19\. If there is a homeomorphism $h : A \to B$ such that

$$h \circ f = g \circ h \quad (\text{or } h \circ f \circ h^{-1} = g)$$

then f and g are *topologically conjugate*. h is called a *topological conjugacy*.

[Think of h as a coordinate transformation]

Cf. Exercise 1.

$$Q_c(x) = x^2 + c \qquad (c < \tfrac{1}{4})$$

$$F_\mu(x) = \mu x(1-x) \qquad (\mu > 1)$$

are topologically conjugate (for some c, μ) and h is of the form

$$h(x) = \alpha x + \beta$$

Look **at** F_μ, $\mu > 2 + \sqrt{5}$.

<u>Def. 7.1</u> The <u>itinerary</u> of $x \in \Lambda$ is a seq. $S(x) = s_0 s_1 s_2 s_3 \cdots$ where $s_i = 0$ if $F_\mu^i(x) \in I_0$ and $s_i = 1$ if $F_\mu^i(x) \in I_1$, $i = 0, 1, 2, \ldots$

<u>Theorem 7.2.</u> $S : \Lambda \to \Sigma_2$ is a homeomorphism.

Proof:

- S is 1-1

If x and y have the same itinerary then the interval $[x, y] \subset \Lambda$ $\lightning$

- $S(\Lambda) = \Sigma_2'$

Denote $\{ x \in I \mid F_\mu^n(x) \in J \}$ by $F_\mu^{-n}(J)$

If $J \subset I$ then $F_\mu^{-1}(J)$ consists of two subintervals, one in I_0, the other one in I_1.

Let $s = s_0 s_1 s_2 s_3 \dots s_n s_{n+1} \dots \in \Sigma_2$. Define

$$I_{s_0 s_1 s_2 \dots s_n} = \{ x \in I \mid x \in I_{s_0}, F_\mu(x) \in I_{s_1}, \dots, F_\mu^n(x) \in I_{s_n} \}$$
$$= I_{s_0} \cap F_\mu^{-1}(I_{s_1}) \cap F_\mu^{-2}(I_{s_2}) \dots \cap F_\mu^{-n}(I_{s_n})$$

I_{s_0} is a single interval

- if $I_{s_0 s_1 \dots s_n}$ is a single interval then so is $I_{s_0 s_1 s_2 \dots s_n s_{n+1}}$,

because $I_{s_0} \cap \underbrace{F_\mu^{-1}(\underbrace{I_{s_1 s_2 \dots s_n s_{n+1}}}_{\text{one interval}})}_{\text{2 interval}}$ is

one interval and $= I_{s_0 s_1 s_2 \dots s_{n+1}}$

$\therefore$ $I_{s_0 s_1 \dots s_n}$ single interval for all n

$I_{s_0 s_1 \dots s_{n+1}} \subset I_{s_0 s_1 \dots s_n}$ $\therefore$ nested intervals

Nested, closed intervals have a non-empty intersection

$$\bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n} \neq \emptyset$$

For any such point x, $S(x) = s_0 s_1 s_2 \dots s_n \dots$

$\therefore$ S is onto

$\therefore$ x unique because S was 1-1, shown before.

$\therefore$ $| I_{s_0 s_1 \dots s_n} | \to 0$, $n \to \infty$

- S cont's

Assume $S(x) = s$. Take $\varepsilon > 0$. Choose n with $2^{-n} < \varepsilon$. Need to find δ.

Look at $| I_{t_0 t_1 \dots t_n} |$. Take the min of these lengths. If this number is called δ, then

$|x - y| < \delta \Rightarrow x, y$ lie in same subinterval

$I_{s_0 \dots s_n}$ $\therefore$ $S(y) = s_0 s_1 s_2 \dots s_n t_{n+1} \dots$

$\therefore$ $d(S(x), S(y)) \leq 2^{-n} < \varepsilon$.

- S^{-1} cont's

Assume $S^{-1}(s) = x$. Take $\varepsilon > 0$.

Take an n with $\delta_n = \min | I_{t_0 t_1 \cdots t_n} | < \varepsilon$.

If s, t are such that $d(s,t) < 2^{-n}$ then

$S^{-1}(s)$, $S^{-1}(t)$ lie in same subinterval of level n.

$= x$

$\therefore$ $| S^{-1}(s) - S^{-1}(t) | < \varepsilon$ if $d(s,t) < 2^{-n}$.

Theorem 7.3. $\quad S \circ F_\mu = \sigma \circ S$

Pf. $x \in \Lambda$ is determined by $\bigcap_{n \geq 0} I_{s_0 s_1 \cdots s_n}$.

$$I_{s_0 s_1 \cdots s_n} = I_{s_0} \cap F_\mu^{-1}(I_{s_1}) \cap \cdots \cap F_\mu^{-n}(I_{s_n})$$

Look at $F_\mu(x)$:

$$F_\mu(x) \in \underbrace{F_\mu(I_{s_0})}_{= I} \cap I_{s_1} \cap F_\mu^{-1}(I_{s_2}) \cap \cdots \cap F_\mu^{-n+1}(I_{s_n})$$

$\therefore$ The itinerary of $F_\mu(x)$ is

$$s_1 s_2 \cdots s_n s_{n+1} \cdots$$

$\therefore$

$$S(F_\mu(x)) = \sigma(S(x)) \qquad \square$$

Topological conjugacy preserves fixed points and periodic points:

General situation $\qquad h \circ f = g \circ h$.

where $f: A \to A$, $g: B \to B$, $h: A \to B$.

If p fixed point for f

$$f(p) = p$$

$$h(f(p)) = h(p)$$

$$g(h(p)) = h(p) \qquad \therefore \; h(p) \text{ fixed pt for } g$$

Period n:

$$f^n(p) = p$$

$$h \circ f \circ f^{n-1}(p) = h(p)$$

$$g \circ h \circ f^{n-1}(p) = h(p)$$

$$g \circ g \circ h \circ f^{n-2}(p) = h(p)$$

$$g \circ g \circ g \circ h \circ f^{n-3}(p) = h(p)$$

$$\vdots$$

$$g^{n-1} \circ h \circ f(p) = h(p)$$

$$g^{n-1} \circ g \circ h(p) = h(p)$$

$$g^n(h(p)) = h(p)$$

$$\therefore \; h(p) \text{ periodic for } g$$

Theorem 7.5. $F_\mu(x) = \mu x(1-x)$, $\mu > 2+\sqrt{5}$. Then

- $|\mathrm{Per}_n(F_\mu)| = 2^n$ on Λ
- $\mathrm{Per}(F_\mu)$ is dense in Λ
- F_μ has a dense orbit in Λ.

Chaos (Section 1.8)

Def 8.1. $f: J \to J$ is topologically transitive if $\forall U, V \subset J$, U, V open sets $\exists k > 0$

$$f^k(U) \cap V \neq \emptyset$$

Def 8.2. $f: J \to J$ has sensitive dependence on initial conditions if

$\exists \delta > 0 \quad \forall x \in J, \ \forall N \ni x, N$ open $\quad \exists y \in N, n \geq 0$

$$|f^n(x) - f^n(y)| > \delta$$

Ex. 8.3. $\quad F_\mu(x)$, $\mu > 2 + \sqrt{5}$

$F_\mu : \Lambda \to \Lambda$ has sensitive dep. on initial cond's

Note, however, Λ is not an interval.

F_μ is topologically transitive since there is a dense orbit.

Ex. 8.4. $\quad T_\lambda$, λ irrational

$$T_\lambda(\theta) = \theta + 2\pi\lambda, \quad \theta \in S^1$$

T_λ is topologically transitive but it has no sens. dependence.

Def. 8.5. Let V be a set. $f : V \to V$ is <u>chaotic</u> on V if

1. f has sensitive dep. on initial cond's
2. f is topologically transitive
3. periodic points are dense in V.

<u>Remark</u>. Vellekoop and Berglund showed that if $f : I \to I$ (I an <u>interval</u>) and f is continuous then

(2) f topologically transitive <u>implies</u> both

(1) f has sensitive dependence on initial conditions

and

(3) $\overline{Per(f)} = V$.

The implication cannot be reversed.

The result is not true if V is not an interval $\subset \mathbb{R}$.

Ex. 6 $\quad f(\theta) = 2\theta$ is chaotic on S^1.

[This is not so easy to show on a computer!]

Def. 8.7. f is <u>expansive</u> if

$$\exists \nu > 0 : \forall x, y \in J,\ x \neq y \quad \exists n :$$
$$|f^n(x) - f^n(y)| > \nu .$$

Ex. F_μ is expansive on Λ if $\mu > 2+\sqrt{5}$

Thus these maps are chaotic.

Ex. 8.9. $F_4(x) = 4x(1-x)$ is chaotic on $[0,1]$.

$$g(\theta) = 2\theta, \quad \theta \in S'$$

Def. $h_1 : S' \to [-1,1]$ by $h_1(\theta) = \cos(\theta)$

Let $q(x) = 2x^2 - 1, \quad x \in [-1,1]$

We have

$$(h_1 \circ g)(\theta) = \cos(2\theta)$$

$$= \cos^2\theta - 1$$

$$= q(\cos\theta)$$

$$= (q \circ h_1)(\theta)$$

$\therefore \quad h_1 \circ g = q \circ h_1$ but h_1 is not 1-1

$q \sim F_4$ since, if $h_2(t) = \frac{1}{2}(1-t)$,

then $h_2 : [-1,1] \to [0,1]$

and

$$F_4(h_2(t)) = h_2(q(t))$$

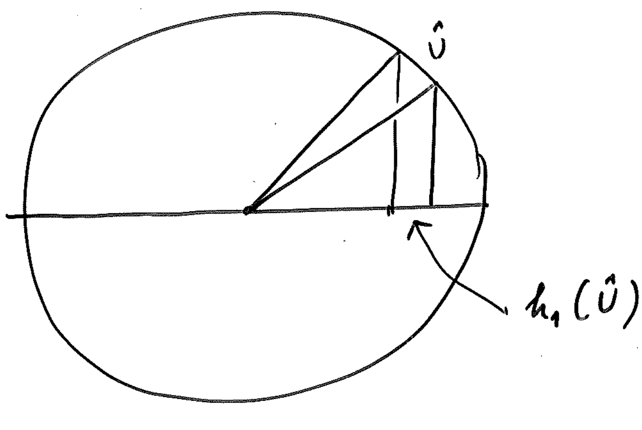
(Elementary verification)

$$\begin{tikzcd} S^1 \arrow[r, "g"] \arrow[d, "h_1"'] & S^1 \arrow[d, "h'"] \\ {[-1,1]} \arrow[r, "q"'] \arrow[d, "h_2"'] & {[-1,1]} \arrow[d, "h_2"] \\ {[0,1]} \arrow[r, "F_4"] & {[0,1]} \end{tikzcd}$$

semi-conjugate

conjugate

F_4 is chaotic iff q is. To show that q is chaotic we need the properties of g.



$$(h_1 \circ g \circ g)\theta = (q \circ h_1 \circ g)(\theta) = (q \circ q \circ h_1(\theta))$$

$$\therefore \quad h_1 \circ g^2 = q^2 \circ h_1 \qquad = (q^2 \circ h_1)(\theta)$$

$$\therefore \quad h_1 \circ g^n = q^n \circ h_1$$

p periodic for g $\Rightarrow$ $h_1(p)$ periodic for q.

top. trans., density of periodic pts + sens. dep. may be shown from the corresp. properties of g.