In this case, solutions may be unbounded. However, assume when one component goes to zero, $x_i(t) \to 0$, then all solutions $x_j(t)$ are bounded, $x_j(t) < K$ for $j = 1, 2, \ldots, n$. [For a proof see Gard and Hallam (1979).]

31. Consider the system of differential equations,

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = -x + 2\mu y - x^2 y,$$

where $\mu < 1$.

(a) Find the Jacobian matrix evaluated at $(0, 0)$ and show that the eigenvalues satisfy $\mu \pm i\sqrt{1 - \mu^2}$.

(b) Show that this system satisfies the Hopf Bifurcation Theorem. This system is related to van der Pol's equation, discussed in Chapter 6.

32. For Example 5.21, apply the criteria in the Appendix to show that the bifurcation at $r = 0$ is a supercritical bifurcation.

33. The following system has been transformed so that the bifurcation value is at $r = 0$ and the equilibrium is at the origin,

$$\frac{dx}{dt} = rx + y + x^3$$

$$\frac{dy}{dt} = -x + ry + yx^2.$$

Show that the Hopf Bifurcation Theorem holds. Then apply the criteria in the Appendix to show that the bifurcation is subcritical.

5.14 References for Chapter 5

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5.15 Appendix for Chapter 5

5.15.1 Subcritical and Supercritical Hopf Bifurcations

A computational method can be applied to test if the Hopf bifurcation is supercritical or subcritical. First, a transformation is made to put the system in canonical form. The canonical form for the differential system is as follows:

$$\frac{dx}{dt} = \alpha(r)x + \beta(r)y + f_1(x, y, r) = f(x, y, r),$$

$$\frac{dy}{dt} = -\beta(r)x + \alpha(r)y + g_1(x, y, r) = g(x, y, r).$$

The Jacobian matrix evaluated at the origin is

$$J(r) = \begin{pmatrix} \alpha(r) & \beta(r) \\ -\beta(r) & \alpha(r) \end{pmatrix},$$

where the eigenvalues are $\alpha(r) \pm i\beta(r)$, $\alpha(0) = 0$, and $\beta(0) > 0$. Thus at $r = 0$ (bifurcation value) the Jacobian matrix $J(0)$ has two purely imaginary eigenvalues $\pm\beta(0)i$.

To test whether the bifurcation is supercritical or subcritical, the signs of two quantities must be checked, $d\alpha(0)/dr$ and a quantity denoted here as C. The partial derivatives of f and g are calculated, then evaluated at the bifurcation value $r = 0$ and at the equilibrium $x = 0$ and $y = 0$. Then the value of the following expression C is computed (Hale and Koçak, 1991):

$$C = (f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{\beta(0)}[-f_{xy}(f_{xx} + f_{yy}) + g_{xy}(g_{xx} + g_{yy}) + f_{xx}g_{xx} - f_{yy}g_{yy}].$$

In the particular case $d\alpha(0)/dr > 0$, the following additional conditions result in either a subcritical or a supercritical bifurcation (see Figure 5.13):

1. If $C < 0$, then for $r < 0$ the origin is a stable spiral but for $r > 0$ there exists a stable periodic solution and the origin is unstable—a supercritical bifurcation.
2. If $C > 0$, then for $r < 0$ there exists an unstable periodic solution and the origin is stable but for $r > 0$ the origin is unstable—a subcritical bifurcation.
3. If $C = 0$, the test is inconclusive.

5.15.2 Strong Delay Kernel

In the continuous-delay logistic model with a strong delay kernel, we show that there do not exist any complex eigenvalues with positive real part when $T < 2/r$. Let

$$\frac{dx(t)}{dt} = rx(t)\left(1 - \frac{1}{K}\int_{-\infty}^{t} k(t - s)x(s)\,ds\right),$$

where $k(t) = tT^{-2}\exp(-t/T)$. The integral expression is known as a *Volterra integral*. A change of variable, linearization about K, and substitution of $x = e^{\lambda t}$ lead to the characteristic equation

$$\lambda = -\frac{r}{T^2}\int_0^{\infty} e^{-\lambda\tau}\tau e^{-\tau/T}\,d\tau.$$

Suppose $\lambda = a + ib$, where $a > 0$ and $b \neq 0$. Then, applying $e^{-\lambda\tau} = e^{-a\tau}(\cos(b\tau) - i\sin(b\tau))$, the two equations satisfied by the real and imaginary parts are

$$a = -\frac{r[(aT + 1)^2 - b^2T^2]}{[(aT + 1)^2 + b^2T^2]^2}$$

and

$$b = \frac{2rbT(aT + 1)}{[(aT + 1)^2 + b^2T^2]^2}.$$

Simplifying these two equations leads to a fifth-degree polynomial in a and a fourth-degree polynomial in b,

$$p(a) = T^4a^5 + 4T^3a^4 + \cdots + r(1 - b^2T^2) = 0$$

and

$$[(aT + 1)^2 + T^2b^2]^2 - 2rT(aT + 1) = 0. \tag{5.26}$$