

fixed point (k^2, k) , k real, has linearization $\dot{y} = Ay$, where

$$A = \begin{bmatrix} 1 & -2k \\ k & -2k^2 \end{bmatrix}. \tag{3.30}$$

Clearly, $\det(A) = 0$ for all k and so the fixed points are all non-simple. The phase portrait for (3.29) is shown in Fig. 3.11.

In view of the above observations, it is not surprising that there is no detailed classification of non-simple fixed points. However, the following definitions of stability (which apply to both simple and non-simple fixed points) do provide a coarse classification of qualitative behaviour.

3.5 STABILITY OF FIXED POINTS

It can be shown that the local phase portrait in the neighbourhood of any fixed point falls into one, and only one, of three stability types: asymptotically stable, neutrally stable or unstable. The following definition of a stable fixed point plays a central role in distinguishing these stability types.

Definition 3.5.1

A fixed point \mathbf{x}_0 of the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is said to be **stable** if, for every neighbourhood N of \mathbf{x}_0 , there is a smaller neighbourhood $N' \subseteq N$ of \mathbf{x}_0 such that every trajectory which passes through N' remains in N as t increases.

This characterization of a stable fixed point is associated with the Russian mathematician Liapunov; indeed, it is often referred to as 'stability in the sense of Liapunov'.

Definition 3.5.2

A fixed point \mathbf{x}_0 of the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ is said to be **asymptotically stable** if it is stable and there is a neighbourhood N of \mathbf{x}_0 such that every trajectory passing through N approaches \mathbf{x}_0 as t tends to infinity.

We have encountered asymptotic stability in connection with linear systems whose coefficient matrix A satisfies $\text{tr}(A) < 0, \det(A) > 0$ (illustrated in Fig. 2.7). For the canonical systems of this type, the trajectory of every point outside the disc $D_r = \{(y_1, y_2) | (y_1^2 + y_2^2)^{1/2} < r\}$ enters the disc once and subsequently remains within it. Let N be any neighbourhood of the origin and D_r be a disc, centred on the origin, that is wholly contained in N . Then Definition 3.5.1 is satisfied with $N' = D_r$ and the origin is a stable fixed point. What is more, the trajectories of all points within D_r approach the origin as t tends to infinity and therefore the origin is an asymptotically stable fixed point by Definition 3.5.2.

We can widen this result to linear systems of the above kind that are not in canonical form by using the non-singular matrix, M , relating the system

satisfy Definitions 3.5.1 and 3.5.2 for the canonical system provide corresponding neighbourhoods for the non-canonical system when mapped into the non-canonical phase plane. This is an example of the more general result that qualitatively equivalent fixed points must have the same stability type. Simple, non-linear fixed points can be tackled in a similar manner by making use of the local qualitative equivalence that is given by the linearization theorem. For example, the non-linear system

$$\dot{x}_1 = -x_1 + x_2 - x_1^3, \quad \dot{x}_2 = -x_1 - x_2 + x_2^2 \tag{3.31}$$

has an asymptotically stable fixed point at the origin. This follows because: the linearized system

$$\dot{x}_1 = -x_1 + x_2, \quad \dot{x}_2 = -x_1 - x_2 \tag{3.32}$$

has eigenvalues $-1 \pm i$, so that the origin is a stable focus; and the existence of neighbourhoods of the kind required by Definitions 3.5.1 and 3.5.2 in the phase plane of (3.31) can be demonstrated by using the continuous bijection that relates the local phase portrait of (3.31) to that of (3.32) at the origin.

It is tempting to think that a fixed point must be asymptotically stable if the trajectories of the system tend to it as $t \rightarrow \infty$. However, this is not the case. The non-simple system

$$\begin{aligned} \dot{x}_1 &= \frac{x_1^2(x_2 - x_1) + x_2^2}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}, \\ \dot{x}_2 &= \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)(1 + (x_1^2 + x_2^2)^2)}. \end{aligned} \tag{3.33}$$

shows that such fixed points may not be stable in the sense of Definition 3.5.1 (as shown in Exercise 3.12) and, consequently, they fail to be asymptotically stable according to Definition 3.5.2.

Every asymptotically stable fixed point is stable by Definition 3.5.2; however, the converse is not true.

Example 3.5.1

Show that the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1^3 \tag{3.34}$$

is stable at the origin but not asymptotically stable.

Solution

The fixed point at the origin of (3.34) is non-simple (the linearized system is $\dot{x}_1 = x_2, \dot{x}_2 = 0$) so that the linearization theorem does not provide a local phase portrait. However, the shape of the trajectories is given by

$$dx_2 = x_1^3$$

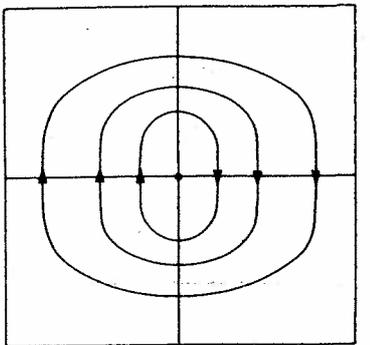


Fig. 3.12. Phase portrait for the system (3.34). Trajectories satisfy $\frac{1}{2}x_1^2 + x_2^2 = C$. Orientation given by $\dot{x}_1 > 0$ for $x_2 > 0$.

which has solutions satisfying

$$\frac{1}{2}x_1^2 + x_2^2 = C, \quad (3.36)$$

where C is a real constant. The phase portrait is shown in Fig. 3.12.

None of the trajectories approach the origin as $t \rightarrow \infty$, so the fixed point is not asymptotically stable. However, as Fig. 3.13 shows, for every disc N centred on the origin, there is a smaller disc N' such that every trajectory passing through N' remains in N . Thus the origin is stable. \square

Definition 3.5.3

A fixed point of the system $\dot{x} = X(x)$ which is stable but not asymptotically stable is said to be **neutrally stable**.

There are many examples of neutrally stable fixed points similar to Example 3.5.1. For instance, the non-trivial fixed point of the Volterra-Lotka equations

$$\begin{aligned} \dot{x}_1 &= x_1(a - bx_2), & \dot{x}_2 &= -x_2(c - dx_1), \end{aligned} \quad (3.37)$$

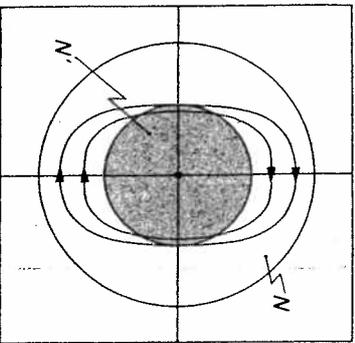


Fig. 3.13. Typical neighbourhoods N and N' (shaded) or Definition 3.5.1. Observe all trajectories passing through N' remain in N .

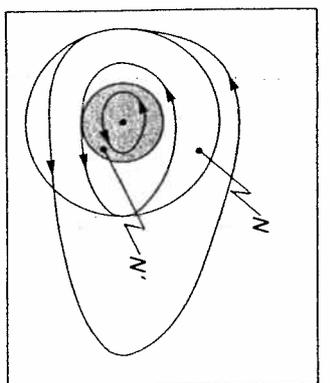


Fig. 3.14. Typical neighbourhoods N and N' for the Volterra-Lotka equations showing neutral stability.

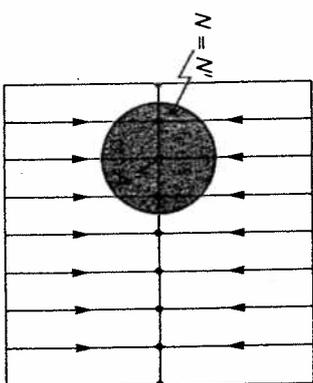


Fig. 3.15. Neutral stability of the system $\dot{x}_1 = 0$, $\dot{x}_2 = -x_2$ at A follows with $N = N'$.

$a, b, c, d > 0$, is neutrally stable. The phase portrait for these equations was given in Fig. 1.33. Neutral stability of the fixed point at $(c/d, a/b)$ follows from the existence of neighbourhoods N and N' satisfying Definition 3.5.1 as indicated in Fig. 3.14. Clearly, the fixed point is not asymptotically stable.

Another example is the non-simple linear fixed point shown in Fig. 3.15. The particular fixed point A is not asymptotically stable because there are trajectories passing through N of Fig. 3.15 which do not approach A as $t \rightarrow \infty$. However, with $N' = N$, every trajectory passing through N' remains in N , so A is stable.

Definition 3.5.4

A fixed point of the system $\dot{x} = X(x)$ which is not stable is said to be **unstable**.

This means that there is a neighbourhood N of the fixed point such that for every neighbourhood $N' \subseteq N$ there is at least one trajectory which passes through N' and does not remain in N . For example, the saddle point is unstable because there is a separatrix, containing points arbitrarily close to the origin, which escapes to infinity with increasing t .

The examples discussed above do not provide a practical procedure for determining the stability type of any given fixed point. An approach that can work for both simple and non-simple fixed points is to find a **Liapunov function** for the system.

For example, suppose we wish to investigate the nature of the fixed point at the origin of the system

$$\dot{x}_1 = -x_1^3, \quad \dot{x}_2 = -x_2^3. \quad (3.38)$$

The linearization theorem is of no use here as the linearized system is clearly non-simple. However, we can show that the origin is asymptotically stable by examining how the function $V(x_1, x_2) = x_1^2 + x_2^2$ changes along the trajectories of (3.38).

Let $\mathbf{x}(t) = (x_1(t), x_2(t))$ be any solution curve of system (3.38). Then

$$\dot{V}(\mathbf{x}(t)) = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 = -2(x_1^4 + x_2^4). \quad (3.39)$$

Therefore, $\dot{V}(\mathbf{x}(t))$ is negative at all points other than the origin of the x_1, x_2 -plane and so $V(\mathbf{x}(t))$ decreases as t increases. This means that the phase point $\mathbf{x}(t)$ moves towards the origin with increasing t . In fact, $\dot{V}(\mathbf{x}(t)) < 0$ for $\mathbf{x}(t) \neq \mathbf{0}$ implies that $V(\mathbf{x}(t)) \rightarrow 0$ as $t \rightarrow \infty$ and hence $\mathbf{x}(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$. Thus, the origin is an asymptotically stable fixed point of system (3.38).

The above example is a simple illustration of the use of a Liapunov function. To develop this idea we will need the following definitions.

Definition 3.5.5

A real-valued function $V: N \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, where N is a neighbourhood of $\mathbf{0} \in \mathbb{R}^2$, is said to be **positive (negative) definite** in N if $V(\mathbf{x}) > 0$ ($V(\mathbf{x}) < 0$) for $\mathbf{x} \in N \setminus \{\mathbf{0}\}$ and $V(\mathbf{0}) = 0$.

Definition 3.5.6

A real-valued function $V: N \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, where N is a neighbourhood of $\mathbf{0} \in \mathbb{R}^2$, is said to be **positive (negative) semi-definite** in N if $V(\mathbf{x}) \geq 0$ ($V(\mathbf{x}) \leq 0$) for $\mathbf{x} \in N \setminus \{\mathbf{0}\}$ and $V(\mathbf{0}) = 0$.

Definition 3.5.7

The derivative of $V: N \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ along a parameterized curve given by $\mathbf{x}(t) = (x_1(t), x_2(t))$ is defined by

$$\dot{V}(\mathbf{x}(t)) = \frac{\partial V(\mathbf{x}(t))}{\partial x_1} \dot{x}_1(t) + \frac{\partial V(\mathbf{x}(t))}{\partial x_2} \dot{x}_2(t). \quad (3.40)$$

The function $V(x_1, x_2) = x_1^2 + x_2^2$ used in the introductory example is positive definite on \mathbb{R}^2 . This function is typical of the positive definite functions used in this section. Any continuously differentiable, positive definite function V has a continuum of closed level curves (defined by $V(x_1, x_2) = \text{constant}$)

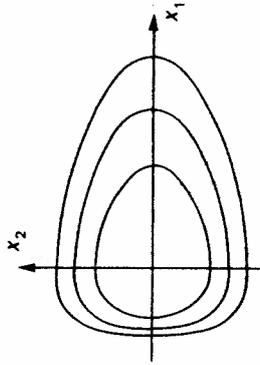


Fig. 3.16. The level curves $V(x_1, x_2) = C$ of the positive definite function $V(x_1, x_2) = x_1 - \log(1 + x_1) + x_2^2$ for $C = 0.5, 1.0, 1.5$.

around the origin. Of course, such curves are not necessarily circular (as shown in Fig. 3.16). However, provided \dot{V} is negative on a trajectory, then that trajectory must still move towards the origin, because V is decreasing. Observe that for any system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$,

$$\dot{V}(\mathbf{x}(t)) = \left(\frac{\partial V}{\partial x_1} X_1 + \frac{\partial V}{\partial x_2} X_2 \right) \quad (3.41)$$

is a function of x_1 and x_2 only and, for this reason, it is often denoted by $\dot{V}(\mathbf{x})$.

Theorem 3.5.1 (Liapunov stability theorem)

Suppose the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$, $\mathbf{x} \in S \subseteq \mathbb{R}^2$ has a fixed point at the origin. If there exists a real-valued function V in a neighbourhood N of the origin such that:

1. the partial derivatives $\partial V/\partial x_1, \partial V/\partial x_2$ exist and are continuous;
 2. V is positive definite;
 3. \dot{V} is negative semi-definite;
- then the origin is a **stable** fixed point of the system.
If 3 is replaced by the stronger condition
- 3'. \dot{V} is negative definite,
- then the origin is an **asymptotically stable** fixed point.

Proof

Properties 1 and 2 imply that the level curves of V form a continuum of closed curves around the origin. Thus, there is a positive k such that $N_k = \{\mathbf{x} \mid V(\mathbf{x}) < k\}$ is a neighbourhood of the origin contained in N . If $\mathbf{x}_0 \in N_k \setminus \{\mathbf{0}\}$, then $V(\phi_t(\mathbf{x}_0)) \leq 0$, for all $t \geq 0$, by 3 and $V(\phi_t(\mathbf{x}_0))$ is a non-increasing function of t . Therefore, $V(\phi_t(\mathbf{x}_0)) < k$, for all $t \geq 0$, and so $\phi_t(\mathbf{x}_0) \in N_k$ for all $t \geq 0$. Consequently, by Definition 3.5.1 the fixed point is stable.

For case 3' we obtain the asymptotic stability of the origin by the following argument. $V(\phi_t(\mathbf{x}_0))$ is a strictly decreasing function of t and $V(\phi_{t_2}(\mathbf{x}_0)) - V(\phi_{t_1}(\mathbf{x}_0)) < k$ for all $t_2 > t_1 \geq 0$. The mean value theorem then gives the existence of a sequence $\{t_i\}_{i=1}^{\infty}$, tending to infinity, such that $V(\phi_{t_i}(\mathbf{x}_0)) \rightarrow 0$ as $t_i \rightarrow \infty$. This, in turn, implies that $\phi_{t_i}(\mathbf{x}_0) \rightarrow \mathbf{0}$ as $t_i \rightarrow \infty$ because \dot{V} is

negative definite. Now, $V(\phi_t(\mathbf{x}_0)) < V(\phi_{\tau_1}(\mathbf{x}_0))$ for all $t > \tau_1$, because $V(\phi_t(\mathbf{x}_0))$ is decreasing. However, V is positive definite and therefore $\{\phi_t(\mathbf{x}_0) | t > \tau_1\}$ lies inside the level curve of V containing $\phi_{\tau_1}(\mathbf{x}_0)$. This is true for every τ_1 , so that ' $\phi_{\tau_1}(\mathbf{x}_0) \rightarrow \mathbf{0}$ as $\tau_1 \rightarrow \infty$ ' implies ' $\phi_t(\mathbf{x}_0) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$ '. Moreover, the above argument is valid for all \mathbf{x}_0 in N_1 and therefore $\mathbf{x} = \mathbf{0}$ is an asymptotically stable fixed point. \square

Definition 3.5.8

A function V satisfying hypotheses 1, 2 and 3 of Theorem 3.5.1 is called a **weak Liapunov function**. If 3 is replaced by 3' then V is a **strong Liapunov function**.

Example 3.5.2

Prove that the function

$$V(y_1, y_2) = y_1^2 + y_1^2 y_2^2 + y_2^4, \quad (y_1, y_2) \in \mathbb{R}^2 \quad (3.42)$$

is a strong Liapunov function for the system

$$\begin{aligned} \dot{x}_1 &= 1 - 3x_1 + 3x_1^2 + 2x_2^2 - x_1^3 - 2x_1 x_2^2 \\ \dot{x}_2 &= x_2 - 2x_1 x_2 + x_1^2 x_2 - x_2^3, \end{aligned} \quad (3.43)$$

at the fixed point $(1, 0)$.

Solution

On introducing local coordinates y_1, y_2 at $(1, 0)$, (3.43) becomes

$$\dot{y}_1 = -y_1^3 - 2y_1 y_2^2, \quad \dot{y}_2 = y_1^2 y_2 - y_2^3. \quad (3.44)$$

The function V in (3.42) is positive definite and

$$\begin{aligned} \dot{V}(y_1, y_2) &= \frac{\partial V}{\partial y_1} \dot{y}_1 + \frac{\partial V}{\partial y_2} \dot{y}_2 \\ &= (2y_1 + 2y_1 y_2^2)(-y_1^3 - 2y_1 y_2^2) \\ &\quad + (2y_1^2 y_2 + 4y_2^3)(y_1^2 y_2 - y_2^3) \\ &= -2y_1^4 - 4y_1^2 y_2^2 - 2y_1^2 y_2^4 - 4y_2^6 \end{aligned}$$

is negative definite. Therefore, V is a strong Liapunov function for (3.43). \square

Example 3.5.3

Investigate the stability of the second-order equation

$$\ddot{x} + \dot{x}^3 + x = 0 \quad (3.45)$$

at the origin of its phase plane.

Solution

If $x_1 = x$ and $x_2 = \dot{x}$, then

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= -x_1 - x_2^3 \end{aligned} \quad (3.46)$$

is the first-order system of (3.45). The derivative of the function $V(x_1, x_2) = x_1^2 + x_2^2$ along the trajectories of (3.46) is

$$\dot{V}(x_1, x_2) = -2x_2^4$$

and so V is only negative semi-definite. Hence by Theorem 3.5.1 the origin is a stable fixed point of system (3.46). \square

In fact, asymptotic stability can be deduced for some systems having a weak Liapunov function similar to that in Example 3.5.3. Observe that $\dot{V}(\mathbf{x})$ only fails to be negative away from the origin on the line $x_2 = 0$. On this line the components of the vector field given by (3.46) are $\dot{x}_1 = 0, \dot{x}_2 = -x_1$. Thus, all trajectories (except the origin) cross the line $x_2 = 0$ and \dot{V} is only momentarily zero. At all other points in the plane it is negative. Under these circumstances the following theorem gives asymptotic stability.

Theorem 3.5.2

If there exists a weak Liapunov function V for the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ in a neighbourhood of an isolated fixed point at the origin, then providing $V(\mathbf{x})$ does not vanish identically on any trajectory, other than the fixed point itself, the origin is asymptotically stable.

Example 3.5.4

Show that all trajectories of the system

$$\begin{aligned} \dot{x}_1 &= x_2, & \dot{x}_2 &= -x_1 - (1 - x_1^2)x_2 \end{aligned} \quad (3.47)$$

passing through points (x_1, x_2) , with $x_1^2 + x_2^2 < 1$, tend to the origin with increasing t .

Solution

The function $V(x_1, x_2) = x_1^2 + x_2^2$ is a weak Liapunov function in the region $x_1^2 + x_2^2 < 1$ ($\dot{V}(x_1, x_2) = -2x_2^2(1 - x_1^2)$). The function V vanishes only on the lines $x_2 = 0$ and $x_1 = \pm 1$. However, there are no trajectories of (3.47) which lie on these lines because on $x_2 = 0, \dot{x}_2 = -x_1 \neq 0$ and on $x_1 = \pm 1, \dot{x}_1 = x_2 \neq 0$. Therefore, by Theorem 3.5.2, the origin is asymptotically stable. Moreover the arguments used to prove Theorem 3.5.1 show that any trajectory $\phi_t(\mathbf{x}_0), |\mathbf{x}_0| < 1$, has the property $\lim_{t \rightarrow \infty} \phi_t(\mathbf{x}_0) = \mathbf{0}$. \square

The fact that the origin is an asymptotically stable fixed point of system (3.47) can be deduced by using the linearization theorem. However, the above solution provides an explicit 'domain of stability' or 'basin of attraction' $N = \{(x_1, x_2) | x_1^2 + x_2^2 < 1\}$. All trajectories through points of N approach the origin as t increases. The linearization theorem gives the existence of a domain of stability but no indication of its size.

Theorem 3.5.3

Suppose the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ has a fixed point at the origin. If a real-valued, continuous function V exists such that:

1. the domain of V contains $N = \{x \mid |x| \leq r\}$ for some $r > 0$;
2. there are points arbitrarily close to the origin at which V is positive;
3. V is positive definite; and
4. $V(0) = 0$,

then the origin is unstable.

Proof

We show that for every point x_0 in N , with $V(x_0) > 0$, the trajectory $\phi_t(x_0)$ leaves N for sufficiently large positive t . By hypothesis, such points can be chosen arbitrarily close to the origin and therefore the origin is unstable.

Given r_1 , such that $0 < r_1 < r$, there is a point $x_0 \neq 0$, with $|x_0| < r_1$ and $V(x_0) > 0$. The function V is positive definite in N and so $V(\phi_t(x_0))$ is an increasing function of t . Therefore, the trajectory $\phi_t(x_0)$ does not approach the origin as t increases. Hence $V(\phi_t(x_0))$ will be bounded away from zero, i.e. there exists a positive K such that $V(\phi_t(x_0)) \geq K$ for all positive t . If we assume the trajectory $\phi_t(x_0)$ remains in N , then

$$V(\phi_t(x_0)) - V(x_0) \geq Kt, \tag{3.48}$$

for all positive t and $V(\phi_t(x_0))$ becomes arbitrarily large in N . This contradicts the hypothesis that V is a continuous function defined on the closed and bounded set N . Thus the trajectory $\phi_t(x_0)$ must leave N as t increases. \square

Example 3.5.5

Show that the system

$$\dot{x}_1 = x_1^2, \quad \dot{x}_2 = 2x_2^2 - x_1x_2 \tag{3.49}$$

is unstable at the origin by using the function

$$V(x_1, x_2) = \alpha x_1^3 + \beta x_1^2 x_2 + \gamma x_1 x_2^2 + \delta x_2^3, \tag{3.50}$$

for a suitable choice of constants $\alpha, \beta, \gamma, \delta$.

Solution

The derivative of V along the trajectories of system (3.49) is

$$\begin{aligned} \dot{V}(x_1, x_2) &= 3\alpha x_1^2 + \beta x_1^3 x_2 \\ &\quad + (2\beta - \gamma)x_1^2 x_2^2 + (4\gamma - 3\delta)x_1 x_2^3 \\ &\quad + 6\delta x_2^4. \end{aligned} \tag{3.51}$$

Observe that if we choose $\alpha = \frac{1}{3}, \beta = 4, \gamma = 2, \delta = \frac{4}{3}$ then the various terms of \dot{V} can be grouped together to form

$$\begin{aligned} \dot{V}(x_1, x_2) &= x_1^4 + 4x_1^3 x_2 + 6x_1^2 x_2^2 + 4x_1 x_2^3 + 8x_2^4 \\ &= (x_1 + x_2)^4 + 7x_2^4 \end{aligned} \tag{3.52}$$

which is clearly positive definite. The function V is given by

$$V(x_1, x_2) = \frac{1}{3}x_1^3 + 4x_1^2 x_2 + 2x_1 x_2^2 + \frac{4}{3}x_2^3. \tag{3.53}$$

This function has the property that $V(x_1, x_2) = \frac{1}{3}x_1^3$ when $x_2 = 0$, and so points arbitrarily close to the origin on the x_1 -axis can be found for which V is positive. It follows that the origin is an unstable fixed point by Theorem 3.5.3. \square

3.6 ORDINARY POINTS AND GLOBAL BEHAVIOUR

3.6.1 Ordinary points

Any point in the phase plane of the system $\dot{x} = X(x)$ which is not a fixed point is said to be an **ordinary point**. Thus, if x_0 is an ordinary point then $X(x_0) \neq 0$ and, by the continuity of X , there is a neighbourhood of x_0 containing only ordinary points. This means that the local phase portrait at an ordinary point has no fixed points. There is an important result concerning the qualitative equivalence of such local phase portraits—the **flow box theorem** (Hirsch and Smale, 1974).

Consider the local phase portraits at a typical ordinary point x_0 shown in Figs 3.17–3.20. In each case, a special neighbourhood of x_0 , called a flow box, is shown. The trajectories of the system enter at one end and flow out through the other; no trajectories leave through the sides. For each phase portrait shown, we can find new coordinates for the plane such that the local phase portrait in the flow box looks like the one shown in Fig. 3.17. For example, in Fig. 3.18 we take polar coordinates (r, θ) . In the r, θ -plane the circles ($r = \text{constant}$) become straight lines parallel to the $r = 0$ axis and the radial lines ($\theta = \text{constant}$) become straight lines parallel to the $r = 0$ axis. Thus, the phase portrait in the flow box in Fig. 3.18 is, in the r, θ -plane, the same as Fig. 3.17.

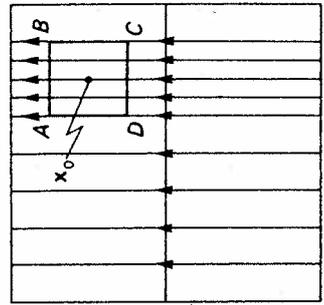


Fig. 3.17. System $\dot{x}_1 = 0, \dot{x}_2 = 1$ with typical flow box.

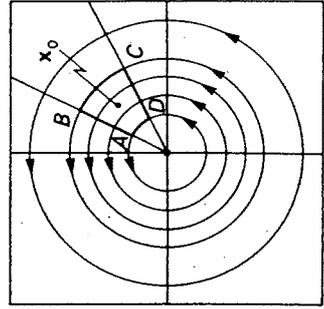


Fig. 3.18. System $\dot{x}_1 = -x_2, \dot{x}_2 = x_1$. In polar coordinates the phase portrait is the same as Fig. 3.17.

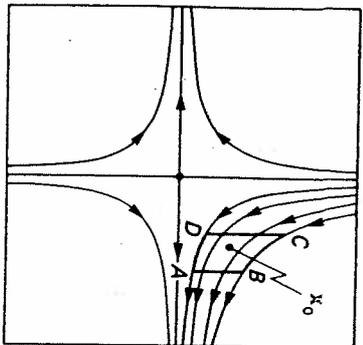


Fig. 3.19. System $\dot{x}_1 = x_1, \dot{x}_2 = -x_2$. The variables $y_1 = x_1 x_2$ and $y_2 = \ln x_1, x_1 > 0$, satisfy $\dot{y}_1 = 0, \dot{y}_2 = 1$.

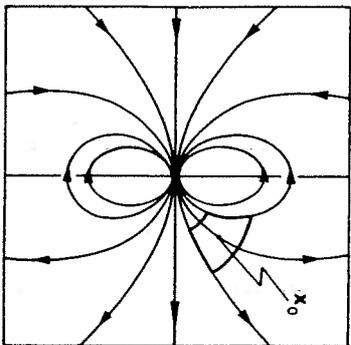


Fig. 3.20. The flow box theorem guarantees the existence of coordinates which transform the local phase portrait at x_0 into the form shown in Fig. 3.17.

For Fig. 3.19, the trajectories in the neighbourhood of x_0 lie on hyperbolae $x_1 x_2 = K > 0$. If we introduce variables $y_1 = x_1 x_2$ and $y_2 = \ln x_1$, then the flow box is bounded by the coordinate lines $y_1 = \text{constant}$ and $y_2 = \text{constant}$ and in the y_1, y_2 -plane the local phase portrait again looks like Fig. 3.17.

Theorem 3.6.1 (Flow box theorem)

In a sufficiently small neighbourhood of an ordinary point x_0 of the system $\dot{x} = X(x)$ there is a differentiable change of coordinates $y = y(x)$ such that $\dot{y} = (0, 1)$.

The flow box theorem guarantees the existence of new coordinates with the above property, at least in some neighbourhood of any ordinary point of any system. Thus, local phase portraits at ordinary points are all qualitatively equivalent.

3.6.2 Global phase portraits

The linearization and flow box theorems provide local phase portraits at most simple fixed points and all ordinary points. However, this information does not always determine the complete phase portrait of a system.

Example 3.6.1

Find and classify the fixed points of the system

$$\dot{x}_1 = 2x_1 - x_1^2, \quad \dot{x}_2 = -x_2 + x_1 x_2. \tag{3.54}$$

Discuss possible phase portraits for the system.

Solution

The system has fixed points at $A = (0, 0)$ and $B = (2, 0)$. The linearized systems are:

$$\dot{x}_1 = 2x_1, \quad \dot{x}_2 = -x_2 \quad \text{at } A; \tag{3.55}$$

and

$$\dot{y}_1 = -2y_1, \quad \dot{y}_2 = y_2 \quad \text{at } B. \tag{3.56}$$

The linearization theorem implies that (3.54) has saddle points at A and B . Furthermore, the non-linear separatrices of these saddle points are tangent to the principal directions at A and B . For (3.55) and (3.56) the principal directions coincide with the local coordinate axes.

This information is insufficient to determine the qualitative type of the global phase portrait. For example, Fig. 3.21 shows two phase portraits that are consistent with the local behaviour. The phase portraits shown are not qualitatively equivalent because the two saddle points have a common separatrix or **saddle connection** in Fig. 3.21(a) whereas in Fig. 3.21(b) they do not. This is a qualitative difference; there is no continuous bijection that relates the two phase portraits.

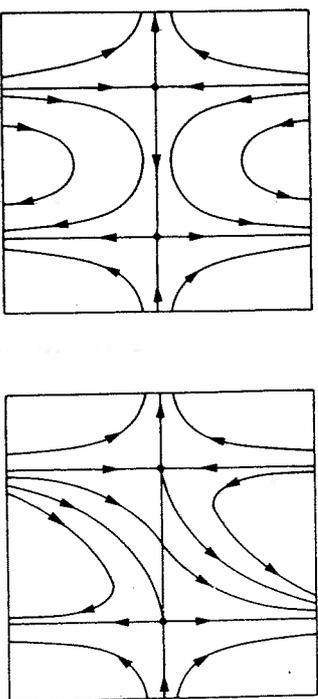


Fig. 3.21. Two qualitatively different phase portraits compatible with the local phase portraits obtained from the linearization theorem.

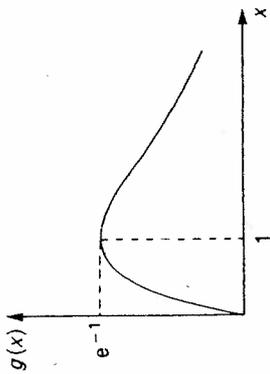


Fig. 3.23. Plot xe^{-x} versus x for $x \geq 0$.

is separable, with solutions satisfying

$$g(x_1)g(x_2) = K, \quad (3.76)$$

where $g(x) = xe^{-x}$ and K is a positive constant. The function g is shown in Fig. 3.23 for $x \geq 0$; it has a single maximum at $x = 1$ where $g(1) = e^{-1}$. It follows that $(x_1, x_2) = (1, 1)$ is a maximum of the first integral $g(x_1)g(x_2)$. This means that there is a neighbourhood of $(1, 1)$ in which the level curves of $g(x_1)g(x_2)$ are closed. Since, these level curves coincide with trajectories, we conclude that $(1, 1)$ is a centre. \square

It is important to realize that first integrals do not give solutions $\mathbf{x}(t)$ for a system; rather they provide the shape of the trajectories.

Example 3.7.4

Show that the systems

$$\dot{x}_1 = x_1, \quad \dot{x}_2 = -x_2 \quad (3.77)$$

and

$$\dot{x}_1 = x_1(1 - x_2), \quad \dot{x}_2 = -x_2(1 - x_2) \quad (3.78)$$

have the same first integral and sketch their phase portraits.

Solution

The trajectories of both systems lie on the solutions of

$$\frac{dx_2}{dx_1} = -\frac{x_2}{x_1}, \quad x_1 \neq 0, \quad (3.79)$$

and both have

$$f(\mathbf{x}) = x_1 x_2 \quad (3.80)$$

as a first integral on \mathbb{R}^2 . The level curves of f are rectangular hyperbolae which, for (3.77), can be oriented by noting the direction of $\dot{\mathbf{x}}$ on the coordinate axes. This is simply the linear saddle familiar from section 2.3.

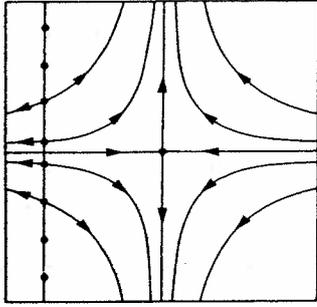


Fig. 3.24. Phase portrait for system (3.78). This system has the same first integral as the linear saddle.

The system (3.78) has fixed points at the origin and everywhere on the line $x_2 = 1$. Furthermore, $\dot{x}_2 > 0$ for $x_2 > 1$ and for $x_2 < 0$ while $\dot{x}_2 < 0$ for $0 < x_2 < 1$. Therefore the phased portrait must be that shown in Fig. 3.24. \square

3.8 LIMIT POINTS AND LIMIT CYCLES

Let \mathbf{x} be a point in the phase portrait of a flow ϕ . The α -(ω -) limit set, $L_\alpha(\mathbf{x})$ ($L_\omega(\mathbf{x})$), of \mathbf{x} contains those points that are approached by the trajectory through \mathbf{x} as t tends to $-\infty$ ($+\infty$). These limit points of \mathbf{x} are defined as follows.

Definition 3.8.1

A point \mathbf{y} is said to be an α -(ω -) limit point of \mathbf{x} if there exists a sequence of times $\{t_n\}_{n=1}^\infty$, tending to minus (plus) infinity as $n \rightarrow \infty$, such that $\lim_{n \rightarrow \infty} \phi_{t_n}(\mathbf{x}) = \mathbf{y}$.

Consider the phase portrait illustrated in Fig. 3.25. For any ordinary point,

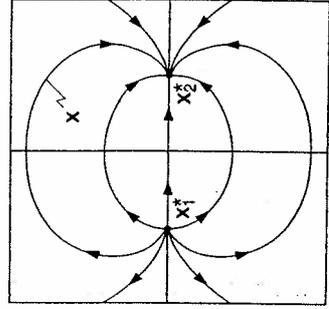


Fig. 3.25. Fixed points are the simplest examples of α - and ω -limit sets. Here $L_\alpha(\mathbf{x}) = \mathbf{x}_1^*$ and $L_\omega(\mathbf{x}) = \mathbf{x}_2^*$ for every ordinary point \mathbf{x} .

x , $\phi_1(x)$ tends to x_1^* as $t \rightarrow -\infty$ and to x_2^* as $t \rightarrow +\infty$, so that $L_\alpha(x) = \{x_1^*\}$ and $L_\omega(x) = \{x_2^*\}$. If $x = x_1^*$ or x_2^* , then $\phi_1(x) = x$ for all t and $L_\alpha(x) = L_\omega(x) = \{x\}$. In both cases, any sequence of times, tending to the appropriate limit, will suffice in Definition 3.8.1.

Example 3.8.1

Let ϕ , be the flow on \mathbb{R}^2 of the differential equation with polar form

$$\dot{r} = ar(1-r), \quad \dot{\theta} = 1, \tag{3.81}$$

where a is a positive constant. Find $L_\alpha(x)$ and $L_\omega(x)$ for $x \neq 0$.

Solution

The phase portrait for the system (3.81) is shown in Fig. 3.26. The main features are the unstable focus at the origin and the closed orbit, C , given by $r \equiv 1$.

To find $L_\omega(x)$ for $x \neq 0$, let y be any point of C and take $\{t_n\}_{n=1}^\infty$ to be the sequence of $t > 0$ at which the trajectory through x crosses the radial line through y (see Fig. 3.26). Clearly, $t_n \rightarrow \infty$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \phi_{t_n}(x) = y$. In particular, if x lies in C then $\phi_{t_n}(x) = y$ for each n . Thus every point of C is an ω -limit point of x by Definition 3.8.1 and $L_\omega(x) = C$ for any $x \neq 0$.

Let us now turn to $L_\alpha(x)$. If $|x| \leq 1$, then a similar definition of $\{t_n\}_{n=1}^\infty$ with $t < 0$ allows us to show that

$$L_\alpha(x) = \begin{cases} \{0\} & \text{if } |x| < 1 \\ C & \text{if } |x| = 1. \end{cases} \tag{3.82}$$

However, if $|x| > 1$ there is no sequence $\{t_n\}_{n=1}^\infty$, with $t_n \rightarrow -\infty$ as $n \rightarrow \infty$, such that $\lim_{n \rightarrow \infty} \phi_{t_n}(x)$ exists. We conclude therefore that $L_\alpha(x)$ is empty for $|x| > 1$. \square

The closed orbit, C , in Example 3.8.1 is an example of what is called a limit cycle.

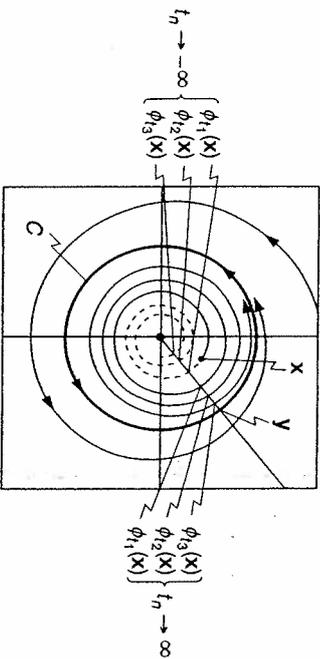


Fig. 3.26. Phase portrait for (3.81) with $a = \frac{1}{2}$. Note that the origin is an unstable focus and the circle $r \equiv 1$ is a closed orbit of period 2π .

Definition 3.8.2

A closed orbit, \mathcal{C} , is said to be a **limit cycle** if \mathcal{C} is a subset of $L_\alpha(x)$ or $L_\omega(x)$ for some x that does not lie in \mathcal{C} .

Observe that Definition 3.8.2 does not require that trajectories approach the limit cycle from both sides, as is the case for C in Example 3.8.1. The limit cycle occurring in Example 3.8.1 has the property that the trajectories of all points, x , with $|x| \neq 0$ or 1 are attracted to it as time increases. It is an example of what is called an **attracting set**. Figure 3.27 shows some limit cycles that are not attracting sets. Thus, while every attracting set is a limit set, not all limit sets are attracting sets. Note also that a closed orbit around a centre is not a limit cycle, because it only contains limit points of points in itself.

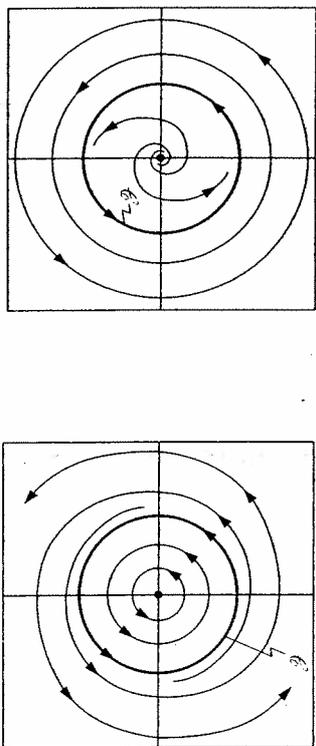


Fig. 3.27. Examples of limit cycles that are not attracting sets. Observe that in: (a) $L_\omega(x) = \mathcal{C}$ for $|x| \leq 1$; (b) $L_\alpha(x) = \mathcal{C}$ for $|x| \geq 1$; and \mathcal{C} is therefore a limit cycle in both cases. However, \mathcal{C} is not an attracting set in either case because trajectories do not approach \mathcal{C} from both sides.

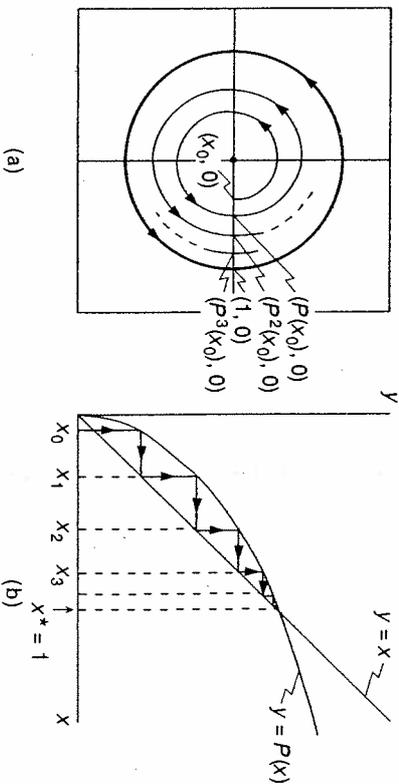


Fig. 3.28. (a) Illustration of the definition of the Poincaré map, P , for the system (3.81). (b) Graphical representation of the fixed point iteration $x_{n+1} = P(x_n)$, $n = 0, 1, \dots$, with attracting fixed point at $x^* = 1$.

The stable nature of the limit cycle in Example 3.8.1 is related to the existence of an attracting fixed point in the corresponding **Poincaré (or first return) map** of the flow. For example, consider the trajectory passing through the point $(x_0, 0)$ with $x_0 > 0$. Suppose we follow this trajectory and that its first return to the positive x -axis occurs at $x = x_1$. Then we define $x_1 = P(x_0)$ (as shown in Fig. 3.28(a)). This procedure allows us to define the Poincaré map $P(x)$ for every $x \in (0, \infty)$. In particular, $P(1) = 1$, because the trajectory passing through $(1, 0)$ is the closed orbit C . Thus the fixed point $x = x^* = 1$ of P corresponds to the limit cycle C in the flow. What is more, the iteration $x_{n+1} = P(x_n)$, $n = 0, 1, \dots$, reflects the convergence of the trajectory through $(x_0, 0)$ to the limit cycle C as shown in Fig. 3.28.

In general, the Poincaré map, P , may only be defined on a line segment—or **local section**—transverse to the closed orbit. However, the stability of the limit cycle is still given by the stability of the associated fixed point iteration of P . In particular, x^* is attracting (repelling), and the corresponding limit cycle is **stable (unstable)** if $|dP/dx|_{x=x^*}$ is less (greater) than unity. If $|dP/dx|_{x=x^*} = 1$ and $d^2P/dx^2 \neq 0$ then the iteration will converge on one side of x^* and diverge from the other. In this case the corresponding limit cycles are said to be **semi-stable**.

Example 3.8.2

Find the limit cycles in the following systems and give their types.

$$1. \dot{r} = r(r-1)(r-2), \quad \dot{\theta} = 1; \quad (3.83)$$

$$2. \dot{r} = r(r-1)^2, \quad \dot{\theta} = 1. \quad (3.84)$$

Solution

1. There are closed trajectories given by

$$r(t) \equiv 1, \quad \theta = t \quad \text{and} \quad r(t) \equiv 2, \quad \theta = t \quad (3.85)$$

corresponding to fixed points in the Poincaré map defined on any radial line. The stability of these fixed points is given by the sign of \dot{r} . Observe

$$\dot{r} \begin{cases} > 0, & 0 < r < 1 \\ < 0, & 1 < r < 2 \\ > 0, & r > 2. \end{cases} \quad (3.86)$$

The system therefore has two circular limit cycles: one stable ($r = 1$) and one unstable ($r = 2$).

2. System (3.84) has a single circular limit cycle of radius one. However, \dot{r} is positive for $0 < r < 1$ and $r > 1$, so the limit cycle is semi-stable. \square

Limit cycles are not always circular and are, therefore, not always revealed by simply changing to polar coordinates. For example, consider the 'Van der Pol' equation

$$\ddot{x} - \dot{x}(1 - x^2) + x = 0 \quad (3.87)$$

with its equivalent first-order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_2(1 - x_1^2) - x_1. \quad (3.88)$$

In polar coordinates this becomes

$$\begin{aligned} \dot{r} &= r \sin^2 \theta (1 - r^2 \cos^2 \theta) \\ \dot{\theta} &= -1 + \cos \theta \sin \theta (1 - r^2 \cos^2 \theta). \end{aligned} \quad (3.89)$$

These equations do not give any immediate insight into the nature of the phase portrait which contains a unique attracting limit cycle (described in section 5.4). In fact, the problem of detecting limit cycles in non-linear systems can be a difficult one which we will have to examine more closely.

3.9 POINCARÉ-BENDIXSON THEORY

We have so far encountered limit sets that are fixed points or closed orbits. What other possibilities can occur?

Example 3.9.1

Consider the phase portrait shown in Fig. 3.29. Find $L_a(\mathbf{x})$ and $L_\omega(\mathbf{x})$ for \mathbf{x} lying in the regions A, A', B, B', C , respectively. Comment on the nature of the limit sets you obtain.

Solution

Take straight lines emanating from each of the fixed points P_1 and P_2 . Suppose \mathbf{x} lies in each of the regions of interest in turn and examine the intersections of the trajectory through \mathbf{x} with these straight lines. In every case, the intersections provide time sequences, $\{t_n\}_{n=1}^\infty$, and limit points, \mathbf{y} , satisfying the requirements of Definition 3.8.1. Rotation of each of the straight lines through 2π radians leads to the limit sets required. The results may be

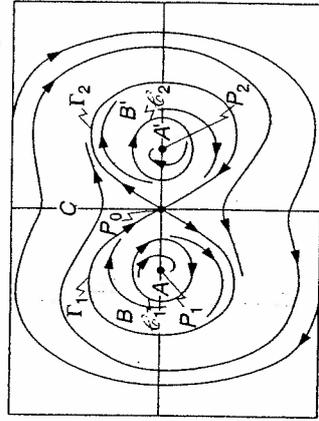


Fig. 3.29. The phase portrait considered in Example 3.9.1. The separatrices at P_0 coincide to form trajectories, Γ_1 and Γ_2 , called **saddle-connections**.

summarized as follows.

$x \in A$	$L_\alpha(x) = \mathcal{G}_1$	$L_\omega(x) = P_1$
$x \in A'$	$L_\alpha(x) = \mathcal{G}_2$	$L_\omega(x) = P_2$
$x \in B$	$L_\alpha(x) = \mathcal{G}_1$	$L_\omega(x) = \Gamma_1 \cup P_0$
$x \in B'$	$L_\alpha(x) = \mathcal{G}_2$	$L_\omega(x) = \Gamma_2 \cup P_0$
$x \in C$	$L_\alpha(x) = \emptyset$	$L_\omega(x) = \Gamma_1 \cup \Gamma_2 \cup P_0$

All the limit sets are closed and bounded and they consist of: (1) fixed points, (2) closed orbits; or (3) unions of fixed points and separatrices. The limit sets of type (3) are closed curves but they are not closed orbits because they are not single trajectories. \square

The following theorem, which only holds for planar phase portraits, essentially states that the types of limit set that are found in Example 3.9.1 are the only compact ones that can occur. The reader will recall that a compact set in the plane is one that is closed and bounded.

Theorem 3.9.1 (Poincaré–Bendixson)

A non-empty, compact limit set of a phase flow in the plane that does not contain a fixed point is a closed orbit.

This result can be used to prove the existence of an attracting/repelling limit cycle provided we can recognize a bounded region of the phase plane which contains a limit set but does not contain a fixed point. Example 3.8.1 illustrates one scenario in which this is possible.

Refer to Fig. 3.26 and consider any closed annulus with inner radius less than one and outer radius greater than one. Observe that the trajectories of the boundary points of the annulus all flow into its interior. Such a region must contain an attracting (limit) set which these trajectories approach as $t \rightarrow \infty$. However, the only fixed point of the system (3.81) is the origin. Consequently, the annulus contains a compact limit set with no fixed point. This set must be a closed orbit by Theorem 3.9.1.

The annulus in the above discussion is an example of what is called a **trapping region**. The phase portraits in Fig. 3.30(a), (b) show that, while the existence of a trapping region guarantees the existence of an attracting set, it is not sufficient to ensure that the limit set is a limit cycle. Figure 3.30(c) highlights the fact that the trapping region may contain more than one limit cycle.

We can formalize these ideas as follows.

Definition 3.9.1

A **trapping region** for the system $\dot{x} = X(x)$, with flow ϕ_t , is a compact, connected set $D \subset \mathbb{R}^2$ such that $\phi_t(D) \subset D$ for all $t > 0$.

In Definition 3.9.1 we have used $\phi_t(D)$ to denote $\{\phi_t(x) | x \in D\}$. The following corollary to Theorem 3.9.1 then encapsulates the argument illustrated above.

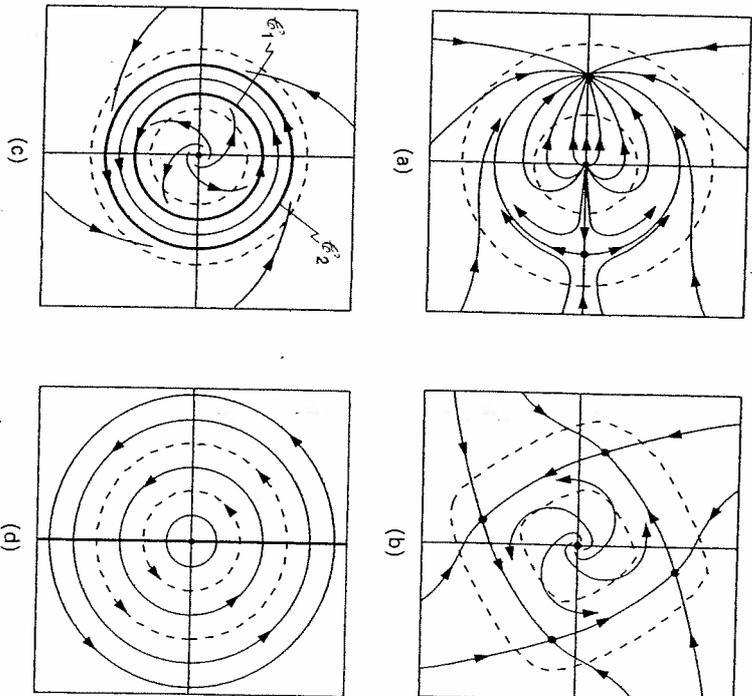


Fig. 3.30. An annular trapping region containing: (a) an attracting fixed point; (b) an attracting set consisting of a union of fixed points and separatrices; (c) two limit cycles \mathcal{G}_1 and \mathcal{G}_2 and a continuum of closed orbits; (d) The annulus shown is a positively invariant set containing no fixed points but the trajectories of the system do not flow into it as t increases. There are no limit cycles in this annulus.

Corollary

If D is a trapping region for the system $\dot{x} = X(x)$ and there are no fixed points in D , then the phase portrait of the system has a limit cycle in D .

In Example 3.8.1 the trajectories of the system are directed into the annulus at every point of its boundary. While this property is sufficient to ensure that a compact, connected set, D , is a trapping region, it is not a necessary one. Of course, it is necessary that no trajectories leave D with increasing time; a set with this property is said to be positively invariant.

Definition 3.9.2

Given the system $\dot{x} = X(x)$ with flow ϕ_t , a subset D of \mathbb{R}^2 is said to be a **positively invariant set** for the system if, for every point x_0 of D , $\phi_t(x_0)$ lies in D for all positive t .

The subset D is called an **invariant set** for the system if the conditions of Definition 3.9.2 are satisfied for all real t .

For positively invariant D the trajectories of boundary points are not required to enter the interior of D with increasing time. To obtain a trapping region steps must be taken to ensure that $\phi_t(\mathbf{x})$ tends to an attracting set inside D . In particular, we must exclude the possibility that the boundary of D is made up of trajectories of the system (like those in 3.30(d)). Thus, the trajectories of the system must be directed into the set at some, but not necessarily all, of its boundary points. This is sufficient to ensure that $\phi_t(D)$ is a proper subset of D and, consequently, such a positively invariant set can form a trapping region.

Example 3.9.2

Show that the phase portrait of

$$\dot{x} - \dot{x}(1 - 3x^2 - 2\dot{x}^2) + x = 0$$

has a limit cycle.

Solution

The corresponding first-order system is

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2), \quad (3.90)$$

which becomes

$$\begin{aligned} \dot{r} &= r \sin^2 \theta (1 - 3r^2 \cos^2 \theta - 2r^2 \sin^2 \theta) \\ \dot{\theta} &= -1 + \frac{1}{2} \sin 2\theta (1 - 3r^2 \cos^2 \theta - 2r^2 \sin^2 \theta) \end{aligned} \quad (3.91)$$

in polar coordinates. Observe:

1. equation (3.91) with $r = \frac{1}{2}$ gives

$$\dot{r} = \frac{1}{4} \sin^2 \theta (1 - \frac{1}{2} \cos^2 \theta) \geq 0 \quad (3.92)$$

with equality only at $\theta = 0$ and π . Thus, $\{\mathbf{x} | r \geq \frac{1}{2}\}$ is positively invariant;

2. equation (3.91) with $r = 1/\sqrt{2}$ implies

$$\dot{r} = -\frac{1}{2\sqrt{2}} \sin^2 \theta \cos^2 \theta \leq 0, \quad (3.93)$$

with equality at $\theta = 0, \pi, \pi/2, 3\pi/2$. Thus $\{\mathbf{x} | r \leq 1/\sqrt{2}\}$ is positively invariant.

Now 1 and 2 imply that the annular region $\{\mathbf{x} | \frac{1}{2} \leq r \leq 1/\sqrt{2}\}$ is positively invariant. What is more, the circles bounding this annulus are not trajectories of the system because \dot{r} does not vanish identically on them. Consequently the trajectories of points on these circles move into the annulus as time increases. Since the only fixed point of (3.90) is at the origin, we conclude that there is a limit cycle in the annulus. \square

The following result gives a condition for there to be no limit cycle in a region D .

Theorem 3.9.2

Let D be a simply connected region of the phase plane in which the vector field $\mathbf{X}(\mathbf{x}) = (X_1(x_1, x_2), X_2(x_1, x_2))$ has the property that

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \quad (3.94)$$

is of constant sign. Then the system $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ has no closed trajectories wholly contained in D .

It will be sufficient for our purpose to recognize that a simply connected region of the plane is a region with no 'holes' in it (as illustrated in Fig. 3.31). The theorem follows from Green's theorem in the plane which may be stated as follows:

Let the real-valued functions $P(x_1, x_2)$ and $Q(x_1, x_2)$ have continuous first partial derivatives in a simply connected region \mathcal{R} of the $x_1 x_2$ -plane bounded by a simple closed curve \mathcal{C} . Then

$$\oint_{\mathcal{C}} P dx_1 + Q dx_2 = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dx_1 dx_2. \quad (3.95)$$

where $\oint_{\mathcal{C}}$ indicates integration along \mathcal{C} in an anticlockwise direction.

To prove Theorem 3.9.2 assume that a limit cycle C of period T exists for the system. Let $P = -X_2, Q = X_1$ in (3.95) and obtain

$$\begin{aligned} \oint_C X_1 dx_2 - X_2 dx_1 &= \int_0^T (X_1 \dot{x}_2 - X_2 \dot{x}_1) dt (= 0) \\ &= \iint_{\mathcal{R}} \left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} \right) dx_1 dx_2 (\neq 0). \end{aligned} \quad (3.96) \quad (3.97)$$

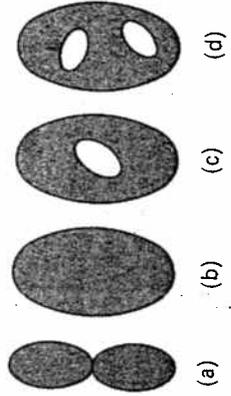


Fig. 3.31. The shaded regions in (a) and (b) have no 'holes' and are simply connected. Those in (c) and (d) have 'holes' and are not simply connected.

Equation (3.96) follows because C is a solution curve, while (3.97) follows from (3.94). Hence, the closed trajectory C cannot exist.

Example 3.9.3

Prove that if the system

$$\dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2), \quad \dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2) + K, \quad (3.98)$$

where K is a constant, has a closed trajectory, then it will either:

1. encircle the origin; or
2. intersect the circle $x_1^2 + x_2^2 = \frac{1}{2}$.

Solution

The quantity

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} = 2 - 4(x_1^2 + x_2^2) \quad (3.99)$$

is positive inside the circle $x_1^2 + x_2^2 = \frac{1}{2}$ and negative outside it. Thus, any closed trajectory cannot be wholly contained in the simply connected region $\{(x_1, x_2) | x_1^2 + x_2^2 < \frac{1}{2}\}$. Therefore, if a closed trajectory exists it is either contained in $\{(x_1, x_2) | x_1^2 + x_2^2 > \frac{1}{2}\}$ or it will intersect $x_1^2 + x_2^2 = \frac{1}{2}$. If the closed orbit is contained in $\{(x_1, x_2) | x_1^2 + x_2^2 > \frac{1}{2}\}$ then it must encircle the origin, otherwise it will bound a region of constant negative sign of (3.99). \square

EXERCISES

Sections 3.1–3.3

- 3.1 Use the method of isoclines to sketch the global phase portraits of the following systems:
 - (a) $\dot{x}_1 = x_1 x_2, \quad \dot{x}_2 = \ln x_1, \quad x_1 > 0;$
 - (b) $\dot{x}_1 = 4x_1(x_2 - 1), \quad \dot{x}_2 = x_2(x_1 + x_1^2);$
 - (c) $\dot{x}_1 = x_1 x_2, \quad \dot{x}_2 = x_2^2 - x_1^2.$
- 3.2 Show that the mapping $(x_1, x_2) \rightarrow (f(r) \cos \theta, f(r) \sin \theta)$, where $x_1 = r \cos \theta, x_2 = r \sin \theta$ and $f(r) = \tan(\pi r / 2r_0)$, is a continuous bijection of $N = \{(x_1, x_2) | r < r_0\}$ onto \mathbb{R}^2 . Does the bijection map the set of concentric circles on $\mathbf{0}$ in N onto the set of concentric circles on $\mathbf{0}$ in \mathbb{R}^2 ? What property of local phase portraits of linear systems in the plane does this result illustrate?
- 3.3 Sketch the local phase portraits of the fixed points in Figs 3.5(a), 3.22(b) and 3.30(a).
- 3.4 Find the linearizations of the following systems, at the fixed points indi-

- (a) introducing local coordinates at the fixed points;
- (b) using Taylor's theorem.

- (i) $\dot{x}_1 = x_1 + x_1 x_2^3 / (1 + x_1^2)^2, \quad \dot{x}_2 = 2x_1 - 3x_2, \quad (0, 0);$
- (ii) $\dot{x}_1 = x_1^2 + \sin x_2 - 1, \quad \dot{x}_2 = \sinh(x_1 - 1) \quad (1, 0);$
- (iii) $\dot{x}_1 = x_1^2 - e^{x_1}, \quad \dot{x}_2 = x_2(1 + x_2), \quad (e^{-1/2}, -1).$

State the preferred method (if one exists) for each system.

- 3.5 Use the linearization theorem to classify, where possible, the fixed points of the systems:

- (a) $\dot{x}_1 = x_2^2 - 3x_1 + 2, \quad \dot{x}_2 = x_1^2 - x_2^2;$
- (b) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1^3;$
- (c) $\dot{x}_1 = \sin(x_1 + x_2), \quad \dot{x}_2 = x_2;$
- (d) $\dot{x}_1 = x_1 - x_2 - e^{x_1}, \quad \dot{x}_2 = x_1 - x_2 - 1;$
- (e) $\dot{x}_1 = -x_2 + x_1 + x_1 x_2, \quad \dot{x}_2 = x_1 - x_2 - x_2^2;$
- (f) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -(1 + x_1^2 + x_1^4)x_2 - x_1;$
- (g) $\dot{x}_1 = -3x_2 + x_1 x_2 - 4, \quad \dot{x}_2 = x_2^2 - x_1^2.$

- 3.6 Linearize the system

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1 - x_1^3$$

at the origin and classify the fixed point of the linearized system. Show that the trajectories of the non-linear system lie on the family of curves $x_1^2 + x_2^2 - x_1^6/3 = C$,

where C is a constant. Sketch these curves to show that the non-linear system and its linearization have qualitatively equivalent local phase portraits at the origin. Why could this conclusion not be deduced from the linearization theorem?

- 3.7 Find the principal directions of the fixed points at the origin of the following systems:
 - (a) $\dot{x}_1 = e^{x_1 + x_2} - 1, \quad \dot{x}_2 = x_2;$
 - (b) $\dot{x}_1 = -\sin x_1 + x_2, \quad \dot{x}_2 = \sin x_2;$
 - (c) $\dot{x}_1 = \ln(x_1 + x_2 + 1), \quad \dot{x}_2 = \frac{1}{2}x_1 + x_2, \quad x_1 + x_2 > -1;$
 and use them to sketch local phase portraits.

Section 3.4

- 3.8 Find the family of solution curves which satisfy

$$\frac{dx_2}{dx_1} = \frac{x_2^2 - x_1^3}{2x_1 x_2}, \quad x_1, x_2 \neq 0,$$

by making the substitution $x_2^2 = u$. Sketch the family of solutions and hence or otherwise sketch the local phase portrait of the non-simple fixed

point of

$$\dot{x}_1 = 2x_1x_2, \quad \dot{x}_2 = x_2^2 - x_1^2.$$

3.9 Are the phase portraits of the systems $\dot{x}_1 = x_1 e^{x_1}$, $\dot{x}_2 = x_2 e^{x_1}$ and $\dot{x}_1 = x_1$, $\dot{x}_2 = x_2$ qualitatively equivalent? If so, state the continuous bijection which exhibits the equivalence.

3.10 Show that the 'straight line' separatrices at the non-simple fixed point of

$$\dot{x}_1 = x_2(3x_1^2 - x_2^2), \quad \dot{x}_2 = x_1(x_1^2 - 3x_2^2)$$

satisfy $x_2 = kx_1$ where

$$k^2(3 - k^2) = 1 - 3k^2.$$

Hence, or otherwise, find these separatrices and by using isoclines sketch the phase portrait.

3.11 Show that the system

$$\dot{x}_1 = x_1^2 - x_2^3, \quad \dot{x}_2 = x_1^2(x_1^2 - x_2^2)$$

has a line of fixed points. Furthermore, show that every fixed point on the line is non-simple. Can this conclusion be reached by using the linearization theorem?

Is the above conclusion true for any system with a line of fixed points?

Section 3.5

3.12 Use the method of isoclines to sketch the phase portrait of (3.33), paying particular attention to the manner in which the trajectories cross the x_2 -axis. Hence show that, while all trajectories approach $(x_1, x_2) = (0, 0)$ as t tends to infinity, the system is *not* asymptotically stable.

3.13 Show that $V(x_1, x_2) = x_1^2 + x_2^2$ is a strong Liapunov function at the origin for each of the following systems:

- $\dot{x}_1 = -x_2 - x_1^3, \quad \dot{x}_2 = x_1 - x_2^3;$
- $\dot{x}_1 = -x_1^3 + x_2 \sin x_1, \quad \dot{x}_2 = -x_2 - x_1^2 x_2 - x_1 \sin x_1;$
- $\dot{x}_1 = -x_1 - 2x_2^2, \quad \dot{x}_2 = 2x_1 x_2 - x_2^3;$
- $\dot{x}_1 = -x_1 \sin^2 x_1, \quad \dot{x}_2 = -x_2 - x_2^2;$
- $\dot{x}_1 = -(1 - x_2)x_1, \quad \dot{x}_2 = -(1 - x_1)x_2.$

3.14 Find domains of stability at the origin for each of the systems even in Exercise 3.13.

3.15 Show that $V(x_1, x_2) = x_1^2 + x_2^2$ is a weak Liapunov function for the following systems at the origin:

- $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2^3(1 - x_1^2);$
- $\dot{x}_1 = -x_1 + x_2^2, \quad \dot{x}_2 = -x_1 x_2 - x_1^2;$
- $\dot{x}_1 = -x_1^3, \quad \dot{x}_2 = -x_1^2 x_2;$
- $\dot{x}_1 = -x_1 + 2x_1 x_2^2, \quad \dot{x}_2 = -x_1^2 x_2^2.$

Which of these systems are asymptotically stable?

3.16 Prove that if V is a strong Liapunov function for $\dot{x} = -X(x)$, in a neighbourhood of the origin, then $\dot{x} = X(x)$ has an unstable fixed point at the origin. Use this result to show that the systems:

- $\dot{x}_1 = x_1^3, \quad \dot{x}_2 = x_2^3.$
- $\dot{x}_1 = \sin x_1, \quad \dot{x}_2 = \sin x_2;$
- $\dot{x}_1 = -x_1^3 + 2x_1^2 \sin x_1, \quad \dot{x}_2 = x_2 \sin^2 x_2;$

are unstable at the origin.

3.17 Prove that the differential equations

- $\ddot{x} + \dot{x} - \dot{x}^3/3 + x = 0;$
- $\ddot{x} + \dot{x} \sin(\dot{x}^2) + x = 0;$
- $\ddot{x} + \dot{x} + x^3 = 0;$
- $\ddot{x} + \dot{x}^3 + x^3 = 0,$

have asymptotically stable zero solutions $x(t) \equiv 0$.

3.18 Prove that $V(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ is positive definite if and only if $a > 0$ and $ac > b^2$. Hence, or otherwise, prove that

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 - (x_1 + 2x_2)(x_2^2 - 1)$$

is asymptotically stable at the origin by considering the region $|x_2| < 1$. Find a domain of stability.

3.19 Find domains of stability for the following systems by using an appropriate Liapunov function:

- $\dot{x}_1 = x_2 - x_1(1 - x_1^2 - x_2^2)(x_1^2 + x_2^2 + 1)$
 $\dot{x}_2 = -x_1 - x_2(1 - x_1^2 - x_2^2)(x_1^2 + x_2^2 + 1);$
- $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + x_2^3 - x_1^5.$

3.20 Use $V(x_1, x_2) = (x_1/a)^2 + (x_2/b)^2$ to show that the system

$$\dot{x}_1 = x_1(x_1 - a), \quad \dot{x}_2 = x_2(x_2 - b), \quad a, b > 0,$$

has an asymptotically stable origin. Show that all trajectories tend to the origin as $t \rightarrow \infty$ in the region

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1.$$

3.21 Given the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_2 - x_1^3$$

show that a positive definite function of the form

$$V(x_1, x_2) = ax_1^4 + bx_1^2 + cx_1x_2 + dx_2^2$$

can be chosen such that $\dot{V}(x_1, x_2)$ is also positive definite. Hence deduce that the origin is unstable.

3.22 Show that the origin of the system

$$\dot{x}_1 = x_2^2 - x_1^2, \quad \dot{x}_2 = 2x_1x_2$$

is unstable by using

$$V(x_1, x_2) = 3x_1x_2^2 - x_1^3.$$

3.23 Show that the fixed point at the origin of the system

$$\dot{x}_1 = x_1^4, \quad \dot{x}_2 = 2x_1^2x_2^2 - x_2^2$$

is unstable by using the function

$$V(x_1, x_2) = \alpha x_1 + \beta x_2$$

for a suitable choice of the constants α and β .

Verify the instability at the fixed point by examining the behaviour of the separatrices.

Section 3.6

3.24 Show that the non-linear change of coordinates

$$y_1 = x_1 + x_2^3, \quad y_2 = x_2 + x_2^2$$

satisfies the requirements of the flow box theorem for the system

$$\dot{x}_1 = -\frac{3x_2^2}{1 + 2x_2}, \quad \dot{x}_2 = \frac{1}{1 + 2x_2}$$

in the neighbourhood of any point (x_1, x_2) with $x_2 \neq -\frac{1}{2}$.

3.25 Prove that the following systems have no fixed points:

(a) $\dot{x}_1 = e^{x_1+x_2}, \quad \dot{x}_2 = x_1 + x_2;$

(b) $\dot{x}_1 = x_1 + x_2 + 2, \quad \dot{x}_2 = x_1 + x_2 + 1;$

(c) $\dot{x}_1 = x_2 + 2x_2^2, \quad \dot{x}_2 = 1 + x_2^2$

and sketch their phase portraits.

3.26 Sketch phase portraits consistent with the following information:

(a) two fixed points, a saddle and a stable node;

(b) three fixed points, one saddle and two stable nodes.

3.27 Find the local phase portraits at each of the fixed points of the system

$$\dot{x}_1 = x_1(1 - x_1^2), \quad \dot{x}_2 = x_2.$$

Use these results to suggest a global phase portrait. Check whether your suggestion is correct by using the method of isoclines.

Section 3.7

3.28 Find first integrals of the following systems together with their domains of definition:

- (a) $\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1^2 + 1;$
- (b) $\dot{x}_1 = x_1(x_2 + 1), \quad \dot{x}_2 = -x_2(x_1 + 1);$
- (c) $\dot{x}_1 = \sec x_1, \quad \dot{x}_2 = -x_2^2, \quad |x_1| < \pi/2;$
- (d) $\dot{x}_1 = x_1(x_1 e^{x_2} - \cos x_2), \quad \dot{x}_2 = \sin x_2 - 2x_1 e^{x_2}.$

3.29 Find a first integral of the system

$$\dot{x}_1 = x_1x_2 - 3x_1^3, \quad \dot{x}_2 = x_2^2 - 6x_1^2x_2 + x_1^4$$

using the substitution $x_2 = \mu x_1^2$. Sketch the phase portrait.

3.30 How do the phase portraits of the two systems

$$\begin{aligned} \dot{x}_1 &= x_1(x_2^2 - x_1), & \dot{x}_2 &= -x_2(x_2^2 - x_1) \\ \dot{x}_1 &= x_1, & \dot{x}_2 &= -x_2 \end{aligned}$$

differ?

3.31 Find a first integral of the system

$$\dot{x}_1 = x_1x_2, \quad \dot{x}_2 = \ln x_1, \quad x_1 > 1$$

in the region indicated. Hence, or otherwise, sketch the phase portrait.

3.32 Find first integrals for the linear systems $\dot{x} = Jx$, where J is a 2×2 Jordan matrix of node, centre and focus type. State maximal regions on which these first integrals exist.

Is a system which has an asymptotically stable fixed point ever conservative?

3.33 Find a Hamiltonian H for a particle moving along a straight line subject to

$$\ddot{x} = -x + \alpha x^2,$$

$\alpha > 0$, where x is the displacement. Sketch the level curves of the Hamiltonian H in the phase plane. Indicate the regions of the phase plane that contain trajectories which give non-oscillatory motion.

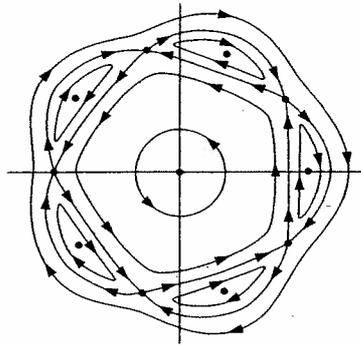
3.34 When expressed in terms of plane polar coordinates (r, θ) given by $x = r \cos \theta, p = r \sin \theta$, Hamilton's equations (3.70) take the form

$$\dot{r} = \frac{1}{r} \frac{\partial \tilde{H}}{\partial \theta}, \quad \dot{\theta} = -\frac{1}{r} \frac{\partial \tilde{H}}{\partial r},$$

where $\tilde{H}(r, \theta) = H(r \cos \theta, r \sin \theta)$. Consider the system with

$$\tilde{H}(r, \theta) = -\mu r^2 + r^4 + r^5 \sin 5\theta,$$

when $0 < \mu \ll \frac{1}{2}$ and $r < \frac{1}{2}$. Show that there are ten fixed points near the circle $r = \sqrt{\mu/2}$ and determine their topological types. Sketch $\tilde{H}(r, \theta)$ as a function of r for $\sin 5\theta = 0, \pm 1$ and verify that the phase portrait takes the form shown below.



Section 3.8

3.35 Sketch phase portraits for each of the following systems:

(a) $\dot{r} = -r(r-1)^2$, $\dot{\theta} = 1$;

(b) $\dot{r} = \begin{cases} r(1-r) & \text{if } r \leq 1; \\ 0 & \text{otherwise} \end{cases}$ $\dot{\theta} = 1$;

(c) $\dot{r} = \begin{cases} 0 & \text{if } r \leq 1; \\ r(r-1) & \text{otherwise} \end{cases}$ $\dot{\theta} = -1$.

In each case, obtain α - and ω -limit sets for all points x with $|x| > 0$.
 3.36 Show that the Poincaré map, P , defined on the positive x -axis, for the system

$$\dot{r} = ar(1-r), \quad \dot{\theta} = 1,$$

is given by

$$P(x) = x/[x + (1-x)\exp(-2\pi a)].$$

Verify that $P(x)$ has a stable fixed point at $x = x^* = 1$. The Poincaré map P_1 is defined on some other r -coordinate line; write down an expression for $P_1(r)$.

3.37 Assume that the Poincaré map, P , defined on the x -axis for the system

$$\dot{r} = r(r-1)^2, \quad \dot{\theta} = 1,$$

satisfies:

- (a) $P(x)$ has a fixed point at $x = x^* = 1$;
- (b) $[dP/dx]_{x=x^*} = 1$;
- (c) $[d^2P/dx^2]_{x=x^*} = 4\pi$.

Sketch the graph of $P(x)$ for x near x^* and illustrate the iteration $x_{n+1} = P(x_n)$, $n = 0, 1, \dots$, for $x_0 \neq 1$. Explain how these diagrams change if the signs of \dot{r} and $\dot{\theta}$ are reversed. Confirm that both systems have a

semi-stable, circular limit cycle of radius unity and describe the difference between them.

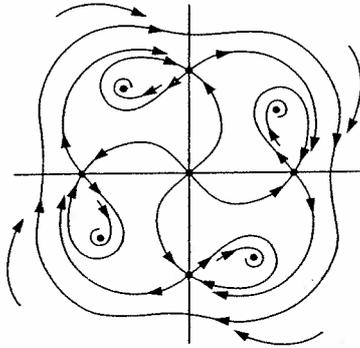
3.38 Consider a planar system with angular equation $\dot{\theta} = 1$ and let P be the Poincaré map defined on the positive x -axis. Suppose:

$$P(x^*) = x^*; \quad |dP/dx|_{x=x^*} = 1; \\ [d^2P/dx^2]_{x=x^*} = 0; \quad [d^3P/dx^3]_{x=x^*} = \varepsilon \neq 0.$$

Draw diagrams to illustrate the form of the iteration $x_{n+1} = P(x_n)$, $n = 0, 1, \dots$, for x_0 near x^* , when: (i) $\varepsilon > 0$; (ii) $\varepsilon < 0$. Sketch corresponding phase portraits for the planar system on an annular neighbourhood of $r = x^*$.

Section 3.9

3.39 Label and list all the non-empty, closed and bounded limit sets in the phase portrait shown below.



3.40 Sketch phase portraits consistent with the following information:

- (a) an unstable limit cycle; three fixed points, one saddle and two stable nodes; and a circular trapping region centred on the origin;
- (b) five foci, one unstable and four stable; four saddle points; a non-empty, closed and bounded limit set containing four fixed points; and a circular trapping region centred on the origin.

3.41 Let D be a closed region of the plane bounded by a simple closed curve ∂D . For each x in ∂D , define $X_{\perp}(x)$ to be the component of $X(x)$ along the inward normal to D . Assume that $X_{\perp}(x)$ satisfies either: (i) $X_{\perp}(x) > 0$; or (ii) $X_{\perp}(x) \geq 0$; for all x in ∂D . Explain why (i) is a sufficient condition for D to be a trapping region for the flow of $\dot{x} = X(x)$ while (ii) is not.

Use the method of isoclines to sketch the phase portrait for the system (3.91). Examine the vector field on the circles $r = \frac{1}{2}$ and $r = 1/\sqrt{2}$

and confirm that the trajectories starting on these circles move into the annulus $\{x \mid \frac{1}{2} \leq r \leq 1/\sqrt{2}\}$ as t increases. What feature of systems satisfying condition (ii) does this example illustrate?

3.42 Show that each of the following regions is a positively invariant set for the system given:

(a) the half-plane $x_2 \geq 0$ for

$$\dot{x}_1 = 2x_1x_2, \quad \dot{x}_2 = x_2^2;$$

(b) the disc $x_1^2 + x_2^2 < 1$ for

$$\dot{x}_1 = -x_1 + x_2 + x_1(x_1^2 + x_2^2), \quad \dot{x}_2 = -x_1 - x_2 + x_2(x_1^2 + x_2^2);$$

(c) the closed region formed by joining the points $(e^{-2\pi}, 0)$ and $(1, 0)$ by a segment of the x_1 -axis and one turn of the spiral with polar form $r = e^{-\theta}$ for

$$\dot{x}_1 = -x_1 - x_2, \quad \dot{x}_2 = x_1 - x_2;$$

(d) the region inside and on the closed curve that is a subset of $\{(x_1, x_2) \mid 3(x_1^2 + x_2^2) - 2x_1^3 = 1\}$ for

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + x_1^2;$$

(e) the closed region bounded by the curves C_1 and C_2 for the system

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)^{\frac{1}{2}}, \quad \dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2)^{\frac{1}{2}},$$

with flow $\phi_t(x_1, x_2)$, where C_x is the closed curve formed by the union of $\{\phi_t(x, 0) \mid 0 \leq t \leq 2\pi\}$ and the segment of the x_1 -axis between $(x, 0)$ and $\phi_{2\pi}(x, 0)$.

Indicate which of these positively invariant sets are trapping regions and specify the attracting set contained within them.

3.43 Prove that there exists a region $R = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq r^2\}$ such that all trajectories of the system

$$\dot{x}_1 = -wx_2 + x_1(1 - x_1^2 - x_2^2), \quad \dot{x}_2 = wx_1 + x_2(1 - x_1^2 - x_2^2) - F,$$

where w and F are constants, eventually enter R . Show that the system has a limit cycle when $F = 0$.

3.44 Prove that the system

$$\dot{x}_1 = 1 - x_1x_2, \quad \dot{x}_2 = x_1$$

has no limit cycles.

3.45 Consider the system

$$\begin{aligned} \dot{x}_1 &= -wx_2 + x_1(1 - x_1^2 - x_2^2) - x_2(x_1^2 + x_2^2), \\ \dot{x}_2 &= wx_1 + x_2(1 - x_1^2 - x_2^2) + x_1(x_1^2 + x_2^2) - F, \end{aligned}$$

where w and F are constants. Prove that if the system has a limit cycle

such that all of its points are at a distance greater than $1/\sqrt{2}$ from the origin, then the limit cycle must encircle the origin.

3.46 Suppose that the region $R = \{(x_1, x_2) \mid x_1, x_2 > 0\}$ is positively invariant for the system $\dot{x}_1 = X_1(x_1, x_2)$, $\dot{x}_2 = X_2(x_1, x_2)$ and that

$$\begin{aligned} \dot{x}_1 &\leq 0 & \text{for } x_2 &\geq -x_1^2 + 3x_1 + 1 \\ \dot{x}_2 &\leq 0 & \text{for } x_2 &\geq x_1, \end{aligned}$$

respectively. Assuming that there are no closed orbits in R prove that the unique fixed point in R is asymptotically stable.