots$$

The general formula for the solution is $x_n = (-2)^n(x_0 - 1) + 1$ (can be proved by induction) and $|x_n| \to \infty$ for $x_0 \neq 0$, otherwise $x_n = 1$.

Example. Let $f(x) = (3 - x)/2$. We get

$$0 \to 3/2 \to 3/4 \to 9/8 \to 15/16 \to \cdots$$

and

$$1 \to 1 \to 1 \to 1 \to 1 \cdots$$

The general formula for the solution is $x_n = (-1/2)^n(x_0 - 1) + 1$ and $x_n \to 1$ for all x_0.

Example. Let $f(x) = 2 - x$. For any x_0 we get $x_0 \to 2 - x_0 \to x_0$ so iterates of all x_0 are 2-periodic except for the iterates of $x_0 = 1$.

We have seen that some iterates do not change but remain the same for all iterates. Such orbits are called fixed points.

Definition of fixed point. If $f(x) = x$ then the point x is called a fixed point.

Thus if x_0 is a fixed point we get $x_n = x_0$ for all n. The fixed points can be calculated by finding the roots of $f(x) - x$.

We have seen that there can also be periodic behaviour and we give the definition of a periodic point and orbit.

Definition of periodic point. If for some $p \in \{0, 1, 2, ...\}$ the point x is a fixed point for f^p but not for f^i, where $0 < i < p$ then x is called p-periodic and the orbit through x is also called p-periodic.

Thus if x_0 is p-periodic then

$$x_0 \to x_1 \to \cdots \to x_{p-1} \to x_0 \to x_1 \to \cdots \to x_{p-1} \to x_0 \to x_1 \to \cdots$$

In the example just above all points are 2-periodic except for $x = 1$ which is a fixed point.

In the examples above we have seen that iterates can be fixed points or periodic orbits or they can tend to infinity or to some fixed point. Later we will see that they can also tend to periodic orbits or have more complicated behaviour. In the complicated case we mostly have a case called chaos.

Let us look at one more example.

Example. Let $f(x) = x^2$. Solving $f(x) - x = 0$ we get that the fixed points are 0 and 1. If $0 < x_0 < 1$ iterates very quickly tend to zero (the square of such numbers decreases rapidly). If $x_0 > 1$ iterates go to plus infinity. If $x_0 = -1$ then $x_1 = 1$ and further iterates are the fixed point $x_n = 1$, $n \geq 1$. If $-1 < x_0 < 0$ then $0 < x_1 < 1$ and thus iterates tend to zero. If $x_0 < -1$ then $x_1 > 1$ and iterates tend to plus infinity.

To know the behaviour of iterates we are clearly interested in knowing the iterates of which points tend to some fixed point or a periodic orbit.

Definition of basin of attraction. Let O_p be a p-periodic orbit $\{x_0, x_1, ..., x_p\}$. The basin of attraction of O_p is the set of points x such that the minimal distance to the set O_p from the iterate n tend to zero for $n \to \infty$.

If $p = 1$ the orbit consists of a fixed point and the basin of attraction consists of the points the iterates of which tend to the fixed point.

So in the example just above the basin of attraction of zero is the interval (-1,1). In the example we also have seen that iterates of points can become fixed points after some iterates even if they where not from the beginning. The same can happen for periodic points.

Definition of eventually periodic point. A point x is said to be eventually periodic if it is not p-periodic but some iterate of x is p-periodic. If $p = 1$ we speak about eventually fixed points.

In the example where $f(x) = x^2$ the point -1 is an eventually fixed point. We also give a name to the points the iterates of which tend to infinity.

Definition of escaping set. The set of points x for which the absolute value of the iterates tend to infinity is called the escaping set.

Initial values in the escaping set often lead to cathastrophic phenomena in the real world model.

The escaping set when $f(x) = x^2$ is the union of the intervals $(-\infty, -1)$ and $(1, \infty)$.

Exercise. Find the fixed points and periodic orbits and basins of attraction and escaping sets when

a) $f(x) = 2x + 3$

b) $f(x) = x^3$

c) $f(x) = -x^3$

d) $f(x) = -x^{1/3}$

In practical applications the system usually depends on some parameters. So the function will depend on a parameter vector $a = (a_1, a_1, ..., a_k)$, that is we consider functions of both state and parameter a and the equations get the form $x_n = f(x_{n-1}, a)$. Let us look at one example.

Example. Let $f(x) = ax$. The general solution is $x_n = a^n x_0$. $x = 0$ is a fixed point for all parameters a. If $a > 1$ and $x_0 \neq 0$ then $|x_n| \to \infty$, but if $0 < a < 1$ then $x_n \to 0$ for $n \to \infty$. So if we have two different a-values $a_* > 1$ and $a_{**} > 1$ there is no difference in the behaviour of iterates , but if $0 < a_* < 1 < a_{**}$ there is clear difference. For parameter values where the behaviour of iterates changes one says that there is a bifurcation for that parameter value. So we can say there is is a bifurcation for $a = 1$.

Let us now look at a practical example.

Example. It is sometimes cold in North Europe so one needs a house where to survive. Not everywhere it is possible to find an apartment for renting so it is necessary to buy a small house for about 500000 SEK. To get such money it is necessary to get a loan. For the loan one has to pay about 10 percentage of the remaining sum every year. The question is what is the size of the unpaid loan every year if L crowns is paid back every year. This leads to a reccurence equation

$$x_n = 1.1x_{n-1} - L,$$

where x_n is the loan at year number n. x_0 is the full loan and the every year payment is a parameter L. The system has a fixed point $x = 10L$ and the solution of the equation is $x_n = (1.1)^n(x_0 - 10L) + 10L$. Clearly if $x_0 < L$ then x_n will decrease and at some time reach zero after which the model is not more valid. If $x_0 > 10L$ the loan will increase but this case is unrealistic because of bank police. If $x_0 = 10L$ the loan will be kept constant which is also unrealistic. Anyhow observe that there is no bifurcation when L is changing. The fixed point only has different size but the behaviour of the system in general is equivalent. Finally if the one needing a home is a teacher in Math this person can probably not pay more than L=60000 SEK in a year. Calculations show that it will take 19 years before the small house is paid for.

Exercise. Show that the solution of a linear system $x_n = ax_{n-1} + b$ is $x_n = a^n(x_0 - b/(1-a)) + b/(1-a)$ if $a \neq 1$ and conclude that $b/(1-a)$ is a fixed point and $|x_n| \to \infty$ if $|a| > 1$, $x_0 \neq b/(1-a)$ and $x_n \to b/(1-a)$ if $|a| < 1$. What is happening in the case $a = \pm 1$?

Most often recurrence equations do not have solutions in form of expressions and we need to find the solutions themselves by numerical methods. Then theory can only give information about the qualitative behaviour of the iterates. We give such an example from population theory en ecology. In such examples it is also more important to know result of qualitative theory. For example, it is more important to know whether a population can survive than to predict the exact size, although exact solutions are useful in comparing with practice to validate the model.

Example. Let $f(x) = rxe^{-x}$. x gives the size of the population at generation n and r is a parameter called growth rate which is kept fixed when we solve the equation. The greater r the higher will the size of the next population be. If x is small the size is growing with a factor r for each population but when x is greater the growth will be less because of competition. As a result of numerical experiments we get the following behaviour for the iterates. We use the intial value $x_0 = 1$. For $r = 5$ we get the fixed point at about 1.6094 after 20 iterates which well coincides with the value $\ln 5$ to be calculated analytically by us later. For $r = 10$ we see a 2-periodic behaviour with the 2-periodic orbit $\{0.9346, 3.6706\}$ after 10 iterates. For $r = 13$ we get a four periodic orbit $\{4.7663, 0.5274, 4.0462, 0.9199\}$ For $r = 14.5$ we get an 8-periodic orbit. after 30 iterates. For $r = 15$ we cannot see any periodicity or regularity in the iterates and there seems to be something called chaos. For $r = 23$ we get the 3-periodic orbit $\{0.4375, 0.042, 0.9269\}$ and for $r = 24$ we get a 6-periodic orbit. For many r-values greater than 15 we can only see chaos.

Exercise. Examine experimentally the behaviour of the iterates of $f(x) = rx/(1+x^3)$ for $r = 2, 3, 5, 14, 20, 20.7$ and 27 supposing x_0 is positive.

Exercise. Examine experimentally the iterates for the Impact map

$$f(x) = \begin{cases} x + \rho & x \leq 0 \\ -\sqrt{x} + x + \rho & x > 0 \end{cases}$$

for $\rho = 1, 0.2, 0.12, 0.1$ and 0.05. For practical meaning we must have $\rho > 0$.

This map was got as a simplification of a problem in vibration mechanics (Nonlinearity 14, 301-321, 2001).

Exercise. Examine experimentally the iterates of the Lorenz model map

$$f(x) = \begin{cases} 1 - b(-x)^a & -1 < x \leq 0 \\ -1 + bx^a & 0 < x < 1 \end{cases}$$

where $a < 1,\ 1 < b < 2,\ ab > 1$. This map is a one dimensional model for modelling the chaos in the Lorenz system giving the famous Lorenz attractor.

Cobweb analysis

An effective graphical tool for examining the behaviour of orbits is the cobweb technics. From there it is usually easy to see all intervals of simple behaviour as when iterates tend to infinity or when they are attracted to a fixed point.

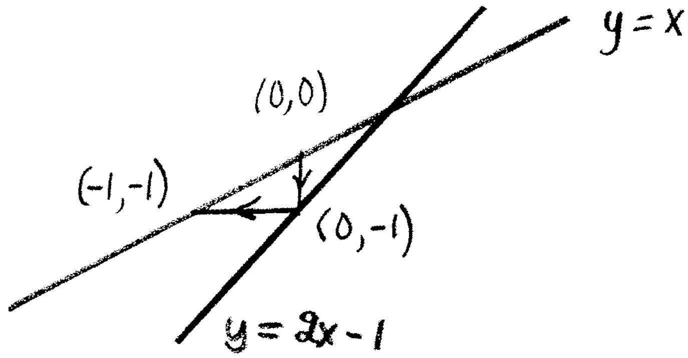
In a cobweb picture we plot in the same figure the diagonal $y = x$ and the graph of the function. Then in the figure the iterates of the point x_0 are seen from the broken line

$$(x_0, x_0) \to (x_0, f(x_0)) = (x_0, x_1) \to (x_1, x_1) \to (x_1, f(x_1)) = (x_1, x_2) \to (x_2, x_2) \to \cdots$$

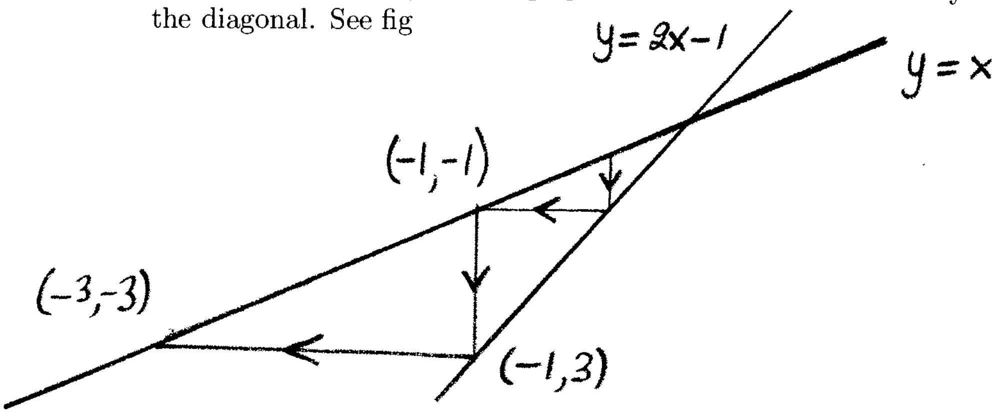
$$\cdots (x_{n-1}, x_{n-1}) \to (x_n, f(x_{n-1})) = (x_{n-1}, x_n) \to (x_n, x_n)$$

We describe some first steps in some examples.

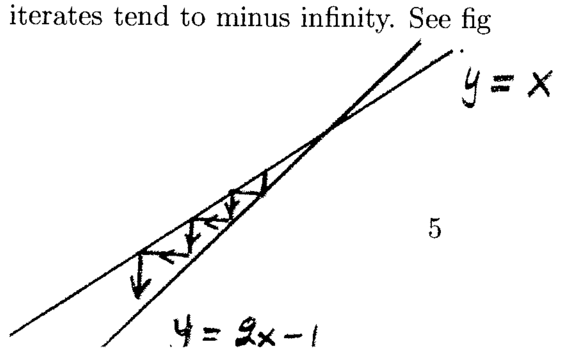
Example. Consider $f(x) = 2x - 1$. We look at the first iterate of 0. Because $f(0) = -1$, to plot the first step of the cobweb giving the first iterate we start at (0,0), go vertically down to the graph to (0,-1) and from there horizontally to the left to the diagonal to (-1,-1). See fig



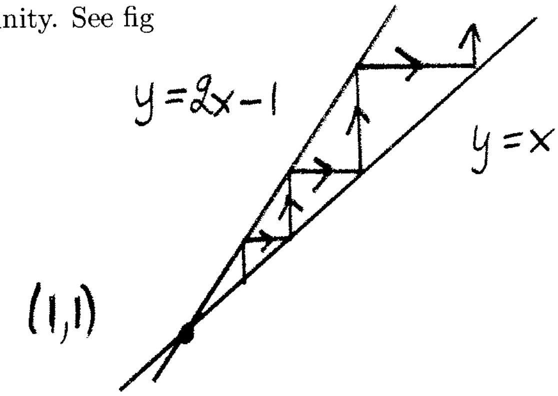
The second iterate is -3, so to plot the second step we continue vertically from (-1,-1) to (-1,3) on the graph and from there horizontally to (-3,-3) on the diagonal. See fig



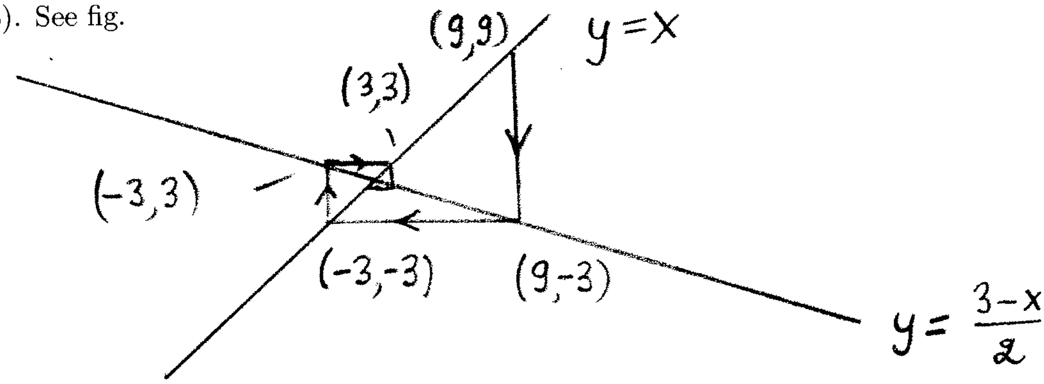
Continuing in this way from (-3,-3) for some iterates we convince ourselves that iterates tend to minus infinity. See fig



The graph and the diagonal intersect at 1, so from there the cobweb will go nowhere and it is a fixed point. Plotting the cobweb for the iterates of 2 in the same way as the cobweb for the iterates of 0, we see that iterates tend to plus infinity. See fig



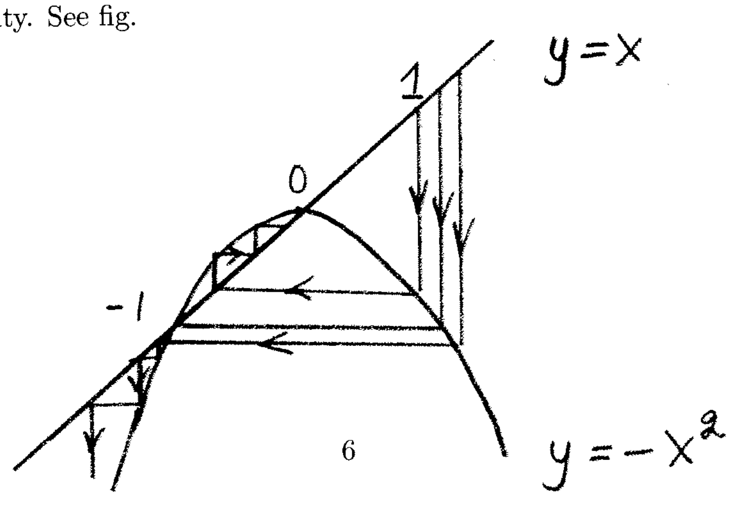
Example. Consider $f(x) = (3 - x)/2$. The diagonal and graph intersect at 1 which is a fixed point. We plot some iterates of the point x=9. The first iterate is -3 so we start at (9,9) and go vertically down to the graph to (9,-3) from where we go horizontally to the left to (-3,-3). The second iterate is 3, so the cobweb continues vertically up to (-3,3) and then horizontally to the right to (3,3). See fig.



Continuing some steps we convince ourselves that iterates of points different from 1 tend to the fixed point and so that every second iterate is to the right and every second to the left of the fixed point. Thus the fixed point 1 is a global attractor.

Let us consider one more example.

Example. Let $f(x) = -x^2$. The diagonal and the graph intersect at 0 and -1 which are fixed points. The point +1 is an eventually fixed point which is mapped to -1 and the cobweb from it ends at the fixed point -1 after one iterate. Iterates of points between -1 and 1 seem to tend to the fixed point zero. Iterates of points greater than one or less than -1 are seen to tend to minus infinity. See fig.



So we have noticed that when we use the cobweb technics the fixed points are the intersections between the diagonal and the graph of the function and starting on the diagonal we go vertically (up or down) to the graph and from the graph we go horizontally (to the left or to the right) to the diagonal.

Exercise. Analyze the dynamics of iterates of the following maps by cobweb technics.

a) $f(x) = x^3$

b) $f(x) = -x^3$

c) $f(x) = -x^{1/3}$

We will give examples from practical applications later when we know more about fixed points.

Type of fixed points

To find the fixed points we have to solve the equation $f(x) - x = 0$. We have seen fixed points of different types. Some of them attracts iterates, other throw them away. Most fixed points are either attractors or repellers given in the following definition.

Definition of attracting fixed point. The fixed point q is an attractor if there is a neighbourhood of the fixed point such that iterates of all points in this neighbourhoods tend to the fixed point. (We also call such fixed points stable).

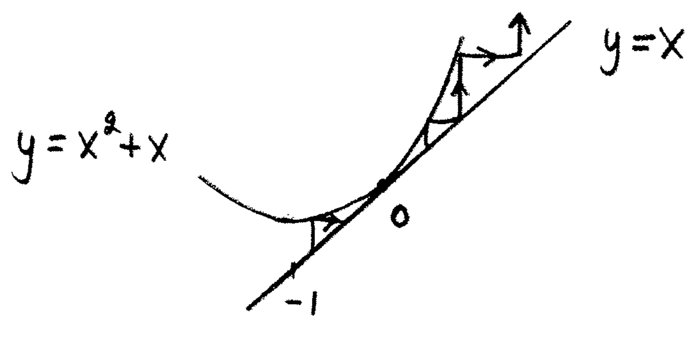
Definition of repelling fixed point. A fixed point q is a repeller if for any neighbourhood of the fixed points iterates of all points leave the neighbourhood. (These points are the most usual kind of unstable points).

If the function has a derivative near to the fixed point then the derivative at the fixed point often gives the type of the fixed point.

Theorem. If $f(q) = q$ and $|f'(q)| < 1$ then q is an attractor and if $|f'(q)| > 1$ then q is a repeller.

Proof is in Appendix.

The theorem does not tell about what happens if $f'(q) = \pm 1$. For example, if $f(x) = x^2 + x$ then zero is attracting iterates from the left in (-1,0) but repelling iterates of all positive points. See fig.



Example. If $f(x) = 1.5x(1-x)$ then 0 and 1/3 are fixed points. $f'(0) = 1,5$ implies that 0 is a repeller and $f'(1/3) = 2/3$ implies that 1/3 is an attractor.

Example. If $f(x) = 2\sin(x) + x + 1$ then $-\pi/6 + 2n\pi$ is a sequence of repellers but $-5\pi/6 + 2n\pi$ is a sequence of attractors, because $f'(-\pi/6 + 2n\pi) = \sqrt{3} + 1 > 1$ and $|f'(-5\pi/6 + 2n\pi)| = |-\sqrt{3} + 1| < 1$.

Example. Consider $f(x) = rxe^{-x}$ for $x \geq 0$ and $r > 0$. We solve $f(x) = x$ for fixed points and we get $x = 0$ or $re^{-x} = 1$. But $re^{-1} = 1$ is equivalent to

$x = \ln r$ which is positive for $r > 1$. For $r < 1$ zero is the only fixed point and it is an attractor because $|f(0)| = |r| < 1$. For $1 < r < e^2$ there are two fixed points 0 and $\ln r$. Zero is repelling because $f'(0) = r > 1$ but $\ln r$ is attracting because $|f'(\ln r)| = |1 - \ln r| < 1$ for $1 < r < e^2$. For $r > e^2$ one finds that both fixed points are repellers.

Exercise. Find the fixed points and their types for

a) $2x - x^2/4 - 3/4$

b) $2x/(1 + x^2)$

c) $(x/2)e^{x+x^2}$

d) Special tent map

$$f(x) = \begin{cases} 2x & x < 1/2 \\ 2(1-x) & x \geq 1/2 \end{cases}$$

Exercise. Find the fixed points and their type depending on parameter r for

a) $x^2 + 2x + r$

b) $r(6x - x^3)$

c) $rx/(1 + x^\gamma)$, $r > 0$, $x \geq 0$, $\gamma \geq 2$

Often we cannot find the fixed points analytically but we can know how many they are and their type and we can estimate their location and calculate them numerically for concrete parameter values.

Example. Let $f(x) = rx^2e^{-x}$. Solving $f(x) = x$ we get $x = 0$ or $r = e^x/x = g(x)$. So $x = 0$ is always a fixed point. It is always an attractor because $f'(0) = 0$. We get $g'(x) < 0$ for $0 < x < 1$ and $g'(x) > 0$ for $x > 1$ so g has a minimum at $x = 1$ and the equation $r = g(x)$ has two solutions for $r > e$ and no for $r < e$. Thus for $r > e$ there is only one fixed point zero. For $r > e$ there are two more fixed points. One x_- in (0,1) and the other x_+ for $x > 1$. Calculations give $f'(x_\pm) = r(2x_\pm - x_\pm^2)e^{-x_\pm} = 2 - x_\pm$ (we used that $e^{x_\pm}/x_\pm = r$) from which follows that x_- is always repeller and x_+ is an attractor for $e < r < e^3/3$ $(1 < x_+ < 3)$ and repeller for $r > e^3/3$. Numerical calculations give the fixed points about 0.26 and 2.54 for $r = 5$.

Exercise. Determine the number of fixed points and their types for the systems given by the function

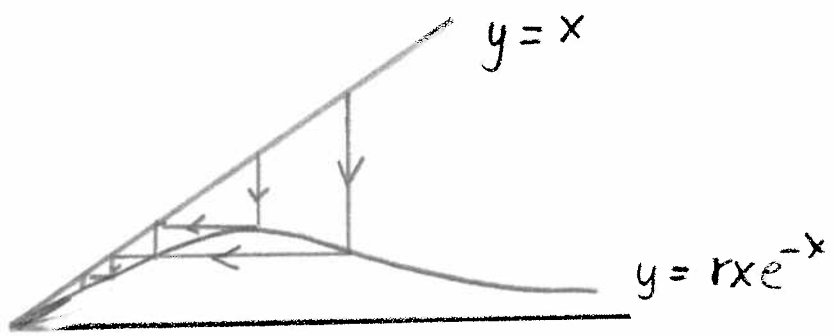
a) $f(x) = rx^2/(1 + x^\gamma)$, $\gamma \geq 3$

b) $f(x) = x^4/8 + x^2 + c$

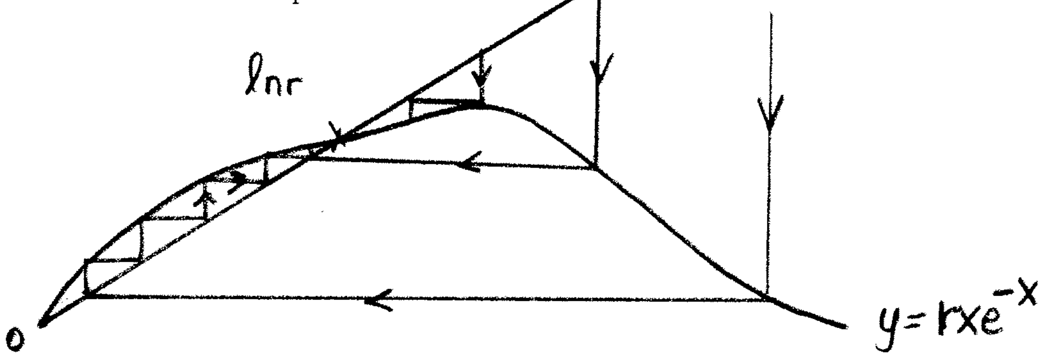
c) $f(x) = cx - x^2 - x^6/5$

We now combine our knowledge with cobweb technics to analyze some systems from population theory treated above.

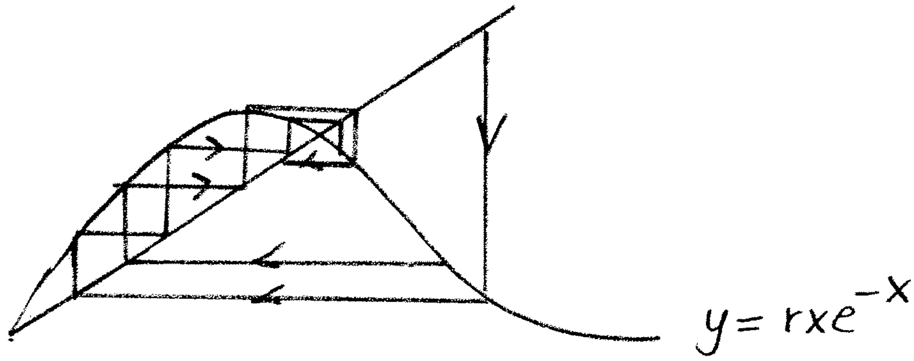
Example. We consider again $f(x) = rxe^{-x}$. If $r < 1$ the graph of f is below the diagonal because $f(x) - x = x(re^{-x} - 1) < 0$ for $x > 0$. We see that the cobweb we try to plot tend to the fixed point zero. This means that the growth rate is too low and the population will go extinct.



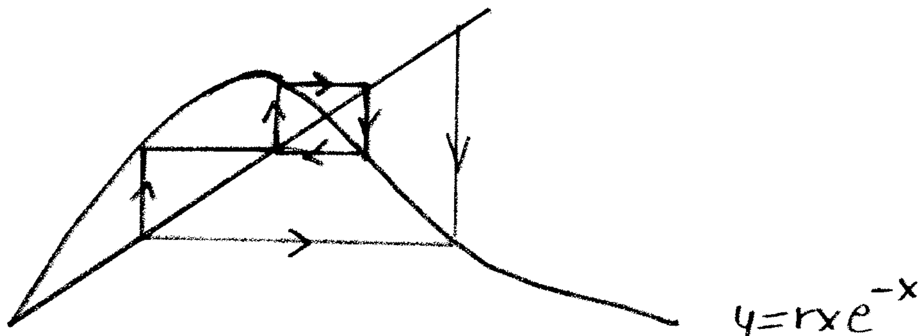
For $r > 1$ the fixed point $\ln r$ exists and it is stable, that is, it is an attractor. We plot the cobweb in two cases when the maximum of the function is below and above the diagonal. The maximum r/e is attained at $x = 1$ and it is below the diagonal if $r < e$ and above otherwise. In both cases the cobwebs of positive points will tend to the fixed point $\ln r$. In the case $r < e$ the cobwebs show that iterates of positive points less than $\ln r$ increase and tend to the fixed point. Iterates of points between $\ln r$ and 1 (including $x = 1$) decrease and tend to the fixed point. Iterates of points greater than one are after one iterate in the interval between zero and one and after that they behave like iterates of points from this interval. Thus we have seen that iterates of all positive points tend to the fixed point $\ln r$.



In the case $e < r < e^2$ the cobweb shows that iterates will tend to the fixed points but now the behaviour is a little bit more complicated because iterates of the same point will tend to the fixed point from both sides and the cobweb is winding around the fixed point. Anyhow it enough to study iterates of points less than $\ln r$ because iterates of all other points will be there after one iterate.



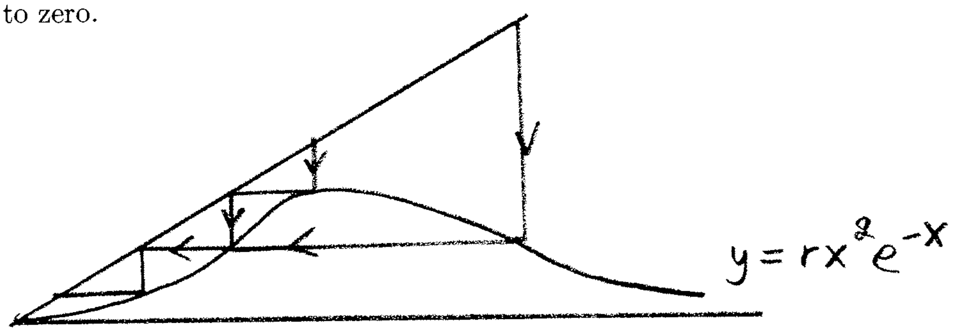
For $r > e^2$ the cobwebs become more complicated. For r a little bit above e^2 the iterates and cobweb stabilizes to a 2-periodic orbit giving a square as the final cobweb intersecting the diagonal at the 2-periodic points.



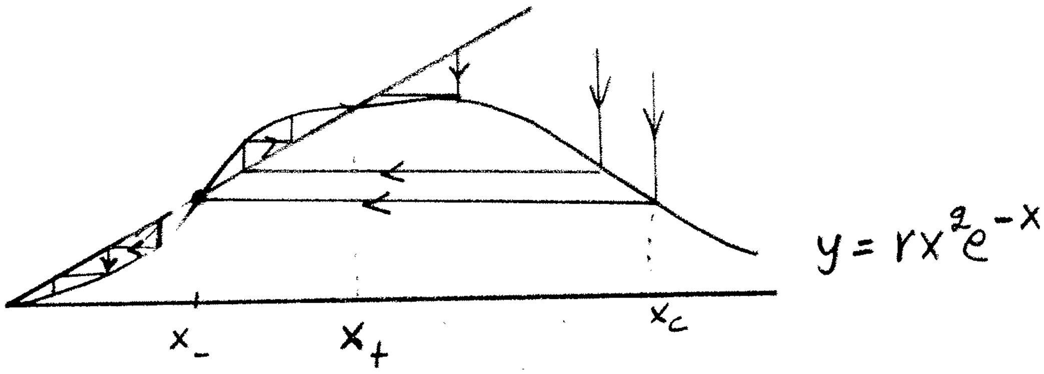
For greater r we mostly have chaos and the cobweb will fill up a great part of the picture with dark belts. For some r however there might be periodic attractors of higher periods as we saw in the experiments in the section describing iterates.

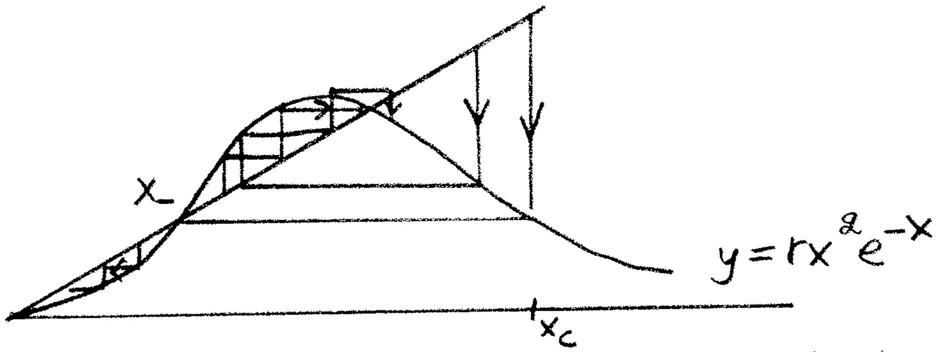
Exercise. Produce such cobwebs.

Example. Consider again $f(x) = rx^2e^{-x}$. For $r < e$ the graph of the function is below the diagonal and the cobweb again shows that the iterates tend to zero.



For $e < r < e^3/3$ there is a unique point $x_c > x_-$ such that $f(x_c) = x_-$. Cobweb examination shows that iterates in $(0, x_-) \cup (x_c, \infty)$ will tend to zero, for points in (x_-, x_c) the iterates will tend to the fixed point x_+. We show cobwebs in one figure when the top of the graphs is below the diagonal $r < e^2/2$ and one figure when the top is above.





For $r > e^3/3$ cobweb analysis show that iterates of points in $(0, x_-) \cup (x_c, \infty)$ tend to zero but the dynamics in $x_-, x_c)$ will be more complicated.

So in this case an important result is that the population cannot survive if it is too small or too big.

Exercise. Produce a cobweb analysis for $f(x) = rx/(1 + x^\gamma)$ and $f(x) = rx^2/(1 + x^\gamma)$ for some values off $r > 0$ and $\gamma \geq 3$ with different behaviours when $x \geq 0$.

Periodic orbits

To find a p-periodic point we have to solve $f^p(x) - x = 0$. But in this equation may be included also periodic points of lower period m. For example, all fixed points are always included because $f^p(x) - x = f(x) - x$ for such a point. If, for example, $p = 4$ for a periodic point q then $f^4(q) = f^2(q) = q$ and q may be 2-periodic instead of 4-periodic if it is a solution to the equation. In general if $m|p$ (m divides $p = km$, where k is an integer) then a solution to $f^p(x) - x = 0$ might be m-periodic because then $f^m(x) = x$ implies $f^p(x) = f^{km}(x) = x$. (If $g = f^m$ then $g^k = f^p$). On the other side if m is not a factor in p then a solution to $f^p(x) - x = 0$ cannot be m-periodic, because if x_0 is m-periodic then only iterates x_{km}, where k is integer can equal x_0. We conclude that we thus have to exclude solutions with period m dividing p from the set of all solutions to $f^p(x) - x = 0$.

Example. Consider the function $f(x) = x^2 + x - 5/4$. Solving $f(x) = x$ we get the fixed points $x = \pm\sqrt{5}/2$. We now form $f^2(x)$. We get $f^2(x) = (x^2 + x - 5/4)^2 + x^2 + x - 5/4 - 5/4 = x^4 + 2x^3 - x^2/2 - 3x/2 - 15/16$ Thus $f^2(x) - x = 0$ is a fourth order equation. Among the roots there also fixed points because for fixed points $f^2(x) = f(f(x)) = f(x) = x$. The pure 2-periodic are obtained if we divide $f^2(x) - x$ with $f(x) - x$ (in this case $f(x) - x = (x - \sqrt{5}/2)(x - (-\sqrt{5}/2)) = x^2 - 5/4$). This is possible because if r is a root to a polynomial it can be factorized so that $x - r$ is a factor according to a known theorem in algebra. We obtain

$$\frac{f^2(x) - x}{f(x) - x} = x^2 + 2x + 3/4$$

The roots are -1/2 and -3/2 which are the 2-periodic points. We check $f(-1/2) = -3/2$ and $f(-3/2) = -1/2$.

Example. Let us consider the tent map

$$f(x) = \begin{cases} 2x & x < 1/2 \\ 2(1-x) & x \geq 1/2 \end{cases}$$

We seek for a three periodic orbit. The orbit of a three periodic point cannot be wholly inside $x < 1/2$ because then it should be an iterate of $2x$ which has only one fixed point. Likewise it cannot be wholly inside $x > 1/2$. We seek for an orbit where two points x_0 and x_1 are in $x < 1/2$ and one x_2 is in $x > 1/2$. Then we get $x_1 = 2x_0$, $x_2 = 2x_1$ and $x_0 = 2(1 - x_2)$ implying $2(1 - 4x_0) = x_0$ with solution $x_0 = 2/9$. The solution is indeed in the right intervals because $x_0 < 1/2$ and $x_1 = 4/9 < 1/2$ and $x_2 = 8/9 > 1/2$. So $\{2/9, 4/9, 8/9\}$ is a 3-periodic orbit.

A periodic orbit can also be an attractor or repeller and the definition can be given in the following way.

Definition of attracting and repelling periodic orbits. A p-periodic orbit is an attractor (repeller) if q is a point on the orbit and q is an attracting (repelling) fixed point for f^p.

If f is differentiable in a neighbourhood of the orbit the type can be determined by the following theorem.

Theorem . Let q be a point on a p-periodic orbit. Then the orbit is attracting if $|(f^p)')(q)| < 1$ and repelling if $|(f^p)')(q)| > 1$.

Note that to calculate $(f^p)'(x_0)$ we can use the cahin rule so that we instead can calculate $f'(x_0)f'(x_1) \cdots f'(x_{p-1})$.

Example. Consider again the function $f(x) = x^2 + x - 5/4$ with the 2-periodic orbit $\{-1/2, -3/2\}$. We get $f'(x) = 2x+1$ giving $f'(-1/2) = 0$ and $f'(-3/2) = -2$ which gives $f'(1/2)f'(-3/2) = 0$. We can check by calculating $(f^2)'(-1/2)$ or $(f^2)'(-3/2)$. We get $(f^2)'(x) = 4x^3 + 6x - x - 3/2$ which implies $(f^2)'(-1/2) = (f^2)'(-3/2) = 0$. So we have a really attracting orbit. The absolute value is even zero.

Example. We try to find 3-periodic orbits for the Impact map (notice that $\rho > 0$) introduced earlier such that $x_0, x_1 < 0 < x_2$. We get $x_2 = x_0 + 2\rho$ and thus we should have $x_0 = -\sqrt{x_0 + 2\rho} + x_0 + 2\rho + \rho$. This equation (taking squares when the squareroot is moved to one side) implies $x_0 + 2\rho = 9\rho^2$. So the three periodic orbit should be $x_0 = 9\rho^2 - 2\rho$, $x_1 = 9\rho^2 - \rho$ and $x_2 = 9\rho^2 > 0$. The condition that x_0, $x_1 < 0$ implies that the orbit exists for $\rho < 1/9$. The derivative at x_0 is $1 \cdot 1 \cdot (1 - 1/(2\sqrt{x_2}) = 1 - 1/(6\rho)$ so it is an attractor for $\rho > 1/12$ but repelling for $\rho < 1/12$.

Exercise. Find the 2-periodic orbit for $f(x) = x^2 + x - 2$ and its type.

Exercise. Determine the constant a and b in the function $f(x) = a + b\cos(x)$ so that $\{\pi/3, \pi/2\}$ is a 2-periodic orbit. Find the type of the orbit. Plot the graph of f^2 for these values for a and b.

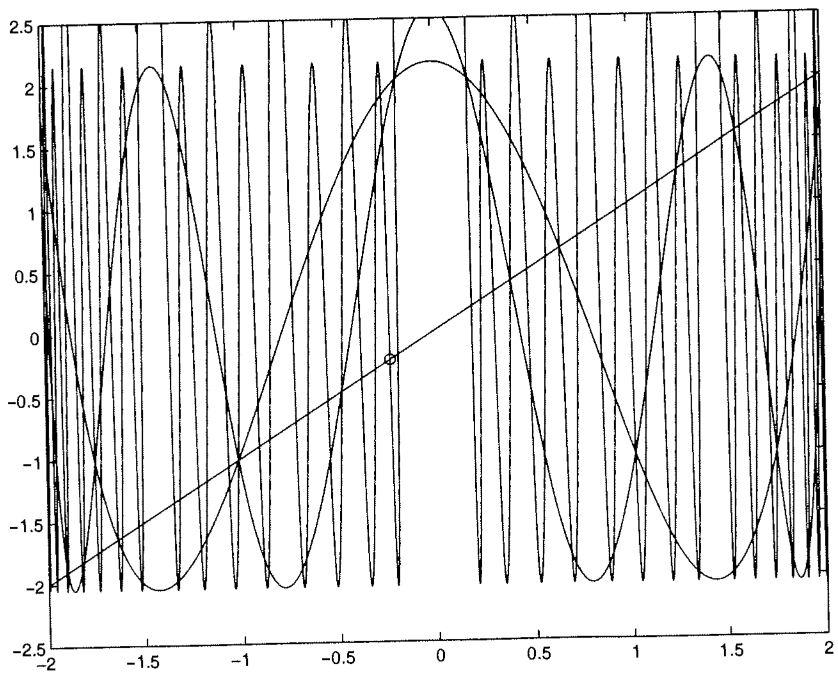
Exercise. Find a 2-periodic orbit and another 3-periodic orbit for the system in the example for the tent map.

Exercise. Prove that $2/(2^m + 1)$ is m-periodic for $m \geq 2$ and $2/(2^m - 1)$ is m-periodic for $m \geq 3$ for the tent map.

Exercise. For the Impact map introduced earlier find the two periodic orbits and determine when they are attracting. Also find a 4-periodic orbit of type $x_0, x_1, x_2 < 0 < x_3$ and determine its type.

Mostly we cannot find periodic points by direct calculations but we are forced to find them numerically. Let us consider an example when $f(x) = x^2 - 2.05$.

Now suppose we wish to find a 6-periodic orbit. We plot the graph of f^2, f^3 and f^6 and the diagonal $y = x$ in the same figure.



f^2, f^3 and f^6 and $y = x$ if $f(x) = x^2 - 2.05$

The graph oscillating most is the graph of f^6. We try to find an intersection with $y = x$. To be sure that it is not 2- or 3-periodic it must be chosen so that f^2 and f^3 do not pass. (We do not need the graphs of the fourth or fifth iterate because a 4- or 5-periodic point cannot have period 6 and be the solution to $f^6(x) = x$) We find such a point at approximately -0.231. Iterating it we get the 6-periodic orbit through -0.231, -1.99657, 1.936, 1.699, 0.837, -1.3486. The product of the derivatives is the same as multiplicating all these numbers and then by $64 = 2^6$. The absolute value is again surely greater than one and the orbit is repelling.

Exercise. Find a 5-periodic point to $3.65x(1 - x)$ and determine its type.

Global behaviour

Up to now the global behaviour was examined by cobwebs and the local behaviour by more exact methods. There are also some statements which can be used to prove the global behaviour and we now introduce some of them to be proved in Appendix.

Statement 1. Let f be a function on a bounded interval I with endpoints a and b and differentiable on $[a, b]$ and satisfying the conditions

a) f has a fixed point $p = f(p)$

b) $|f'(x)| < 1$ for $x \in I$

c) $f(I) \subseteq I$

Then the interval I is in the basin of attraction of p.

Corollary 1. Suppose f is differentiable everywhere and satisfies

a) f has a fixed point $p = f(p)$

b) $|f'(x)| < 1$ everywhere

Then the fixed point p is a global attractor.

We give some examples of how to use the Corollary to see that there is a fixed point which attracts iterates of all other points.

Example. Consider the function $f(x) = (x + \sin(x))/4$. Clearly zero is a fixed point. Calculating the derivative we get $f'(x) = (1 + \cos(x))/4$ and because $-1 \leq \cos(x) \leq 1$ we get $0 \leq f'(x) \leq 1/2$ from which follows that zero is a global attractor.

Example. Consider the function $f(x) = (\sqrt{x^2 + 3})/2$. We seek for fixed points solving $f(x) = x$. Thus a fixed point must be a solution to $(x^2+3)/4 = x^2$. The solutions are -1 and 1. Checking we confirm that the only fixed point is 1. Calculating the first derivative we get $f'(x) = x/(2\sqrt{x^2 + 3})$. Further the second derivative is $f''(x) = 3/(2(x^2+3)^{3/2}) > 0$. Thus f' is an increasing function. Because $f'(x) \to -1/2$ for $x \to -\infty$ and $f'(x) \to 1/2$ for $x \to \infty$ we conclude that the derivative is between -1 and 1 and the fixed point 1 is a global attractor.

Example. Consider the function $f(x) = (x + e^{\sin(x)})/4$. In this case fixed points must be calculated numerically. Because $f(0) = e/4 > 0$ and $f(\pi/2) = (\pi/2 + 1)/4 < \pi/2$ we conclude that must be a fixed point between zero and $\pi/2$ ($f(x) - x$ changes sign in this interval). Numerically we find the fixed point at 0.573. Calculating the derivative we get $f'(x) = (1+\cos(x)e^{\sin(x)})/4$. Because $-1 \leq \sin(x), \cos(x) \leq 1$ we get the estimates $-1 < (1 - e)/4 \leq f'(x) \leq (1 + e)/4 < 1$ and thus the fixed point we found is a global attractor (also meaning that there are no other fixed points).

Exercise. Find out whether there is a fixed point which is a global attractor for the functions below.

a) $f(x) = (x + \cos(2x))/4$

b) $f(x) = 0.9xe^{-x^2}$

c) $f(x) = (\sqrt{x^2 + x + 10})/2$

d) $f(x) = 1/(x^2 + 2)$

e) $f(x) = \ln(x^2 + 2)$

f) $(x + \sin(x) + \cos^2(x))/8$

g) $x/3 + \sin(x)/2 - \cos^2(x)/4$

We proceed to give some statements which can be used to see that the behaviour is simple in some regions and thus helps us to locate intervals to be analyzed more carefully for more complex dynamics.

Statement 2a. Suppose f satisfies the following conditions in an interval $I = [p, a[$:

a) p is fixed point

b) $p < f(x) < x$ for any $x \in I$

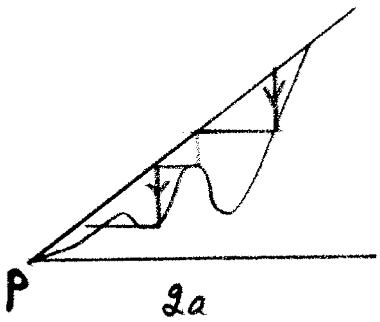
Then the basin of attraction of p contains the interval I.

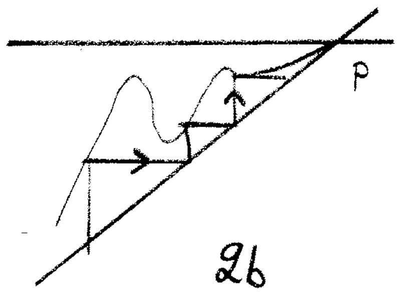
Statement 2b. Suppose f satisfies the following conditions in an interval $I =]a, p]$:

a) p is fixed point

b) $p > f(x) > x$ for any $x \in I$

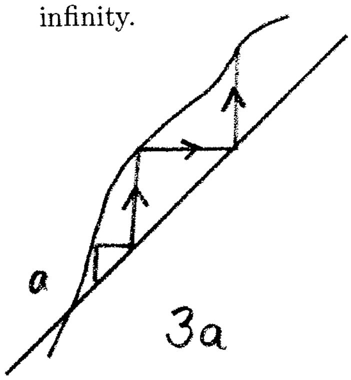
Then the basin of attraction of p contains the interval I.

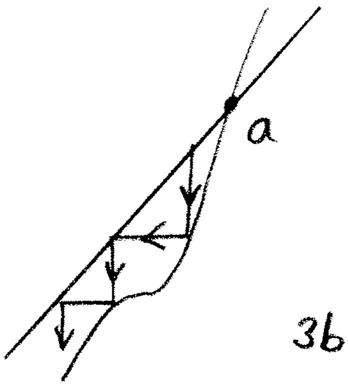




Statement 3a. Suppose f satisifies the conditions $f(x) > x$ in the interval $I =]a, \infty[$. Then the iterates of any point in this interval will tend to plus infinity.

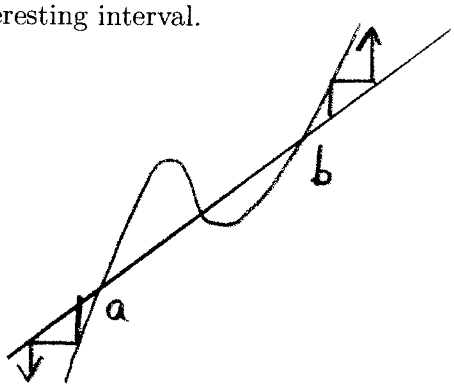
Statement 3b. Suppose f satisifies the conditions $f(x) < x$ in the interval $I =]-\infty, a[$. Then the iterates of any point in this interval will tend to minus infinity.



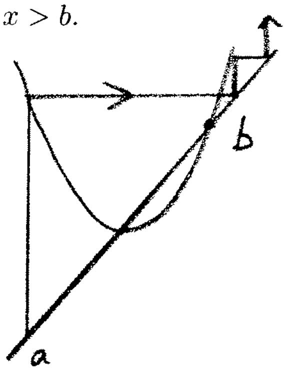


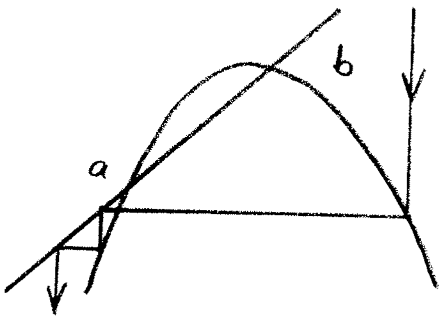
To locate where there might be chaos we find interesting intervals such that the iterates of points outside behave trivially. We dinstinguish four types of such intervals

1) Double repelling interval. In this case there are two fixed points a and b with $a < b$ such that $f(x) > x$ for $x > b$ and $f(x) < x$ for $x < a$. According to Statements 3 the iterates of points greater than b tend to infinity and the iterates of points less than a tend to minus infinity. Thus the interval $[a, b]$ should be an interesting interval.

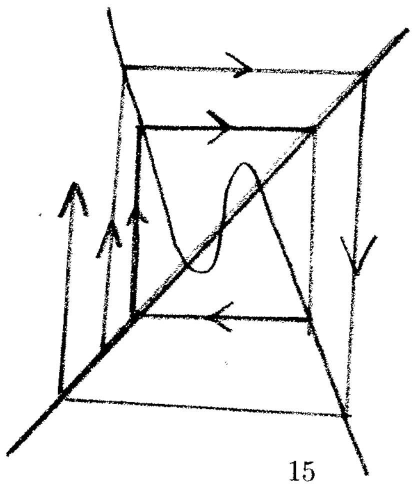


2) Simple repelling interval. In this case there is a fixed point b and a point a such that $f(x) > x$ for $x > b$ and $f(x) > b$ for $x < a$. According Statement 3a the iterates of points greater than b tend to infinity and the first iterate of points less than a is greater than b and further iterates tend to infinity. Thus all iterates outside the interesting interval $[a, b]$ tend to plus infinity. We have an analogous situation where all iterates tend to minus infinity if there is a fixed point a such that $f(x) < x$ for $x < a$ and a point b such that $f(x) < a$ for $x > b$.

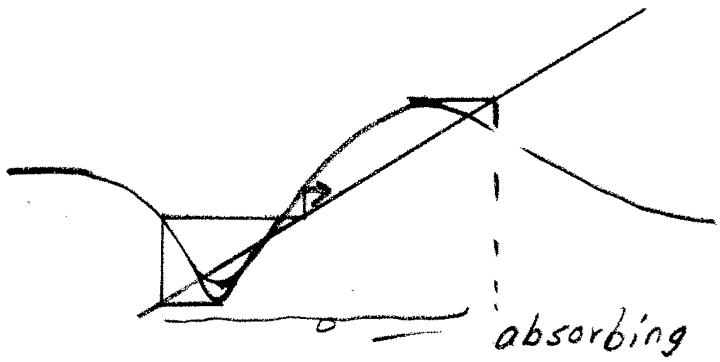




3) Alternate repelling interval. In this case there is a two periodic orbit consisting of $-a$ and $a > 0$ such that $|f(x)| > |x|$ and the absolute value of points with $|x| > a$ tend to infinity. The sign of the iterates will alternate. There is, of course, a more general case when the 2-periodic points are not symmetric with respect to the origin, but we will only consider this symmetric case here.



4) Absorbing interval. The first iterate of all points sometimes enter this interval and after that remain in the interesting interval.



In many cases in practice there should be an absorbing interval where the model is realistic. This concerns, for example, models in vibration mechanics and population dynamics.

We give some examples of how to find these interesting intervals and what can happen inside.

Example. We examine the function $f(x) = x + 0.66(x-1)(x-2)(x+3)$. Solving $f(x) = x$ we get that the function has fixed points at 1,2 and -3. If $x < -3$ then $(x-1)(x-2)(x+3) < 0$ and $f(x) < x$ so iterates of such points will tend to minus infinity. If $x > 2$ then $(x-1)(x-2)(x+3) > 0$ and $f(x) > x$ so iterates of such points will tend to plus infinity. Thus the interval $[-3, 2]$ is an interesting double repelling interval. Iterating the point 1.537 in the interval we get chaos. We also see that this chaotic attractor is not attracting all points in the interesting interval. Often the iterates of points in the interesting interval also tend to plus infinity.

Example. We examine the function $f(x) = x^2 + x - 2.2$. This function has two fixed points $x_\pm = \pm\sqrt{2.2}$. Let us look at the function $g(x) = f(x) - x = x^2 - 2.2$. If $x > x_+$ then $g(x) > 0$ and $f(x) > x$ and the iterates of such points tend to plus infinity. The function has a minumum for $x = 0.5$. For $x < 0.5$ the function is decreasing and for $x \to -\infty$ the values of the function tend to plus infinity. Thus there is a unique point less than 0.5 where $f(x) = x_+$ and this point can be calculated as $u = -1 - \sqrt{2.2}$. If $x < u$ then $f(x) > x_+$ and further iterates will tend to plus infinity. Thus the interval $[u, x_+]$ can be taken as an interesting simple repelling interval. Experiments show that most points in this interval are attracted to a chaotic attractor.

Example. We examine the function

$$f(x) = \begin{cases} 3x/2 & x \le 1/2 \\ 3(1-x)/2 & x \ge 1/2 \end{cases}$$

If $x < 0$ then $f(x) < x$ and iterates of such point tend to minus infinity. If $x > 1$ then $f(x) < 0$ and further iterates will again tend to minus infinity. Thus $[0, 1]$ can be taken as a simple repelling interesting interval.Experiments show that most points in this interval are attracted to a chaotic attractor.

Example. We consider the function $4x\ln(x^2 + 0.5)$. The function has the fixed points zero and $x_\pm = \pm 0.5\sqrt{4e^{1/4} - 2} = \pm 0.886$. If $x > 0$ then $f(x) > x$ for $\ln(x^2 + 0.5) > 1$ which happens exactly for $x > x_+$. Thus iterates of such points tend to plus infinity. If $x < 0$ then $f(x) < x$ for $\ln(x^2 + 0.5) > 1$ which happens exactly for $x < x_-$. Thus iterates of such points tend to minus infinity. Consequently $[x_-, x_+]$ is a double repelling interesting interval. Iterating the point 0.2251 in the interval we get a chaotic attractor.

Example. We consider the function $f(x) = 5/(x^2 - 4x + 5)$. Calculating extrema we get maximum 5 for $x = 2$ and $f(x)$ is always positive. Thus all iterates are in the absorbing interesting interval $[0, 5]$ after first iterate. Iterating the point 4.5498 in the interval we get a chaotic attractor.

Example. We consider the function $f(x) = 6\sin(x)+3\cos(2x)$. Calculating extrema we get maximum 4.5 taken at $x = \pi/6+2n\pi$ and $x = 5\pi/6+2n\pi$ and minimum -9 taken at $x = 3\pi/2 + 2n\pi$. Thus all iterates are in the absorbing interesting interval $[-9, 4.5]$ after first iterate. Iterating points in the interval we get a chaotic attractor.

Example. We consider the function $f(x) = 2xe^{x-x^2}$. Calculating extrema we get maximum 2 at $x = 1$ and minimum $-e^{-3/4}$ at $x = -1/2$. Further $f(x) \to 0$ for $x \to \infty$. Thus all iterates are in the absorbing interesting interval $[-0.47, 2]$ after first iterate. Iterating the point 1.2512 in the interval we get a chaotic attractor. The fixed point -0.47 is also attracting.

Example. We consider the function $f(x) = -4(x^3 - 3x)/5$ Because there is a symmetry we can solve for 2-periodic orbits requiring $f(x) = -x$ giving the 2-periodic points $u_\pm = \pm\sqrt{17}/2$. That they are really 2-periodic can be checked by iterating them. For $|x| < |u_\pm| = \sqrt{17}/2$ we get $x^2 > 17/4$ equivalent with $4/5(x^2 - 3) > 1$ which implies $|f(x)| > |x|$. Thus the absolute value of iterates of all such points tend to infinity. The iterates alternate in sign. We conclude that $[u_-, u_+] = [-2.06, 2.06]$ is an alternate repelling interesting interval. Experiments allow us to find two different chaotic attractors using the intial values ± 1.5647 in the interval.

Exercise. For each of the following functions find an interesting interval where there might be chaos and check whether this a the case by iterating som points in the interesting interval. Of what kind is the interesting interval?

a) $f(x) = (x - 2)^2$
b) $f(x) = -x^2/3 + x + 16/3$
c) $f(x) = -x^2/3 + 5x/3 + 8/3$
d) $f(x) = 2x^3 - 3x$
e) $f(x) = 3x - 2x^3$
f) $f(x) = 10x/(x^2 - 4x + 6)$
g) $f(x) = 3xe^{x-x^2}$
h) $f(x) = x + 1.3(x^2 - 1)(x + 2)$
i) $f(x) = x - 2x^2 + 1.1$
j) $f(x) = 2x(2 - x)$
k) $f(x) = 2.5x\ln(2x^2 + 0.25)$
l) $f(x) - 4x\ln(x^2 + 0.5)$
m) $f(x) = 0.8x^3/(1 - 1.5x^2 + x^4)$
n) $f(x) = 0.77(x^3 - 3x)$

Even if we have found an interesting interval above there need not to be chaos inside. We will look how one can prove that there is no chaos in a special case. In general there can be no chaos if the derivative of the function does not change sign. In our case we have three fixed points a, b and c with $a < b < c$ and $[a, c]$ is a double repelling interesting interval. But the derivative of the function is always positive and thus $x < f(x) < b$ for $a < x < b$ and $b < f(x) < x$ for $b < x < c$ and according to Statements 2 all points in the interesting interval are attracted to b except for endpoints.

We consider some examples.

Example. Consider the function $f(x) = x + 0.4(x^2 - 1)(x - 2)$. Examining the sign of $(x^2 - 1)(x - 2)$ we conclude that $f(x) > x$ for $x > 2$ and $f(x) < x$

for $x < -1$ so $[-1, 2]$ should be a double repelling interval. However it is not so interesting, at least, there is no chaos. Calculating the derivative we get $f'(x) = 1.2x^2 - 1.6x + 0.6$ which is always greater than zero because $4 * 1.2 * 0.6 > 1.6^2$.Thus f is increasing everywhere. The fixed points of f are -1, 1 and 2 and $1 > f(x) > x$ for $-1 < x < 1$ and $1 < f(x) < x$ for $1 < x < 2$. Thus the fixed point 1 is attracting the whole interesting interval except the endpoints.

Example. Consider the function

$$f(x) = \begin{cases} 2x + 3 & x \leq -2 \\ x/2 & -2 < x < 2 \\ 2x - 3 & x \geq 2 \end{cases}$$

The function has three fixed points: repelling -3 and 3 and attracting zero. For $x > 3$ we get $f(x) > x$ and for $x < -3$ we get $f(x) > x$, so the interval between -3 and 3 is a double repelling interesting interval. Anyhow for $-3 < x < 0$ we get $x < f(x) < 0$ and for $0 < x < 3$ we get $0 < f(x) < x$ and thus zero is attracting the whole interesting interval except endpoints.

Exercise. Show that the following functions have a double repelling interesting interval without chaos.

a) $f(x) = x + 0.4(x^2 - 1)(x + 2)$

b)

$$f(x) = \begin{cases} 3x + 5 & x \leq -2 \\ x/2 & -2 < x < 2 \\ 3x - 5 & x \geq 2 \end{cases}$$

We shall look how we can use the statements for global behaviour to examine the behaviour of iterates for $f(x) = rxe^{-x}$ and $f(x) = rx^2e^{-x}$. In the case the graph is below the diagonal it follows from Statement 2a that zero is a global attractor for positive x in both cases. If $1 < r < e$ we can use Statements 2 to prove that $\ln r$ is a global attractor for $f(x) = rxe^{-x}$. We can always use Statement 2 to prove that zero attracts the intevals $(0, x_-)$ and (x_c, ∞). for $f(x) = rx^2e^{-x}$.

Exercise. Examine in the same way the dynamics when $f(x) = rx/(1+x^\gamma)$ and $f(x) = rx^2/(1 + x^\gamma)$ for $\gamma \geq 3$, and $x \geq 0$.

Exercise. Can we use the statements to prove something for the Impact map?