

(Review of We 11.11)

Linear Differential Equations

Thm (Theorem 4.1 (ii), p. 143 in Allen)

$$\text{Let } x'(t) = A(t)x(t) + g(t) \quad (4.9)$$

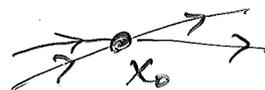
where $x(t) = (x_1(t), \dots, x_n(t))^T$, $g(t) = (g_1(t), \dots, g_n(t))^T$, $A(t) = (a_{ij}(t))_{i,j=1}^n$

$$\text{and } x(t_0) = x_0 = (x_{10}, x_{20}, \dots, x_{n0})^T.$$

If the coefficients $a_{ij}(t)$, $g_i(t)$ ($i, j = 1, \dots, n$) are continuous on some interval $\alpha < t_0 < \beta$ then there exists a unique solution to the initial value problem (4.9) on (α, β) .

Note. The condition is sufficient.

Note. Two different solutions cannot pass through a given point:



Impossible: violates
uniqueness.

11.11/p.2

Matrix exponential (4.14.1 p. 173 in Allen)

For a given matrix A ($n \times n$ real matrix) we define

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \quad (*)$$

We note that for any norm $\|\cdot\|$ on matrices

$$\|A^k\| \leq \|A\|^k, \quad k=1, 2, \dots$$

($\|\cdot\|$ should be a so-called operator norm,
e.g. $\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$)

or $\|A\| = (\text{maximal eigenvalue of } A^T A)^{\frac{1}{2}}$)

The inequality (for real sums)

$$\left| \sum_{k=M}^N \frac{A_{ij}^k t^k}{k!} \right| \leq \sum_{k=M}^N \frac{\|A\|^k t^k}{k!}$$

shows that all the n^2 entries in (*)

11.11/p.3

are Cauchy sequences, i.e., convergent. Moreover, they are uniformly absolutely convergent power series. Hence they can be differentiated infinitely many times inside the radius of convergence, which for the exponential series is ∞ .

Thus, differentiating (*), we see that

$$\frac{d}{dt} e^{At} = A e^{At} = e^{At} A$$

and

$$\frac{d}{dt} e^{At} x_0 = A e^{At} x_0$$

where x_0 is any $n \times 1$ vector.

Hence we have:

The unique solution to

$$x'(t) = A x(t), \quad x(0) = x_0$$

is given by

$$x(t) = e^{At} x_0.$$

11.11./p.4

Cf. 4.14.1. in Allen

We say that A is diagonalizable

if $A = H \Lambda H^{-1}$

for some non-singular (complex in general) matrix H and a diagonal (complex) matrix Λ . The diagonal entries of Λ are $\lambda_1, \lambda_2, \dots, \lambda_n$ the eigenvalues of A .

Note: It is shown in Linear Algebra that A and Λ have the same eigenvalues.

Since

$$\begin{aligned} e^{At} &= H e^{\Lambda t} H^{-1} \\ &= H \begin{pmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \\ & & & e^{\lambda_n t} \end{pmatrix} H^{-1} \end{aligned}$$

we see that for given x_0 each component of $e^{At} x_0$ is a (complex in general) linear combination

11.11. / p.5

of $e^{\lambda_1 t}$, $e^{\lambda_2 t}$, ..., $e^{\lambda_n t}$

Theorem (cf. Theorem 4.3., p. 149, in Allen)

If all eigenvalues of A have negative real parts then

$$\|e^{At} x_0\| \leq M e^{-bt}, \quad t \geq 0$$

for some $M > 0$ and $b > 0$ (depend on the λ_i 's and x_0)

The theorem holds true for all A , diagonalizable or not. This is because A may be written

$$A = P J P^{-1}$$

where P is a non-singular matrix and J is a diagonal matrix is

block

Jordan form:

11.11./p.6

Each block is of size $r_i \geq 1$

$$\begin{pmatrix} \lambda_i & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & & \lambda_i \end{pmatrix}$$

Ex. $r_i = 3$ $\begin{pmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{pmatrix}$ $r_i = 1$ (λ_i)

From this follows that each $e^{At} x_0$ is a linear combination of $e^{\lambda_i t}$, $t e^{\lambda_i t}$, ..., $t^{r_i-1} e^{\lambda_i t}$

Ex. $r_i = 3$
 $e^{\lambda_i t}$, $t e^{\lambda_i t}$, $t^2 e^{\lambda_i t}$

where λ_i , $i = 1, \dots, m$, are the eigenvalues of A and r_i is the so-called geometric multiplicity of λ_i ,

$$r_1 + r_2 + \dots + r_m = n.$$