

1.

$$f: \mathbb{R} \rightarrow \mathbb{R} \quad (\text{or } \mathbb{C} \rightarrow \mathbb{C} \text{ or } \mathbb{R}^2 \rightarrow \mathbb{R}^2)$$

Consider

$$x_{m+1} = f(x_m), \quad x_0 \in \mathbb{R}, \text{ i.e.}$$

$$x_m = (f \circ f \circ \dots \circ f)(x_0) = f^m(x_0)$$

Def.  $O^+(x) = \{x, f(x), f^2(x), \dots, f^m(x), \dots\}$   
forward orbit of  $x$

$$O^-(x) = \{x, f^{-1}(x), f^{-2}(x), \dots\}$$

backward orbit (if  $f$  is invertible)

$$O(x) = \{\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots\}$$

(full) orbit = bana = rata

Def.  $x$  fixed point if  $f(x) = x$  fixpunkt  
kunto piste

$x$  periodic point with period  $n$  if  $f^n(x) = x$

prime period of  $x = \min \{n > 0 \mid n \text{ period of } x\}$

$$\text{Per}_n(f) = \{x \mid x \text{ period for } f, \text{ with period } n\}$$

$$\text{Fix}(f) = \{x \mid x \text{ fixed pt for } f\}$$

periodic orbit =  $O^+(x)$  if  $x$  periodic

period = jakso

2.

Def.  $x$  eventually periodic if  $x$  is not periodic but  $f^i(x)$  is.

Def. Let  $p$  have period  $m$ .  $x$  is forward asymptotic to  $p$  if

$$\lim_{k \rightarrow \infty} f^{km}(x) = p$$

$W^s(p)$ , the stable set of  $p$ , is

$$\{x \mid x \text{ is forward asymptotic to } p\}$$

Def. If  $f$  is invertible (e.g. a homeomorphism)

$W^u(p)$ , the unstable set of  $p$ , is

$$\{x \mid x \text{ is backward asymptotic to } p\}$$

or

$$\{x \mid \lim_{k \rightarrow -\infty} f^{km}(x) = p\}$$

Def.  $x$  is a critical point of  $f$  if  $f'(x) = 0$ .  
 $x$  is a non-degenerate critical point, if, in addition,  $f''(x) \neq 0$ ,  
and degenerate if  $f''(x) = 0$ .

3.

Ex. 3.12 Translations of the circle

$$T_\lambda(\theta) = \theta + 2\pi\lambda \pmod{2\pi}$$

Theorem 3.13 (Jacobi) Each orbit of  $T_\lambda$  is dense in  $S^1$  if  $\lambda$  is irrational.

[Weyl: Orbit is "uniformly distributed"]

Def. 4.1.  $p$  periodic w. prime period  $n$ .

The number  $(f^n)'(p)$  is the multiplier of  $p$ .

If  $|(f^n)'(p)| \neq 1$  the point  $p$  is hyperbolic.

Prop. 4.4. Let  $f \in C^1$  and let  $p$  be a hyperbolic fixed point with  $|f'(p)| < 1$ . Then there is an open interval  $U \ni p$  such that

$$\lim_{n \rightarrow \infty} f^n(x) = p$$

i.e.  $U \subset W^s(p)$ .

Cor. The convergence is exponentially fast.

Jacobi

$$T = T_\lambda$$

$$T(\theta) = 2\pi\lambda + \theta$$

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$\lambda \in \mathbb{Q} \Leftrightarrow$  all points periodic (period  $q$  if  $\lambda = \frac{p}{q}$ )

$\lambda \notin \mathbb{Q} \Leftrightarrow$  each orbit dense in  $S'$

Proof outline:

(1)  $T^n(\theta)$  are all distinct,  $n=0,1,2,\dots$

(2)  $\{T^n(\theta)\}$  has limit point  $\in S'$

(3)  $\exists k = m - n$  so that  $|T^m(\theta) - T^n(\theta)| < \varepsilon$  and

$$|T^k(\theta) - \theta| < \varepsilon$$

(4)  $\theta, T^k(\theta), T^{2k}(\theta), \dots$  partition  $S'$  in arcs shorter than  $\varepsilon$ .

Weyl's theorem on equidistribution

$\forall a, b$  ( $a < b$ )

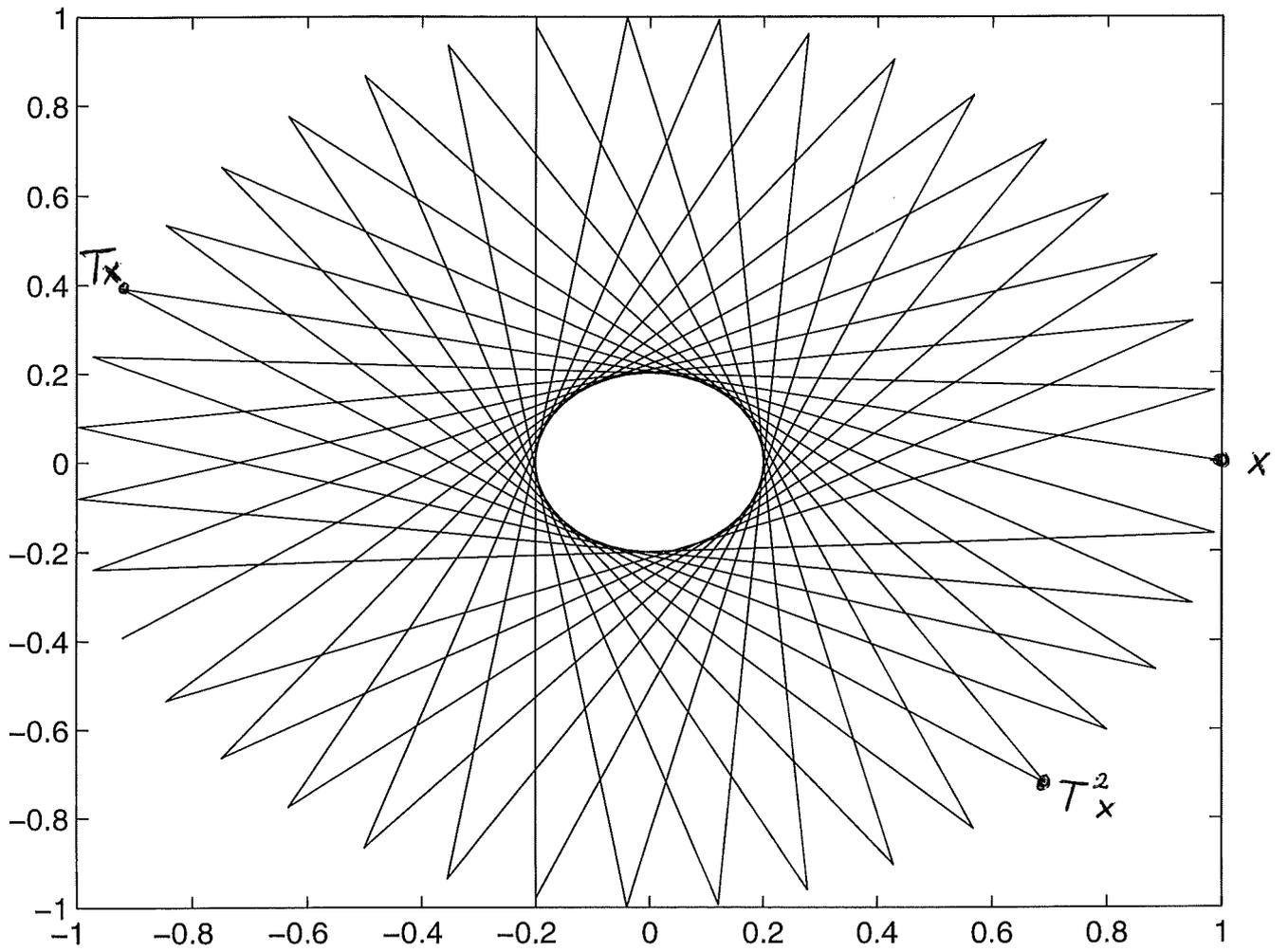
$$\frac{\#\{1 \leq m \leq N \mid T^m(\theta) \in (a, b)\}}{N} \rightarrow \frac{b-a}{2\pi}$$

Cor.  $\{n\lambda - [n\lambda] \mid n=1,2,\dots\}$  dense in  $[0,1)$

if  $\lambda \notin \mathbb{Q}$ .

$$\frac{\#\{1 \leq m \leq N \mid m\lambda - [m\lambda] \in (a, b)\}}{N} \rightarrow b-a$$

$$T_{\frac{17}{39}}^m(0), m = 0, \dots, 38$$



Def. 4.5.  $p$  hyperbolic periodic point

If  $|(f^m)'(p)| < 1$  then  $p$  is called an attractor (attracting periodic point, sink)

Note:  $\exists U \ni p : f^m(U) \subset U$

Also  $U \subset W^s(p)$ .

We call  $U$  a local stable set  $W_{loc}^s(p)$ .

Prop. 4.6.  $p$  hyperbolic fixed point

If  $|f'(p)| > 1$  then  $\exists U \ni p$ , interval,

such that  $x \in U \setminus \{p\} \Rightarrow \exists k > 0 : f^k(x) \notin U$ .

N.B.  $k = k(x)$ !

Def 4.7.  $p$  is called a repellor (repelling fixed point, source).

Note. Similar def. and results for  $f^m$ .

Note.  $U \setminus \{p\} \subset W^u(p)$

$U \setminus \{p\}$  is a local unstable set  $W_{loc}^u(p)$ .

Non-hyperbolic points:

weakly attracting, weakly repelling

one-sided weakly attr., weakly rep.

Ex. 4.9.

$$Q_c(x) = x^2 + c$$

Fix p. ?

$$x^2 + c = x \iff x^2 - x + c = 0$$

has solution iff  $c \leq \frac{1}{4}$

$c = \frac{1}{4}$  Fixed point at  $x = \frac{1}{2}$ . Note:  $Q_c(x) = (x - \frac{1}{2}) + (x - \frac{1}{2})^2 + \frac{1}{2}$  when  $c = \frac{1}{4}$ .

$c < \frac{1}{4}$  Fixed points at  
 $x_1 = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$   
 $x_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - c}$

Multiplicities:  $1 + \sqrt{1 - 4c}$  at  $x_1$   
 $1 - \sqrt{1 - 4c}$  at  $x_2$

Also  $Q_c(x) = (1 + \sqrt{1 - 4c})(x - x_1) + (x - x_1)^2 + x_1$

Hence we can write the dynamical system

$$y_{n+1} = Q_c(y_n)$$

in the form

$$y_{n+1} - x_1 = (1 + \sqrt{1 - 4c})(y_n - x_1) + (y_n - x_1)^2$$

or, with a change of coordinates

$$\bar{y}_{n+1} = (1 + \sqrt{1 - 4c}) \bar{y}_n + \bar{y}_n^2$$

(Here  $\bar{y}_n = y_n - x_1$ )

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$$\text{Ex. } f_{\beta}(x) = \beta x + (1 - \beta x)x, \quad \beta > 0$$

$$f_{\beta}(0) = 0$$

$$f_{\beta}(1) = 1$$

$$f'_{\beta}(x) = 1 + \beta - 2\beta x$$

$$f'_{\beta}(0) = 1 + \beta > 1$$

$$f'_{\beta}(1) = 1 - \beta$$

$$\begin{aligned} f_2(x) &= 2x + (1 - 2x)x = 3x - 2x^2 \\ &= -(x-1) - 2(x-1)^2 + 1 \end{aligned}$$

Dyn. syst. for  $\beta = 2$  of the form

$$x_{n+1} = -(x_n - 1) - 2(x_n - 1)^2 + 1$$

or

$$\bar{x}_{n+1} = -\bar{x}_n - 2\bar{x}_n^2$$

where  $\bar{x}_n = x_n - 1$ .

LOGISTIC MAP (Ex. 4.10, Ch. 1.5)

$$F_{\mu}(x) = \mu x(1-x), \quad \mu > 1$$

$$F_{\mu}(0) = 0, \quad F'_{\mu}(0) = \mu > 1.$$

$$F_{\mu}(p_{\mu}) = p_{\mu} \quad \text{where } p_{\mu} = \frac{\mu-1}{\mu} \in (0, 1)$$

$$F'_{\mu}(x) = \mu - 2\mu x, \quad F''_{\mu}(x) = -2\mu$$

$$F'_{\mu}(p_{\mu}) = 2 - \mu \quad \therefore p_{\mu} \text{ attr. for } \mu \in (1, 3).$$

Prop. 5.1. Let  $F_\mu(x) = \mu x(1-x)$ ,  $\mu > 1$ .

1.  $F_\mu(0) = 0$ ,  $F_\mu\left(\frac{\mu-1}{\mu}\right) = \frac{\mu-1}{\mu}$ ,  $F_\mu(1) = 0$

2.  $0 < \frac{\mu-1}{\mu} < 1$  if  $\mu > 1$ .

$$P_\mu := \frac{\mu-1}{\mu}$$

Prop. 5.2. If  $\mu > 1$ , then

$$F_\mu^n(x) \rightarrow -\infty \text{ if } x < 0 \text{ or } x > 1.$$

Pf. Alt. 1. For  $x < 0$  we have  $F_\mu(x) = \mu x - \mu x^2 < \mu x$ ,  
 $F_\mu^2(x) < \mu^2 x$  [because  $F_\mu$  is increasing]

$\vdots$

$$F_\mu^n(x) < \mu^n x \rightarrow -\infty$$

Alt. 2. For  $x < 0$  we have  $F_\mu(x) < \mu x < x$ .

Also  $F'_\mu(x) = \mu - 2\mu x > 0$ , i.e.  $F_\mu$  increasing on  $(-\infty, 0)$ . Hence  $F_\mu^2(x) < F_\mu(x) < x < 0$ .

By induction

$$F_\mu^{n+1}(x) < F_\mu^n(x) < 0.$$

The seq.  $F_\mu^n(x)$  is decreasing. If it has

a limit  $p$ , then  $p$  is necessarily a fixed point:  $F_\mu^n(x) \rightarrow p$ ,  $F_\mu^{n+1}(x) \rightarrow$

$F_\mu(p)$  means  $F_\mu(p) = p$ . But  $F_\mu$  has no

fixed point  $< 0$ .  $\therefore \lim F_\mu^n(x) \rightarrow -\infty$ .

For  $x > 1$ , we have  $F_\mu(x) < 0$ , so

$F_\mu^n(x) \rightarrow -\infty$  also in this case  $\square$

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Prop. 5.3. Let  $1 < \mu < 3$ .

1.  $F_\mu$  has a repelling fixed point at 0,  
 $F_\mu$  has an attracting fixed point at  $p_\mu$ .

2.  $\lim_{n \rightarrow \infty} F_\mu^n(x) \rightarrow p_\mu$  for all  $x \in (0, 1)$ .

Pf. of 2.

$$\underline{1 < \mu \leq 2}$$

$F_\mu$  is increasing on  $[0, \frac{1}{2}]$  and decreasing on  $[\frac{1}{2}, 1]$ .

$$\text{Also } 0 < x < \frac{\mu-1}{\mu} \left( \leq \frac{1}{2} \right) \implies x < F_\mu(x) < \frac{\mu-1}{\mu}$$

$\mu x - \mu x^2 - x = 0$  at  $x=0$   
and  $x = \frac{\mu-1}{\mu}$   
and positive between  
its zeroes.

Iterating:

$$x < F_\mu(x) < F_\mu^2(x) < \frac{\mu-1}{\mu} \quad \text{and, by induction,}$$

$$F_\mu^n(x) < F_\mu^{n+1}(x) < \frac{\mu-1}{\mu}.$$

The increasing sequence  $F_\mu^n(x)$  converges to

some  $p \leq \frac{\mu-1}{\mu}$ . But, again,  $p$  must be a fixed point, since  $\lim F_{\mu}^n(x) = \lim F_{\mu}^{n+1}(x) = p$ , and  $p_{\mu}$  is the only available fixed point.

$$\therefore \lim_{n \rightarrow \infty} F_{\mu}^n(x) = \frac{\mu-1}{\mu}.$$

If

$$\frac{\mu-1}{\mu} \leq x \leq \frac{1}{2} \quad \text{then} \quad \frac{\mu-1}{\mu} \leq F_{\mu}(x) \leq x$$

$$\leq \frac{\mu}{4} \leq \frac{1}{2}$$

see above:

$$\mu x - \mu x^2 - x \leq 0$$

outside the zeroes

We get

$$\frac{\mu-1}{\mu} \leq F_{\mu}^2(x) \leq F_{\mu}(x) \leq x \leq \frac{1}{2}$$

and

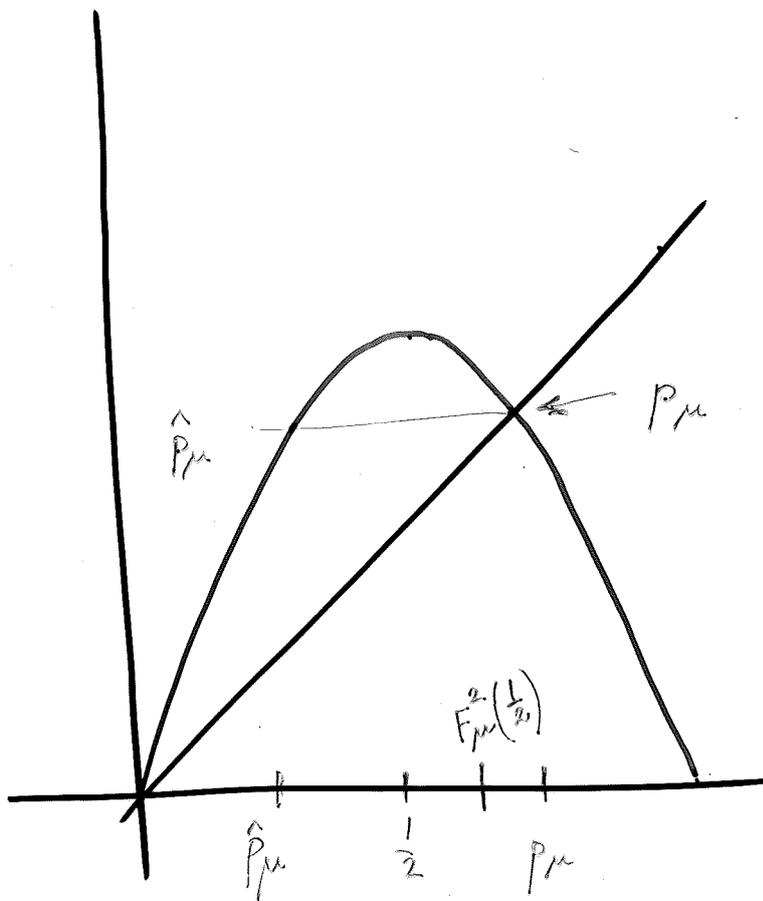
$$F_{\mu}^{n+1}(x) \leq F_{\mu}^n(x) \text{ all } n.$$

The decreasing sequence  $F_{\mu}^n(x)$  must have a limit  $p \geq \frac{\mu-1}{\mu}$ . Only fixed point available

is  $\frac{\mu-1}{\mu}$  itself.  $\therefore F_{\mu}^n(x) \rightarrow \frac{\mu-1}{\mu}$ , as  $n \rightarrow \infty$

Finally,  $\frac{1}{2} < x < 1 \Rightarrow F_{\mu}(x) < \frac{1}{2}$  so the previous cases apply. Thus  $0 < x < 1 \Rightarrow \lim_{n \rightarrow \infty} F_{\mu}^n(x) = \frac{\mu-1}{\mu}$ .

$$\underline{2 < \mu \leq 3}$$



$$P_\mu \equiv \frac{\mu-1}{\mu} > \frac{1}{2}$$

$$\hat{P}_\mu = \frac{1}{\mu} < \frac{1}{2}$$

$$F_\mu(\hat{P}_\mu) = P_\mu$$

$$\frac{\mu}{4} > \frac{\mu-1}{\mu} \text{ because}$$

$$\frac{\mu^2}{4} - \mu + 1 > 0$$

$$\frac{1}{4}(\mu^2 - 4\mu + 4)$$

$$\frac{1}{4}(\mu-2)^2$$

$$0 < x < \hat{P}_\mu \Rightarrow x < F_\mu(x)$$

so  $x, F_\mu(x), \dots, F_\mu^n(x)$  increasing until  $\in [\hat{P}_\mu, P_\mu]$

$P_\mu < x < 1 \Rightarrow 0 < F_\mu(x) < \hat{P}_\mu$ , so "can't stop because no fixed point in  $(0, \hat{P}_\mu)$ "

$F_\mu(x), F_\mu^2(x), \dots, F_\mu^n(x)$  increasing until  $\in [\hat{P}_\mu, P_\mu]$

$$\hat{P}_\mu \leq x \leq P_\mu \Rightarrow F_\mu(x) \in [P_\mu, \frac{\mu}{4}]$$

Further  $F_\mu^2(x) \in (\frac{1}{2}, P_\mu]$  where  $F_\mu$  is decreasing

and  $F_\mu^2(x)$  is increasing. We have, then,

$$\frac{1}{2} \leq x \leq P_\mu \Rightarrow \frac{1}{2} < F_\mu^2(\frac{1}{2}) \leq F_\mu^2(x) \leq F_\mu^2(P_\mu) = P_\mu$$

Since  $x < F_\mu^2(x)$  we get an increasing sequence  $x, F_\mu^2(x), F_\mu^4(x), \dots \rightarrow p \leq \frac{\mu-1}{\mu}$ . 11

As usual, there is no choice but  $p = p_\mu$ . Then the odd powers converge to  $p_\mu$ , too.

Why is  $F_\mu^2[\hat{p}_\mu, p_\mu] = F_\mu^2[\frac{1}{2}, p_\mu] \subset [\frac{1}{2}, p_\mu]$ ?

look at

$$F_\mu^2\left(\frac{1}{2}\right) = F_\mu\left(\frac{\mu}{4}\right) = \frac{\mu}{4} \cdot \mu \cdot \left(1 - \frac{\mu}{4}\right) = \frac{\mu^2}{4} - \frac{\mu^3}{16}$$

This expression is  $\frac{1}{2}$  for  $\mu = 2$ ,  $\frac{9}{16}$  for  $\mu = 3$ , maximal at  $\mu = \frac{8}{3}$  with value  $\frac{16}{27}$ , minimal at  $\mu = 2$  (for  $\mu \in [2, 3]$ ).

Why is  $x < F_\mu^2(x)$  for  $x \in [\frac{1}{2}, p_\mu]$ .

Answer 1. If not we'd have  $x = F_\mu^2(x)$  somewhere in the interval and  $p_\mu$  is the only fixed point for  $F_\mu^2$ . (look at graph!)

Answer 2. Careful analysis of 4th degree expression

$F_\mu^2(x) - x$  to show that there are no zeroes in  $[\frac{1}{2}, p_\mu]$  other than right endpoint  $p_\mu$ .

12 If  $F_\mu^2(x) - x$  has a zero in  $[\frac{1}{2}, p_\mu)$  then, necessarily, its second derivative must be 0 somewhere between  $\frac{1}{2}$  and  $p_\mu$ . Let us show that this is impossible!

$$\begin{aligned} (F_\mu^2(x) - x)'' &= F_\mu''(F_\mu(x)) \cdot F_\mu'(x)^2 + F_\mu'(F_\mu(x)) \cdot F_\mu''(x) \\ &= (-2\mu) \{ (\mu - 2\mu x)^2 + \mu - 2\mu(\mu x - \mu x^2) \} \\ &= (-2\mu) \{ 6\mu^2 x^2 - 6\mu^2 x + \mu^2 + \mu \}. \end{aligned}$$

This parabola has its maximal value in  $x = \frac{1}{2}$  (where the derivative  $(-2\mu)(12\mu^2 x - 6\mu^2)$  is 0). The value is  $\mu^2(\mu - 2) > 0$  (recall:  $2 < \mu \leq 3$ ).



Thus we need to show that the expression is  $> 0$  at  $p_\mu$ , too.

For  $x = p_\mu$  we get the value

$$(-2\mu)(\mu^2 - 5\mu + 6)$$

which is positive for  $2 < \mu < 3$  (and 0 for  $\mu = 2$  and  $\mu = 3$ )

$\therefore 2 < \mu \leq 3 \implies F_\mu^2(x) - x$  has no zeroes in  $[\frac{1}{2}, p_\mu)$ .

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$F_\mu$  with  $\mu > 3$  has a 2-period in  $(0, 1)$ :

Proof: The fixed points are 0 and  $p_\mu = \frac{\mu-1}{\mu}$

$F_\mu^2$  has derivative  $(2-\mu)^2 > 1$  in the fixed point:

$$\begin{aligned} (F_\mu^2)'(p_\mu) &= F_\mu'(F_\mu(p_\mu)) \cdot F_\mu'(p_\mu) \\ &= (F_\mu'(p_\mu))^2 = (2-\mu)^2 \end{aligned}$$

Thus the eq.  $F_\mu^2(x) - x$  has at least (actually exactly) 4 roots in  $[0, 1]$ : 0,  $p_\mu$  and two further roots.

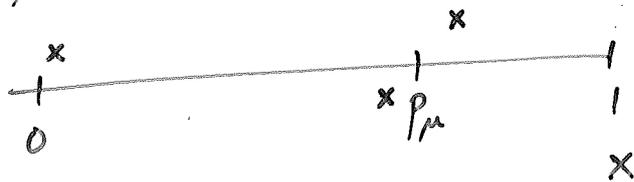
The derivative of  $F_\mu^2$  being  $> 1$  in the fixed points we get

$$F_\mu^2(x) - x > 0 \quad \text{for } x > 0, x \approx 0^+$$

$$F_\mu^2(x) - x < 0 \quad \text{for } x < p_\mu, x \approx p_\mu^-$$

$$F_\mu^2(x) - x > 0 \quad \text{for } x > p_\mu, x \approx p_\mu^+$$

$$F_\mu^2(1) - 1 = -1$$



Use the Intermediate Value Theorem!

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Now take  $\mu > 4$ .

Write  $F$  instead of  $F_\mu$ .

Consider  $\mathcal{O}^+(x) = \{x, F(x), F^2(x), \dots\}$

When  $\mu > 4$  "most" orbits go to  $-\infty$ , but some do not.

Put  $\Lambda = \{x \in [0, 1] \mid F^n(x) \not\rightarrow -\infty \text{ as } n \rightarrow \infty\}$

$0, 1, p_\mu, \hat{p}_\mu \in \Lambda$

Let  $A_0 = \{x \in (0, 1) : F(x) > 1\}$ .  $A_0 \subset \Lambda^c$ .

$x \in A_0$  has an orbit which escapes from  $(0, 1)$  in one step.

$A_0$  is a symmetric interval around  $\frac{1}{2}$  of the form  $(a, b)$  where  $F(a) = F(b) = 1$ .

Write  $I = [0, 1]$ ,  $I_0 = [0, a]$ ,  $I_1 = [b, 1]$ .

$I = I_0 \cup A_0 \cup I_1$ .

Note that  $F$  is monotone on  $I_0$  ( $I_1$ ).

$F(I_0) = F(I_1) = I$ .

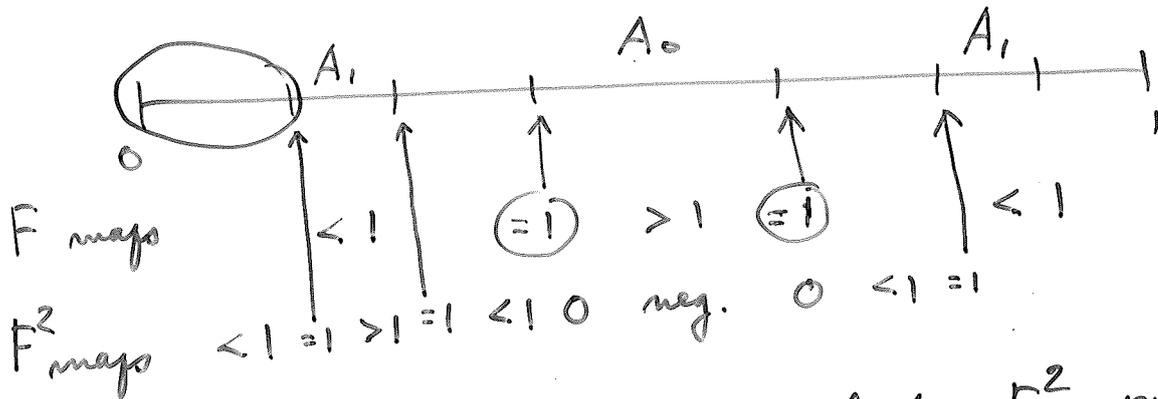
Hence there is an interval in  $I_0$  ( $I_1$ ) mapped onto  $A_0$ .

Let  $A_1 = \{x \in I_0 \cup I_1 \mid F(x) \in A_0\}$

12c  $A_1$  consists of two open intervals, one is the middle interval of  $I_0$ , the other one is the middle interval of  $I_1$ .

$F^2$  maps these intervals out of  $[0,1]$ .

$$A_1 = \{x \in I_0 \cup I_1 \mid F^2(x) > 1\}$$



The red interval is mapped by  $F^2$  into  $[0,1]$ .  
 It has "a middle interval" mapped by  $F^2$  into  $A_0$ . This is part of  $A_2$

Def.  $A_2 = \{x \in I \setminus (A_0 \cup A_1) \mid F^2(x) \in A_0\}$   
 or  $F^3(x) > 1$

$A_2$  consists of 4 open intervals

$I \setminus (A_0 \cup A_1 \cup A_2)$  consists of 8 closed intervals

Note The interval endpoints are eventually fixed  $F^k(x) = 1, F^{k+1}(x) = 0$ .

12d

$F^3$  maps these 8 intervals onto  $[0, 1]$ . The graph of  $F^3$  intersects the main diagonal at least 8 times.

$\therefore$  There are at least 8 points of period 3.

Generalizing:

$$I \setminus (A_0 \cup A_1 \cup \dots \cup A_n)$$

contains all  $x \in I$  such that  $F^{n+1}(x) \in I$ .

It consists of  $2^{n+1}$  intervals. On each of these intervals  $F^{n+1}$  is strictly monotone. Each of these closed intervals is mapped onto  $[0, 1]$  by  $F^{n+1}$ .

$\therefore$  There are, at least  $2^{n+1}$  points in  $\text{Per}_{n+1}(F)$ .

$$\Lambda := I \setminus \bigcup_{k=0}^{\infty} A_k$$

is a non-empty closed set, a so-called Cantor set

Georg Cantor  
1845 - 1918

closed, totally disconnected, perfect,  
contains no interval, contains no isolated points