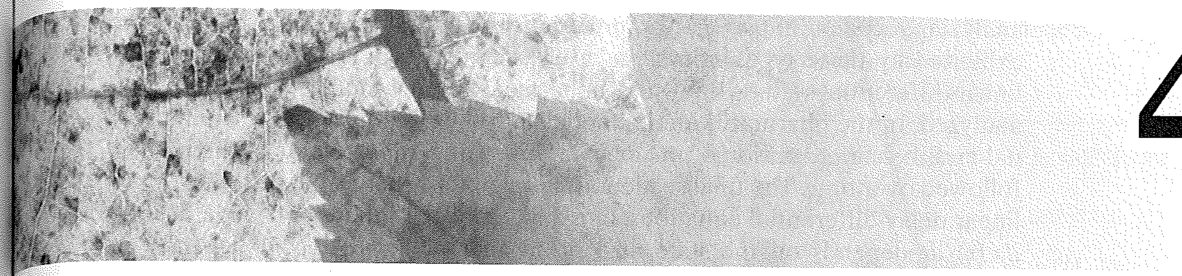
Table 3.4 Breeding population estimates (in thousands) for total ducks in Minnesota from 1968 through 2004 (United States Fish and Wildlife Service, 2004).

Year	Size	Year	Size	Year	Size
1968	368.5	1981	515.2	1994	1320.1
1969	345.3	1982	558.4	1995	912.2
1970	343.8	1983	394.2	1996	1062.4
1971	286.9	1984	563.8	1997	953.0
1972	237.6	1985	580.3	1998	739.6
1973	415.6	1986	537.5	1999	716.5
1974	332.8	1987	614.9	2000	815.3
1975	503.3	1988	752.8	2001	761.3
1976	759.4	1989	1021.6	2002	1224.1
1977	536.6	1990	886.8	2003	748.9
1978	511.3	1991	868.2	2004	1099.3
1979	901.4	1992	1127.3		
1980	740.7	1993	875.9		

Chapter

4



Linear Differential Equations: Theory and Examples

4.1 Introduction

When changes such as births and deaths occur continuously, then generations overlap and a continuous-time model (differential equations) is more appropriate than a discrete-time model (difference equations). In differential equations, the time interval is continuous and can be either finite or infinite in length $[t_0, T)$ for $t_0 < T < \infty$ or $[t_0, \infty)$, respectively, as opposed to difference equations, where time is a set of discrete values, $t = 0, 1, 2, \ldots$.

The most well-known population growth model and one of the simplest is due to Malthus (1798). The model for Malthusian growth is a differential equation. The Malthusian model assumes the rate of growth is proportional to the size of the population. Hence, if $x(t)$ is the population size, then

$$\frac{dx}{dt} = rx, \quad x(t_0) = x_0, \tag{4.1}$$

where $r > 0$ is referred to as the *per capita growth rate* or the *intrinsic growth rate*. The solution to this differential equation is found by separating variables,

$$\frac{dx}{x} = r\,dt, \quad \int_{x_0}^{x(t)} \frac{dy}{y} = \int_{t_0}^{t} r\,d\tau, \quad \ln[x(t)/x_0] = r(t - t_0).$$

Finally,

$$x(t) = x_0 \exp(r(t - t_0))$$

or $x(t) = x_0 e^{rt}$ when $t_0 = 0$. The population grows exponentially over time. Note also that the differential equation (4.1) is linear in x. The exponential growth exhibited by the solution of the differential equation (4.1) is comparable to the geometric growth exhibited by the solution to the linear difference equation, $x_{t+1} = ax_t$, where $x_t = x_0 a^t$. The constant $a = e^r$. If $a > 1$ (or $r > 0$), then there is exponential growth.

In this chapter, basic notation and definitions are given for first- and higher-order differential equations, as well as first-order systems. We concentrate on linear differential equations. Criteria are stated for solutions to approach the zero solution. These criteria are known as the Routh-Hurwitz criteria and are the analogue of the Jury conditions for difference equations. Techniques for analyzing a system of two first-order equations (in the plane) and the behavior exhibited by these types of systems are discussed. A biological example of a linear differential system, known as a pharmacokinetics model, is presented and analyzed. In the pharmacokinetics model, a drug is administered to an individual and the concentration of the drug in different compartments of the body is followed over time. The final section of this chapter gives a brief introduction to linear delay differential equations. In a delay model, the rate of change of state $dx(t)/dt$ depends on the state at a prior time, $t - \tau$, that is, it depends on $x(t - \tau)$. Thus, the dynamics of $x(t)$ are delayed by τ time units.

4.2 Basic Definitions and Notation

Differential equations are classified according to their order, whether they are linear or nonlinear, and whether they are autonomous or nonautonomous. These classifications schemes are similar to the ones defined for difference equations.

Definition 4.1. A *differential equation of order n* is an equation of the form

$$f(x, dx/dt, d^2x/dt^2, \ldots, d^nx/dt^n, t) = 0.$$

If this differential equation does not depend explicitly on t, then it is said to be *autonomous*; otherwise it is *nonautonomous*.

If an nth-order differential equation can expressed as follows:

$$\frac{d^nx}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = g(t), \tag{4.2}$$

then it is referred to as linear.

Definition 4.2. The differential equation (4.2) is said to be *linear* if the coefficients a_i, $i = 1, \ldots, n$, and g are either constant or functions of t but not functions of x or any of its derivatives. Otherwise, the differential equation (4.2) is said to be *nonlinear*. The linear differential equation (4.2) is said to be *homogeneous* if $g(t) \equiv 0$ and *nonhomogeneous* otherwise.

It will always be assumed that the functions f, g, and a_i are real valued. Analogous definitions can be stated for systems. A first-order system of differential equations satisfies

$$\frac{dX}{dt} = F(X(t), t), \tag{4.3}$$

where the vector $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $F = (f_1, f_2, \ldots, f_n)^T$, and

$$f_i \equiv f_i(x_1(t), x_2(t), \ldots, x_n(t), t).$$

Definition 4.3. The system of differential equations (4.3) is said to be *autonomous* if the right-hand side of (4.3) does not depend explicitly on t; otherwise it is said to be *nonautonomous*.

Definition 4.4. The first-order system (4.3) is said to be *linear* if it can be expressed as

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + g_i(t), \tag{4.4}$$

$i = 1, \ldots, n$. If not, then it is *nonlinear*. If the system is linear and $g_i(t) \equiv 0$, then the system is said to be *homogeneous*; otherwise it is *nonhomogeneous*.

Definition 4.5. A *solution* of a differential equation or system is a scalar function $x(t)$ or vector function $X(t)$, respectively, which when substituted into the differential equation or system makes it an identity.

Suppose, in addition to the nth-order differential equation, the solution satisfies n initial conditions. That is,

$$x(t_0) = x_0, \quad \frac{dx(t_0)}{dt} = x_1, \quad \cdots \quad \frac{d^{n-1}x(t_0)}{dt^{n-1}} = x_{n-1}. \tag{4.5}$$

Then the solution satisfying the differential equation and the initial conditions is known as the solution to an *initial value problem* (IVP). For a first-order system, an initial value problem has the form

$$\frac{dX}{dt} = F(X(t), t), \quad t > t_0, \quad X(t_0) = X_0. \tag{4.6}$$

The notation $dx(t_0)/dt$ means differentiation of x, then evaluation at t_0.

It is important to know conditions on the coefficients so that solutions to initial value problems exist and are unique. In the case of linear differential equations, the existence and uniqueness conditions are straightforward.

Theorem 4.1

(i) Let

$$\frac{d^nx}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = g(t), \tag{4.7}$$

$$x(t_0) = x_0, \quad \frac{dx(t_0)}{dt} = x_1, \quad \cdots \quad \frac{d^{n-1}x(t_0)}{dt^{n-1}} = x_{n-1}. \tag{4.8}$$

If the coefficients a_i and g, $i = 0, 1, \ldots, n-1$ are continuous on some interval containing t_0, $\alpha < t_0 < \beta$, then there exists a unique solution to the initial value problem (4.7) and (4.8) on this interval.

(ii) Let

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)x_j + g_i(t), \quad x_i(t_0) = x_{i0}. \tag{4.9}$$

for $i = 1, 2, \ldots, n$. If the coefficients a_{ij} and g_i, $i, j = 1, 2, \ldots, n$ are continuous on some interval $\alpha < t_0 < \beta$, then there exists a unique solution to the initial value problem (4.9) on this interval. □

Example 4.1 Consider the initial value problem

$$\frac{d^2x}{dt^2} + \frac{3}{t}\frac{dx}{dt} + \frac{x}{t^2} = 0,$$

where $x(1) = 0$ and $dx(1)/dt = 1$. This differential equation is second order, linear, nonautonomous, and homogeneous. The coefficients are continuous on $(0, \infty)$. Hence, applying Theorem 4.1, there exists a unique solution to this initial value problem. Two linearly independent solutions (see Section 4.4) to the differential equation are $x_1(t) = t^{-1}$ and $x_2(t) = t^{-1}\ln|t|$. The general solution to the differential equation is $x(t) = c_1x_1(t) + c_2x_2(t)$. Applying the initial conditions leads to the unique solution to the initial value problem: $x(t) = t^{-1}\ln|t|$. The differential equation in this example is a special type of equation known as a Cauchy-Euler differential equation. (See Exercise 4.) ■

For additional information on the theory of differential equation, consult a textbook on ordinary differential equations listed in the references (Brauer and Nohel, 1969; Cushing, 2004; Sánchez, 1968; Waltman, 1986).

4.3 First-Order Linear Differential Equations

In this section, we review how to solve first-order linear differential equations. This method involves an integrating factor.

An initial value problem for a first-order linear differential equation has the following form:

$$\frac{dx}{dt} + a_1(t)x = g(t), \quad x(t_0) = x_0.$$

Assume that a_1 and g are continuous for all $t \geq t_0$. The solution to this differential equation can be found with the use of an integrating factor. Let $I(t) = \exp(\int_{t_0}^t a_1(\tau)\,d\tau)$. The function $I(t)$ is known as an *integrating factor*. There are an infinite number of integrating factors, since any constant multiple of $I(t)$ is also an integrating factor. Multiplying both sides of the differential equation by the integrating factor $I(t)$ yields

$$I(t)\frac{dx\,(t)}{dt} + a_1(t)I(t)x(t) = I(t)g(t).$$

The left-hand side is an exact derivative,

$$\frac{d[x(t)I(t)]}{dt} = I(t)g(t).$$

Integrating both sides and solving for x gives the unique solution to the initial value problem,

$$x(t) = e^{\left(-\int_{t_0}^t a_1(\tau)\,d\tau\right)}\left[x_0 + \int_{t_0}^t e^{\left(\int_{t_0}^\tau a_1(u)\,du\right)}g(\tau)\,d\tau\right]. \tag{4.10}$$

Note that if $t = t_0$, then the solution satisfies the initial condition $x(t_0) = x_0$.

If $a_1 =$ constant and $t_0 = 0$, then the solution simplifies to

$$x(t) = e^{-a_1t}\left[x_0 + \int_0^t e^{a_1\tau}g(\tau)\,d\tau\right].$$

In addition, if the equation is homogeneous, $g \equiv 0$, then the solution is given by

$$x(t) = x_0e^{-a_1t}.$$

If $a_1 < 0$, then the preceding solution represents Malthusian or exponential growth. If $a_1 > 0$, then $\lim_{t\to\infty} x(t) = 0$.

Example 4.2 Let

$$\frac{dx}{dt} - \frac{x}{t} = te^{3t}, \quad x(1) = 2.$$

An integrating factor for this differential equation is $I(t) = e^{-\ln t} = t^{-1}$. Then

$$\frac{d(xt^{-1})}{dt} = e^{3t}.$$

Integrating both sides and solving for x yields the general solution

$$x(t) = ct + t\frac{e^{3t}}{3}. \tag{4.11}$$

Applying the initial condition gives the unique solution to the initial value problem,

$$x(t) = \left(2 - \frac{e^3}{3}\right)t + t\frac{e^{3t}}{3}.$$ ■

The solution (4.11) is the sum of two terms. The first term is the general solution to the homogeneous differential equation; that is, $x_h(t) = ct$ is the general solution to $dx/dt = x/t$. The second term is a particular solution to the nonhomogeneous differential equation, that is, $x_p(t) = te^{3t}/3$ is a solution to $dx/dt - x/t = te^{3t}$. The sum of these two solutions, $x(t) = x_h(t) + x_p(t)$, forms the *general solution* to the nonhomogeneous differential equation. These ideas form the basis of the solution method for higher-order linear differential equations.

4.4 Higher-Order Linear Differential Equations

The general solution to an nth-order, linear nonhomogeneous differential equation is the sum of two solutions, a general solution to the homogeneous differential equation and a particular solution to the nonhomogeneous differential equation,

$$x(t) = x_h(t) + x_p(t).$$

The general solution to an nth-order, homogeneous differential equation is the sum of n linearly independent solutions, $\phi_1(t), \dots, \phi_n(t)$, that is,

$$x_h(t) = \sum_{i=1}^{n} c_i\phi_i(t).$$

The linearly independent set $\{\phi_i(t)\}_{i=1}^n$ is called a *fundamental set of solutions* to (4.2). [Recall that solutions $\phi_1(t), \dots, \phi_n(t)$ are *linearly independent* if $\sum_{i=1}^n \alpha_i\phi_i(t) = 0$ implies $\alpha_i = 0$, $i = 1, 2, \dots, n$.] Therefore, the general solution to the nonhomogeneous differential equation (4.2) is $x(t) = \sum_{i=1}^n c_i\phi_i(t) + x_p(t)$. Various methods can be used to find the particular solution (e.g., variation of constants, method of undetermined coefficients).

4.4.1 Constant Coefficients

The special case of a linear homogeneous differential equation, where the coefficients are constant, is discussed in more detail. For this special case, there is a well-known method for solving homogeneous differential equations.

Suppose the coefficients of a linear homogeneous differential equation are constant. Then the differential equation (4.2) has the following form:

$$\frac{d^n x}{dt^n} + a_1\frac{d^{n-1}x}{dt^n} + \cdots + a_{n-1}\frac{dx}{dt} + a_n x = 0. \tag{4.12}$$

There exist n linearly independent solutions to this differential equation that exist for all time $(-\infty, \infty)$. To find these solutions, assume that $x(t) = e^{\lambda t}$. Note that λ can be real or complex. Substituting $x(t) = e^{\lambda t}$ into the homogeneous differential equation (4.12) yields

$$e^{\lambda t}(\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n) = 0.$$

The resulting polynomial

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

is known as the *characteristic polynomial* and the equation $P(\lambda) = 0$ is known as the *characteristic equation* of the differential equation (4.12). The roots of $P(\lambda)$ are the *eigenvalues*. The solution form taken by $e^{\lambda t}$ depends on whether the eigenvalues are real or complex.

In the case of a second-order linear differential equation, $n = 2$, the forms of the solution are summarized.

1. The eigenvalues λ_i, $i = 1, 2$ are real and distinct. The general solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$

2. The eigenvalues $\lambda_1 = \lambda_2$ are real and equal. The general solution is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}.$$

3. The eigenvalues $\lambda_{1,2} = u \pm iv$ are complex conjugates. The general solution is

$$x(t) = e^{ut}[c_1\cos(vt) + c_2\sin(vt)].$$

In the case of complex conjugates, $u \pm iv$, the identity $e^{ivt} = \cos(vt) + i\sin(vt)$ is used to express the solutions in terms of sines and cosines.

Higher-order, linear differential equations with constant coefficients may have a characteristic polynomial with an eigenvalue λ repeated r times (a root of multiplicity r). There must be r linearly independent solutions associated with that root. It can be shown that additional solutions are found by multiplying by powers of t. When the eigenvalue is real, then there are r linearly independent solutions given by

$$e^{\lambda t}, te^{\lambda t}, \dots, t^{r-1}e^{\lambda t}.$$

When the eigenvalue is complex, $\lambda = u + iv$, and is a root of multiplicity r, the complex conjugate $u - iv$ is also a root of multiplicity r. There must be $2r$ linearly independent solutions. Additional solutions independent from $e^{ut}\cos(vt)$ and $e^{ut}\sin(vt)$ are found by multiplying by powers of t:

$$te^{ut}\cos(vt),\quad te^{ut}\sin(vt),\quad \cdots,\quad t^{r-1}e^{ut}\cos(vt),\quad t^{r-1}e^{ut}\sin(vt).$$

The Wronskian can be used to verify that the n solutions of an nth-order linear differential equation are independent.

Definition 4.6. If $x_1(t), \dots, x_n(t)$ are n functions with $n - 1$ continuous derivatives, then the determinant

$$W(x_1, \dots, x_n)(t) = \det\begin{pmatrix} x_1(t) & \cdots & x_n(t) \\ x_1'(t) & \cdots & x_n'(t) \\ \vdots & \cdots & \vdots \\ x_1^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{pmatrix}$$

is called the *Wronskian* of $x_1, \dots, x_n$. The primes (′) denote differentiation with respect to t.

The following theorem states that if the Wronskian is nonzero for some t on the interval of existence, then the n solutions are linearly independent.

Theorem 4.2 *Suppose $\phi_1(t), \dots, \phi_n(t)$ are n solutions of the nth-order linear differential equation on the interval I,*

$$\frac{d^n x}{dt^n} + a_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dx}{dt} + a_n(t)x = 0.$$

Then the n solutions are linearly independent iff the Wronskian $W(\phi_1, \dots, \phi_n)(t) \neq 0$ for some $t \in I$. □

A proof of this result can be found in many ordinary differential equation texts.

Example 4.3 Consider the differential equation $x'''(t) - 4x''(t) = 0$, where $x'' = d^2x/dt^2$, and so on. The characteristic equation is given by

$$\lambda^3 - 4\lambda^2 = 0.$$

Hence, the roots or eigenvalues are 0, 0, and 4 and the three linearly independent solutions are $1, t$, and e^{4t}, respectively. The general solution is

$$x(t) = c_1 + c_2 t + c_3 e^{4t}.$$

To verify that these three solutions are linearly independent, we compute the Wronskian,

$$W(1, t, e^{4t}) = \det\begin{pmatrix} 1 & t & e^{4t} \\ 0 & 1 & 4e^{4t} \\ 0 & 0 & 16e^{4t} \end{pmatrix} = 16e^{4t} \neq 0.$$

■

Example 4.4 Suppose the characteristic polynomial for a seventh-order, linear homogeneous differential equation is

$$P(\lambda) = (\lambda^2 - 3\lambda + 4)^2(\lambda - 2)^3 = 0.$$

Then the roots are $\lambda_{1,2,3,4} = 3/2 \pm i\sqrt{7}/2$ and $\lambda_{5,6,7} = 2$. The general solution of the differential equation is

$$x(t) = e^{3t/2}[c_1\cos(\sqrt{7}t/2) + c_2\sin(\sqrt{7}t/2) + c_3 t\cos(\sqrt{7}t/2) + c_4 t\sin(\sqrt{7}t/2)] + e^{2t}[c_5 + c_6 t + c_7 t^2].$$

The coefficients c_i can be uniquely determined from the initial conditions. ■

Example 4.5 Consider the fourth-order linear differential equation,

$$\frac{d^4x}{dt^4} + 6\frac{d^3x}{dt^3} + 10\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0.$$

The characteristic polynomial satisfies

$$\lambda^4 + 6\lambda^3 + 10\lambda^2 + 6\lambda + 9 = (\lambda^2 + 1)(\lambda + 3)^2 = 0.$$

The eigenvalues are $\pm i, -3, -3$. Hence, the general solution satisfies

$$x(t) = c_1\cos(t) + c_2\sin(t) + c_3 e^{-3t} + c_4 t e^{-3t}.$$

If the initial conditions are $x(0) = 1$, $dx(0)/dt = 2$, $d^2x(0)/dt^2 = 1$, and $d^3x(0)/dt^3 = 0$, then the four constants c_1, c_2, c_3, and c_4 can be found by solving the following linear system:

$$\begin{aligned} c_1 + c_3 &= 1, \\ c_2 - 3c_3 + c_4 &= 2, \\ -c_1 + 9c_3 - 6c_4 &= 1, \\ -c_2 - 27c_3 + 27c_4 &= 0. \end{aligned}$$

The constants are $c_1 = 8/25$, $c_2 = 81/25$, $c_3 = 17/25$, and $c_4 = 4/5$. ■

It is important to note that an nth-order, linear homogeneous differential equation always has a solution equal to zero, $x(t) \equiv 0$. If all of the initial conditions are zero, then this is the unique solution to the initial value problem. In the case that all of the coefficients of the homogeneous, linear differential equation are constant, then we can determine whether the zero solution is "stable," that is, whether a solution to the initial value problem will tend to zero. Stability of the zero solution depends on the eigenvalues, the roots λ_i of the characteristic equation $P(\lambda) = 0$. Because solutions have the form $e^{\lambda_i t}$, it follows that solutions to initial value problems approach zero if the λ_i are negative real numbers or are complex numbers having negative real part.

The distinction in behavior between linear difference and linear differential equations lies in the form of their solution. In difference equations, the solutions are linear combinations of λ_i^t whereas in differential equations they are linear combinations of $e^{\lambda_i t}$. Solutions to a linear homogeneous *difference equation* with constant coefficients tend to zero if the eigenvalues λ_i have magnitude less than one, $|\lambda_i| < 1$, whereas solutions to the linear homogeneous *differential equation* with constant coefficients tend to zero if the eigenvalues λ_i are negative or have negative real part, $\lambda_i < 0$ or $u \pm iv$, $u < 0$. The following theorem shows that solutions approach zero at an exponential rate if the eigenvalues lie in the left half of the complex plane.

Theorem 4.3 *If all of the roots of the characteristic polynomial $P(\lambda)$ are negative or have negative real part, then given any solution $x(t)$ of the homogeneous differential equation (4.2), there exist positive constants M and b such that*

$$|x(t)| \leq Me^{-bt} \text{ for } t > 0$$

and

$$\lim_{t\to\infty}|x(t)| = 0.$$

Proof Let the roots of $P(\lambda)$ be denoted as $\lambda_k = u_k + iv_k$, where $u_k < 0$, $k = 1, \ldots, n$. There exists a positive constant b such that $u_k < -b$ or $u_k + b < 0$ for all $k = 1, \ldots, n$. Then

$$|e^{\lambda_k t}e^{bt}| = e^{(u_k+b)t},$$

which approaches zero as $t \to \infty$. Also, $|t^{r_k}e^{\lambda_k t}e^{bt}| = |t^{r_k}e^{(u_k+b)t}|$, where r_k is a nonnegative integer. This latter expression approaches zero as $t \to \infty$. Thus, there exists a constant $M_k > 0$ such that $|t^{r_k}e^{\lambda_k t}e^{bt}| \leq M_k$ or $|t^{r_k}e^{\lambda_k t}| \leq M_k e^{-bt}$ for $t > 0$. Any solution $x(t)$ is the the sum of terms of the form $\phi_k(t) = t^{r_k}e^{\lambda_k t}$,

$$x(t) = \sum_{k=1}^{n} c_k\phi_k(t),$$

where $\phi_k(t)$ are the fundamental set of solutions and c_k are constants, $k = 1, \ldots, n$. If $M_0 = \max_k|c_k|$ and $M = M_0[\sum_{k=1}^n M_k]$, then for $t \geq 0$,

$$|x(t)| \leq \sum_{k=1}^{n}|c_k||\phi_k(t)| \leq M_0\sum_{k=1}^{n}|\phi_k(t)|$$

$$\leq M_0\left[\sum_{k=1}^{n} M_k\right]e^{-bt} = Me^{-bt}.$$

It follows that $\lim_{t\to\infty}|x(t)| = 0$. □

Theorem 4.3 shows that the rate of convergence to zero is exponential and is determined by the root with the largest negative real part.

positive, which implies that if the characteristic equation is expanded and simplified,

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0,$$

then all of the coefficients must satisfy $a_i > 0, i = 1, \ldots, n$. ◀

As a consequence of this corollary, it follows that if any coefficient is zero in the characteristic polynomial, then at least one eigenvalue is either zero, is purely imaginary, or lies in the right half of the complex plane. For example, if $a_n = 0$, then there is a zero eigenvalue.

Example 4.6 Consider the differential equation,

$$\frac{d^3x}{dt^3} + a_2\frac{dx}{dt} + a_3x = 0, \quad a_2, a_3 > 0.$$

Because $a_1 = 0$, it follows from the corollary that at least one eigenvalue is zero, is purely imaginary, or lies in the right half of the complex plane. For example, when $a_2 = 2$ and $a_3 = 1$, the roots of the characteristic polynomial

$$\lambda^3 + 2\lambda + 1 = 0$$

are approximately -0.453 and $0.227 \pm 1.468i$. There are two complex roots with positive real part. ■

Example 4.7 Consider the linear differential equation

$$\frac{d^3x}{dt^3} + 4\frac{d^2x}{dt^2} + \frac{dx}{dt} + ax = 0.$$

The characteristic polynomial is

$$P(\lambda) = \lambda^3 + 4\lambda^2 + \lambda + a.$$

According to the Routh-Hurwitz criteria for the roots to have negative real part and the solution to approach zero, the coefficients must satisfy, $a_1 > 0$, $a_3 > 0$, $a_1a_2 > a_3$. But $a_1 = 4$, $a_2 = 1$, and $a_3 = a$ so that a must satisfy $4 > a > 0$. ■

4.6 Converting Higher-Order Equations to First-Order Systems

A linear differential equation of the form (4.2),

$$\frac{d^nx}{dt^n} + a_1\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-1}\frac{dx}{dt} + a_nx = g(t), \tag{4.13}$$

can be expressed as an equivalent first-order system. Define n new variables, $x_1, \ldots, x_n$, as follows:

$$\begin{aligned} x_1 &= x, \\ x_2 &= \frac{dx}{dt}, \\ &\vdots \\ x_n &= \frac{d^{n-1}x}{dt^{n-1}}. \end{aligned}$$

Then

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{dx}{dt} = x_2, \\ \frac{dx_2}{dt} &= \frac{d^2x}{dt^2} = x_3, \\ &\vdots \\ \frac{dx_n}{dt} &= \frac{d^nx}{dt^n} = -a_1x_n - a_2x_{n-1} - \cdots - a_{n-1}x_2 - a_nx_1 + g(t), \end{aligned}$$

where the last equation follows from the differential equation (4.13). Written in matrix form, the first-order linear system can be expressed as

$$\frac{dX}{dt} = AX + G(t),$$

where $X(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}$$

and $G(t) = (0, 0, 0, \ldots, g(t))^T$. Matrix A is the called the *companion matrix* associated with the differential equation (4.13).

Example 4.8 The following second-order equation can be converted to an equivalent first-order system of the form $dX/dt = AX + G(t)$:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = \sin(t).$$

The matrices A and G satisfy

$$A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \quad \text{and} \quad G(t) = \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}.$$ ■

A first-order system can sometimes be converted to a higher-order equation. This conversion cannot be done for all systems because first-order systems are more general than higher-order equations. We shall see where some of the problems lie in the next example. (A first-order differential system is *not* equivalent to a higher-order differential equation.)

Consider the case of a first-order system with constant coefficients,

$$\frac{dx}{dt} = a_{11}x + a_{12}y,$$

$$\frac{dy}{dt} = a_{21}x + a_{22}y.$$

In matrix notation, $dX/dt = AX$, where $A = (a_{ij})$ and $X = (x, y)^T$. Differentiating dy/dt with respect to t,

$$\frac{d^2y}{dt^2} = a_{21}\frac{dx}{dt} + a_{22}\frac{dy}{dt}.$$

If $a_{21} = 0$, then our technique fails and we try differentiating dx/dt with respect to t (which then requires $a_{12} \neq 0$). Suppose $a_{21} \neq 0$. Then substituting $dx/dt = a_{11}x + a_{12}y$ and $a_{21}x = dy/dt - a_{22}y$ leads to a second-order differential equation in y,

$$\frac{d^2y}{dt^2} - (a_{11} + a_{22})\frac{dy}{dt} + (a_{11}a_{22} - a_{12}a_{21})y = 0.$$

Note that the coefficient of dy/dt is $-\mathrm{Tr}(A)$ and the coefficient of y is $\det(A)$. The characteristic equation for the differential equation in y is

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{22} = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0.$$

Since the coefficients a_{ij} are constants, the solution to y can be obtained by finding the roots of the characteristic equation. Once y is known, the solution to x can be obtained from one of the original differential equations.

4.7 First-Order Linear Systems

The nonhomogeneous linear system has the form

$$\frac{dX}{dt} = A(t)X(t) + G(t).$$

The elements of the coefficient matrix $A(t)$ and the elements of the vector $G(t)$ are continuous on some interval containing the initial point t_0 so that there exists a unique solution to an IVP (Theorem 4.1). It follows from the theory for linear differential equations that the general solution to the nonhomogeneous system is the sum of the general solution to the homogeneous system and a particular solution to the nonhomogeneous system,

$$X(t) = X_h(t) + X_p(t).$$

The general solution to the homogeneous system consists of n linearly independent solutions, $\phi_i(t)$, $i = 1, \ldots, n$. A *fundamental matrix* of solutions is $\Phi(t) = (\phi_1(t), \ldots, \phi_n(t))$, where the columns of $\Phi(t)$ are the vectors $\phi_i(t)$. Because the solutions are linearly independent, $\det\Phi(t) \neq 0$ for all t on the interval of existence. (Compare with the Wronskian.) Hence, the inverse $\Phi^{-1}(t)$ exists for all t on the interval of existence. The unique solution to the IVP for the linear nonhomogeneous system can be expressed in the form

$$X(t) = \Phi(t)\Phi^{-1}(t_0)X_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)G(s)\,ds,$$

where $X(t_0) = X_0$. For a proof of this result see Brauer and Nohel (1969) or Waltman (1986). Compare the solution $X(t)$ to the unique solution of the first order equation $x(t)$ given in (4.10).

4.7.1 Constant Coefficients

There are many methods that can be applied to find solutions to first-order linear, homogeneous systems with constant coefficients. This type of system will be especially important in the study of nonlinear autonomous systems of differential equations in the next chapter. Let

$$\frac{dX}{dt} = AX, \tag{4.14}$$

where $A = (a_{ij})$ is a constant matrix with real elements a_{ij}. First note that the zero solution, $X = \mathbf{0}$, is a fixed point of the differential equation. The zero solution is also referred to as an *equilibrium point*, a *steady state*, or a *critical point*.

In the simplest case of (4.14) when the system reduces to a scalar equation, $A = (a)$ is a 1×1 matrix, then the differential equation is

$$\frac{dx}{dt} = ax. \tag{4.15}$$

The general solution to (4.15) is $x(t) = ce^{at}$. If the initial condition $x(0) = x_0$, then $x(t) = x_0e^{at}$ (as already shown in the Introduction). If $a < 0$, then $\lim_{t\to\infty} x(t) = 0$, but if $a > 0$, then the limit is infinite ($\pm\infty$ depending on the sign of x_0).

When the dimension of A is greater than one, the general solution of (4.14) can be expressed in terms of the exponential of a matrix A. The general solution is

$$X(t) = e^{At}C,$$

where e^{At} is an $n \times n$ matrix and C is an $n \times 1$ vector. Matrix $\Phi(t) = e^{At}$ is known as the fundamental matrix with the property that $\Phi(0) = I$, the $n \times n$ identity matrix (the columns of Φ are n linearly independent solutions of the differential equation). The matrix exponential e^{At} is defined as follows:

$$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots = \sum_{k=0}^{\infty} A^k\frac{t^k}{k!},$$

Where the series converges for all t. There are many methods for computing the matrix exponential (see, e.g., Leonard, 1996; Moler and Van Loan, 1978, 2003; Waltman, 1986). Some of these methods are discussed in the Appendix for Chapter 4. If the elements of A are known real values, then one may use computer algebra systems or numerical methods to compute e^{At}. In the next examples, the matrix exponential is computed directly from the definition of e^{At}.

Example 4.9 Suppose A is diagonal,

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

Then $A = \begin{pmatrix} a_{11}^k & 0 \\ 0 & a_{22}^k \end{pmatrix}$ and

$$e^{At} = \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(a_{11}t)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(a_{22}t)^k}{k!} \end{pmatrix} = \begin{pmatrix} e^{a_{11}t} & 0 \\ 0 & e^{a_{22}t} \end{pmatrix}.$$

The solution to the system $dX/dt = AX$ is

$$X(t) = e^{At}X_0 = x_0 e^{a_{11}t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0 e^{a_{22}t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x_0 e^{a_{11}t} \\ y_0 e^{a_{22}t} \end{pmatrix},$$

where $X_0 = (x_0, y_0)^T$. ∎

Example 4.10 Suppose $A = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$. Then $A^k = \begin{pmatrix} 0 & 2^{k-1} \\ 0 & 2^k \end{pmatrix}$ and

$$e^{At} = \begin{pmatrix} 1 & \frac{1}{2} \sum_{k=1}^{\infty} \frac{(2t)^k}{k!} \\ 0 & \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}[e^{2t} - 1] \\ 0 & e^{2t} \end{pmatrix}.$$

Thus, the solution to the linear system $dX/dt = AX$ is

$$X(t) = e^{At}X_0 = \begin{pmatrix} x_0 + \frac{1}{2} y_0 [e^{2t} - 1] \\ y_0 e^{2t} \end{pmatrix}.$$ ∎

The Maple commands for computing the exponential of a matrix are available in the linear algebra package. They are given below for the matrix in Example 4.10.

```
> with(linalg):
> A:=matrix(2,2,[0,1,0,2]);
```

$$A := \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

```
> eA:=exponential(A,t);
```

$$eA := \begin{bmatrix} 1 & \frac{1}{2}e^{2t} - \frac{1}{2} \\ 0 & e^{2t} \end{bmatrix}$$

A straightforward method to compute the general solution to $dX/dt = AX$, instead of computing e^{At}, is the same method that was used for the higher-order, constant coefficient differential equations. We need to find n linearly independent solutions, $\phi_1(t), \ldots, \phi_n(t)$, which make up the fundamental matrix, $\Phi(t)$. Let $X = e^{\lambda t}V$. Then it follows that $AV = \lambda V$, where λ is an

eigenvalue of A and V is an eigenvector corresponding to λ. We summarize the form taken by the general solution in the case of a 2×2 matrix A. The eigenvalues are the solutions to the characteristic polynomial, $\det(A - \lambda I) = 0$. If $A = (a_{ij})$, then the characteristic polynomial has the following form:

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21},$$

where the coefficients of the polynomial are the negative of the trace and the determinant of the matrix A,

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0.$$

The form taken by the solutions is summarized in the following three cases.

1. Eigenvalues λ_i, $i = 1, 2$ of A are real and there exist two linearly independent eigenvectors $V_i, i = 1, 2$ corresponding to $\lambda_i, i = 1, 2$. The general solution to $dX/dt = AX$ has the form
$$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}.$$
2. Eigenvalues $\lambda_i, i = 1, 2$ of A are real and equal ($\lambda_1 = \lambda_2$) and there exists only one linearly independent eigenvector V_1. The general solution to $dX/dt = AX$ has the form
$$X(t) = c_1 V_1 e^{\lambda_1 t} + c_2[V_1 t e^{\lambda_1 t} + P e^{\lambda_1 t}],$$
where the equation $(A - \lambda_1 I)P = V_1$ can be solved for P (vector P is known as a *generalized eigenvector*).
3. Eigenvalues λ_i, $i = 1, 2$ of A are complex conjugate pairs, $\lambda_{1,2} = a \pm ib$, $b \neq 0$. The general solution to $dX/dt = AX$ has the form
$$X(t) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} e^{at} \sin(bt) + \begin{pmatrix} \alpha_3 \\ \alpha_4 \end{pmatrix} e^{at} \cos(bt).$$
Actually, the solution involves only two independent arbitrary constants, c_1 and c_2. The four constants α_i, $i = 1, 2, 3, 4$, depend on the associated eigenvalues and eigenvectors and can be found by substitution into the differential equation.

For more information about solutions to first-order linear systems, please consult a textbook on ordinary differential equations.

4.8 Phase Plane Analysis

The solution behavior for two-dimensional linear systems is studied in the phase plane, that is, in the x-y plane. Let $X = (x, y)^T$ and $A = (a_{ij})$ so that $dX/dt = AX$ can be expressed as follows:

$$\frac{dx}{dt} = a_{11}x + a_{12}y,$$

$$\frac{dy}{dt} = a_{21}x + a_{22}y. \tag{4.16}$$

The origin, $x = 0$ and $y = 0$, is an equilibrium solution of system (4.16). Assume that $\det(A) \neq 0$. Then the origin is the unique equilibrium solution, an *isolated equilibrium*.

Solutions to the linear system (4.16) are characterized by the eigenvalues of the matrix A, which in turn depend on the trace and determinant of A. The origin will be classified as a node, saddle, spiral, or center. The origin is further classified as stable or unstable. We begin by defining these latter terms for a linear system. We distinguish between stable and asymptotically stable.

The origin is *asymptotically stable* if the eigenvalues of A are negative or have negative real part. The origin is *stable* if the eigenvalues of A are nonpositive or have nonpositive real part. The origin is *unstable* if the eigenvalues of A are positive or have positive real part. Solutions approach the origin if the origin is asymptotically stable, $\lim_{t\to\infty}(x(t), y(t)) = (0, 0)$. Based on these definitions, it is easy to determine the stability of the origin once the eigenvalues are known. In addition, even without calculating the eigenvalues, the stability can be determined by applying the Routh-Hurwitz criteria to the characteristic polynomial of A:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0.$$

Asymptotic stability is determined only by the trace and determinant because these two quantities are the coefficients of the characteristic polynomial. According to the Routh-Hurwitz criteria, the eigenvalues lie in the left half of the complex plane iff the coefficients are positive.

Corollary 4.2 *Suppose $dX/dt = AX$, where A is a constant 2×2 matrix with $\det(A) \neq 0$. The origin is asymptotically stable iff*

$$\mathrm{Tr}(A) < 0 \quad \text{and} \quad \det(A) > 0.$$

The origin is stable iff $\mathrm{Tr}(A) \leq 0$ *and* $\det(A) > 0$. *The origin is unstable iff* $\mathrm{Tr}(A) > 0$ *or* $\det(A) < 0$. ◀

Now, we give specific criteria for the origin of a general linear differential system to be classified into one of four types: node, saddle, spiral, or center. Then we apply the previous results to classify the origin as stable or unstable. This classification scheme is based on the fact that the origin is the only fixed point or equilibrium solution of the linear system, $\det(A) \neq 0$. Matrix A has no zero eigenvalues. The classification scheme depends on whether the eigenvalues are real or complex, whether the real eigenvalues are positive or negative, and whether the complex eigenvalues have negative real part. References for the qualitative theory of differential equations can be found in many textbooks (see, e.g., Brauer and Nohel, 1969; Cushing, 2004; Sánchez, 1968).

Real Eigenvalues:

In the case of real eigenvalues, λ_1 and λ_2, the corresponding eigenvectors V_1 and V_2 are directions along which solutions travel toward or away from the origin. For example, if λ_1 is positive, solutions will travel along V_1, away from the origin. If λ_2 is negative, solutions will travel along V_2, toward the origin. In general, solutions travel in a direction which is a linear combination of V_1 and V_2. The origin is classified as either a node or a saddle.

1. ***Node:*** Both eigenvalues have the same sign and may be distinct or equal, $\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$. The origin can be further classified as proper or improper (Brauer and Nohel, 1969; Sánchez, 1968). A node is called

proper when the eigenvalues are equal and there are two linearly independent eigenvectors; otherwise it is called *improper*. A proper node is also referred to as a *star point* or *star solution* (Cushing, 2004; Gulick, 1992). The reason for this latter name can be seen in Figure 4.1 (solutions approach the origin in all directions). The term *degenerate node* is also used to refer to a node when the two eigenvalues of matrix A are equal (Gulick, 1992). In Figure 4.1, the improper node in the upper left corner has two distinct eigenvalues; it is not degenerate. But the other two nodes, to the right of the node in the upper left corner, are degenerate nodes (the eigenvalues are equal). If there is only one independent eigenvector, the dynamics are illustrated in the center figure and if there are two independent eigenvectors, the dynamics are illustrated in the upper right corner (star solution).

2. ***Saddle:*** Eigenvalues λ_1 and λ_2 have opposite signs, $\lambda_1\lambda_2 < 0$ (e.g., $\lambda_1 < 0 < \lambda_2$).

Complex Eigenvalues:

In the case of complex eigenvalues, $\lambda_{1,2} = a \pm ib, b \neq 0$. Because solutions to the linear system $dX/dt = AX$ include factors with $\cos(bt)$ and $\sin(bt)$, solutions spiral around the equilibrium. If the real part $a < 0$, then the solutions with $e^{at}\cos(bt)$ or $e^{at}\sin(bt)$ spiral inward, toward the origin. But if the real part $a > 0$, then solutions spiral outward, away from the origin. Finally,

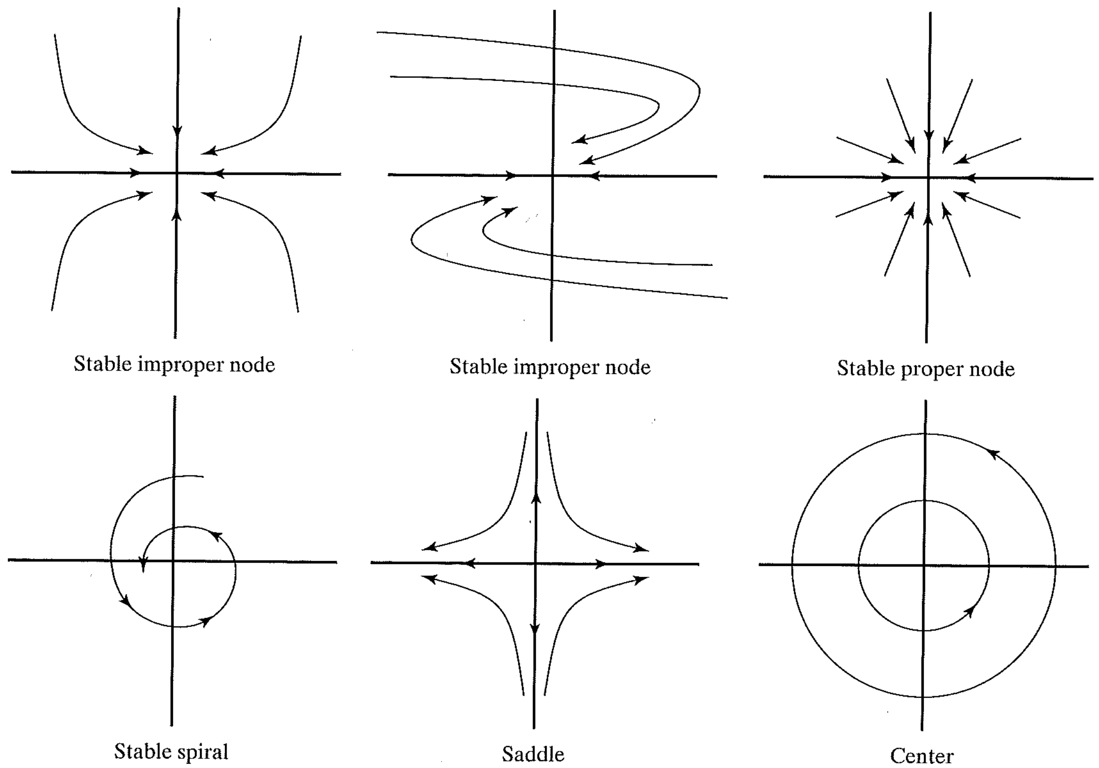


Figure 4.1 Graphs of solutions for an improper and proper node, spiral, saddle, and center.

if the real part $a = 0$, then solutions are closed curves, encircling the origin. The origin is classified as a *spiral (or focus)* if $a \neq 0$ and a *center* if $a = 0$.

3. ***Spiral or Focus:*** Eigenvalues have nonzero real part ($a \neq 0$).
4. ***Center:*** Eigenvalues are purely imaginary ($a = 0$), $\lambda_{1,2} = \pm ib$.

A node or spiral can be classified as either asymptotically stable or unstable depending on whether the real part of the eigenvalue is negative or positive, respectively. A saddle point is always unstable and a center is neither asymptotically stable nor unstable. A center is sometimes called *neutrally stable* (it is stable, but not asymptotically stable).

Example 4.11 Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The eigenvalues of A are $\pm i$, so that the origin is a center. The solution to $dX/dt = AX$ can be found directly by noting $dx/dt = -y$ and $dy/dt = x$ so that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{x}{y}.$$

Separating variables and integrating,

$$\frac{y^2}{2} + \frac{x^2}{2} = c.$$

This latter equation is a circle centered at the origin. Solutions travel in a counterclockwise direction on circles surrounding the origin. ■

Example 4.12 Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The eigenvalues of A are ± 1, so that the origin is a saddle. The system $dX/dt = AX$ can be written as $dy/dx = x/y$. Separating variables and integrating,

$$\frac{y^2}{2} - \frac{x^2}{2} = c.$$

This latter equation is a hyperbola with center at the origin. See Figure 4.2. ■

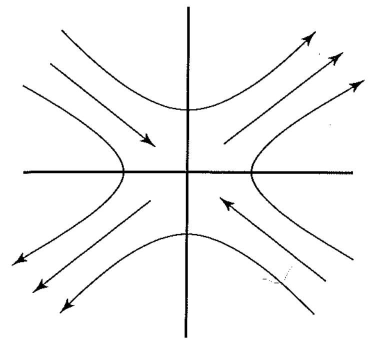


Figure 4.2 Solutions to Example 4.12 are graphed in the phase plane. The origin is a saddle.

Example 4.13 Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$. The eigenvalues of A are 1 and 3. The origin is an unstable node. Solutions can be found by integrating $dx/dt = x$ and $dy/dt = 3y$ so that $x(t) = x_0e^t$ and $y(t) = y_0e^{3t}$. Solutions in the phase plane have the form

$$X(t) = x_0e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y_0e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ ■

The classification schemes can be related to the signs of the trace and determinant of A and the discriminant of the characteristic polynomial. Let

$$\tau = \mathrm{Tr}(A) \quad \text{and} \quad \delta = \det(A).$$

Recall that the eigenvalues, the roots of the characteristic polynomial $\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - \tau\lambda + \delta$, satisfy

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}.$$

The discriminant is denoted as γ and defined as follows:

$$\gamma = \tau^2 - 4\delta.$$

The following classification scheme summarizes the dynamics according to the sign of the discriminant, γ, that is, according to whether the eigenvalues are real or complex conjugates. Improper and proper nodes are not distinguished.

Eigenvalues are real ($\gamma \geq 0$):

Unstable node if $\tau > 0$ and $\delta > 0$ ($\lambda_{1,2} > 0$)
Saddle point if $\delta < 0$ ($\lambda_1 < 0 < \lambda_2$)
Stable node if $\tau < 0$ and $\delta > 0$ ($\lambda_{1,2} < 0$).

Eigenvalues are complex conjugates $a \pm bi$ ($\gamma < 0$):

Unstable spiral if $\tau > 0$ ($a > 0$)
Neutral center if $\tau = 0$ ($a = 0$)
Stable spiral or stable focus if $\tau < 0$ ($a < 0$)

The classification scheme is illustrated in the τ-δ plane in Figure 4.3. Note that asymptotic stability requires $\tau < 0$ and $\delta > 0$ (the trace is negative and the determinant is positive, Corollary 4.2).

Example 4.14 Determine the conditions on a so that the zero equilibrium of the following system is a stable spiral:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x + ay.$$

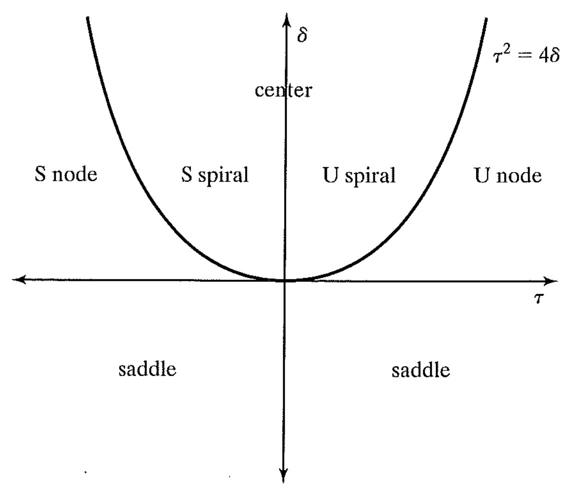


Figure 4.3 Stability diagram in the τ-δ plane.

17. Show that the solution to the pharmacokinetics model is

$$x(t) = \frac{1}{a}(1 - e^{-at}),$$
$$y(t) = \frac{1}{b} + \frac{e^{-at}}{a - b} - \frac{ae^{-bt}}{b(a - b)}.$$

18. Suppose the drug in the pharmacokinetics model is administered periodically, $d(t) = 2$ for $t \in [6t, 6t + 0.5)$, $t = 0, 1, \ldots,$ and $d(t) = 0$ elsewhere. Let $a = 2/\ln(2)$ and $b = \ln(2)/5$. Let the initial conditions satisfy $x(0) = 0$ and $y(0) = 0$. The differential equation for $x(t)$ can be expressed in terms of the Heaviside function, $H(t) = 1$ for $t \geq 0$ and $H(t) = 0$ for $t < 0$. For example, on the interval $t \in [0, 48)$,

$$\frac{dx(t)}{dt} = -ax(t) + 2\sum_{k=0}^{7} H(t - 6k) - 2\sum_{k=0}^{7} H(t - 6k - 0.5).$$

(a) Use a differential equation solver to graph the solution during the first 48 hours, $t \in [0, 48)$. See the Maple program in the Appendix.

(b) How does the drug concentration change (maximum and minimum after 24 hours) if the dose is changed from every 6 hours to every 12 hours?

(c) How does the drug concentration change (maximum and minimum after 24 hours) if the dose is reduced to $d(t) = 1$?

19. Show that the solutions to the delay differential equation on $[0, 2]$ ($\tau = 1$) in Example 4.18 are

$$\phi_1(t) = \frac{1}{2}e^{-t} + \frac{1}{2} \quad \text{and} \quad \phi_2(t) = \frac{1}{4}[1 + te^{1-t}] + \frac{1}{2}e^{-t}.$$

20. Solve the following delay differential equations on the interval $[0, 2]$ by the method of steps.

(a) $\dfrac{dx(t)}{dt} = x(t - 1) + 2t$, with initial condition $x(t) = 0$ for $t \in [-1, 0]$.

(b) $\dfrac{dx(t)}{dt} = x(t - 1) + x(t)$ with initial condition $x(t) = 1$ for $t \in [-1, 0]$.

(c) $\dfrac{d^2x(t)}{dt^2} = 2x(t - 1) + 1$ with initial conditions $x(t) = 0$ for $t \in [-1, 0]$ and $dx/dt = 0$ for $t = 0$. This is a second-order delay differential equation, so the derivative at zero needs to be specified.

21. The whooping crane data in the Appendix for Chapter 3 appears to increase exponentially. Fit the whooping crane data to the exponential curve $x(t) = x_0e^{at}$ using a least squares approximation. Use *polyfit* in MATLAB with the curve $\ln(x(t)) = \ln(x_0) + at$, where $x(t)$ is the whooping crane winter population size. What are the estimates for x_0 and a? These estimates should agree with those for the discrete model $x(t) = x_0\lambda^t$, where $\lambda = e^a$. See Example 3.1 in Chapter 3.

4.13 References for Chapter 4

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4.14 Appendix for Chapter 4

4.14.1 Exponential of a Matrix

The matrix exponential e^{At} can be computed in several ways. We discuss two methods here.

If matrix A is an $n \times n$ diagonalizable matrix, then, in theory, it is straightforward to compute the exponential of matrix A. Recall that a matrix A is diagonalizable iff A has n linearly independent eigenvectors (Ortega, 1987). In addition, if all of the eigenvalues of A are distinct, then the eigenvectors are linearly independent. Suppose matrix A is diagonalizable and the eigenvalues of A are λ_i, $i = 1, 2, \ldots, n$. Then A^k can be expressed in terms of the eigenvectors of A,

$$A^k = H\Lambda^k H^{-1},$$

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the columns of H are the right eigenvectors of A, ordered corresponding to their associated eigenvalues. Then e^{At} simplifies to

$$e^{At} = H\sum_{k=0}^{\infty} \Lambda^k \frac{t^k}{k!} H^{-1} = H\,\text{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})H^{-1}. \tag{4.18}$$

Another method for computing e^{At} is due to Leonard (1996). This method does not require A to be diagonalizable. However, it requires solving an nth-order differential equation. This is sometimes easier than solving the linear system $dX/dt = AX$ consisting of n equations. Suppose A is an $n \times n$ matrix with characteristic equation

$$\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0.$$

This polynomial equation is also a characteristic equation of an nth-order scalar differential equation of the form

$$x^{(n)}(t) + a_1x^{(n-1)}(t) + \cdots a_nx(t) = 0.$$

To find a formula for e^{At} it is necessary to find n linearly independent solutions to this nth-order scalar differential equation, $x_1(t), x_2(t), \ldots, x_n(t)$, with initial conditions

$$\left.\begin{array}{l} x_1(0) = 1 \\ x_1'(0) = 0 \\ \vdots \\ x_1^{(n-1)}(0) = 0 \end{array}\right\}, \quad \left.\begin{array}{l} x_2(0) = 0 \\ x_2'(0) = 1 \\ \vdots \\ x_2^{(n-1)}(0) = 0 \end{array}\right\}, \quad \ldots, \quad \left.\begin{array}{l} x_n(0) = 0 \\ x_n'(0) = 0 \\ \vdots \\ x_n^{(n-1)}(0) = 1 \end{array}\right\}.$$

Then

$$e^{At} = x_1(t)I + x_2(t)A + \cdots + x_n(t)A^{n-1}, \quad -\infty < t < \infty. \tag{4.19}$$

Verification of equation (4.19) can be found in Leonard (1996). These latter two methods are illustrated in the following example.

Example 4.19. Suppose matrix A is given by

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$ with corresponding eigenvectors $(1,1)^T$ and $(-1,1)^T$, respectively. Matrix A is diagonalizable. Both of the methods discussed previously can be applied. Matrices

$$H = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad H^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Then

$$\begin{aligned} e^{At} &= H\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{pmatrix}H^{-1} \\ &= \begin{pmatrix} \frac{1}{2}[e^{-t} + e^{3t}] & \frac{1}{2}[e^{-t} - e^{3t}] \\ \frac{1}{2}[e^{-t} - e^{3t}] & \frac{1}{2}[e^{-t} + e^{3t}] \end{pmatrix} \end{aligned} \tag{4.20}$$

To apply the method of Leonard given by (4.19), we find the characteristic polynomial of A: $\lambda^2 - 2\lambda - 3 = 0$. Then the corresponding second-order differential equation, $x''(t) - 2x'(t) - 3x(t) = 0$, has a general solution $x(t) = c_1e^{-t} + c_2e^{3t}$. Applying the initial conditions to find the constants c_1 and c_2, the solutions $x_1(t)$ and $x_2(t)$ are

$$x_1(t) = \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \quad \text{and} \quad x_2(t) = -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t},$$

respectively. Then applying the identity (4.19) gives the solution

$$\begin{aligned} e^{At} &= x_1(t)I + x_2(t)A \\ &= \begin{pmatrix} \frac{1}{2}[e^{-t} + e^{3t}] & \frac{1}{2}[e^{-t} - e^{3t}] \\ \frac{1}{2}[e^{-t} - e^{3t}] & \frac{1}{2}[e^{-t} + e^{3t}] \end{pmatrix} \end{aligned}$$

which agrees with (4.20).

4.14.2 Maple Program: Pharmacokinetics Model

The following Maple commands use the DEtools package to numerically approximate the solution to the pharmacokinetics model in Exercise 18.

```
> with(DEtools):
> a:=2*ln(2); b:=ln(2)/5;
> xeq:=diff(x(t),t)=-a*x(t)+2*sum(Heaviside(t-6*k),k=0..7)
      -2*sum(Heaviside(t-6*k-0.5),k=0..7);
> yeq:=diff(y(t),t)=a*x(t)-b*y(t);
> ic:=x(0)=0, y(0)=0;
> DEplot([xeq,yeq],[x(t),y(t)],t=0..48,[[ic]],
      stepsize=0.025,scene=[t,x]);
> DEplot([xeq,yeq],[x(t),y(t)],t=0..48,[[ic]],
      stepsize=0.025,scene=[t,y]);
```