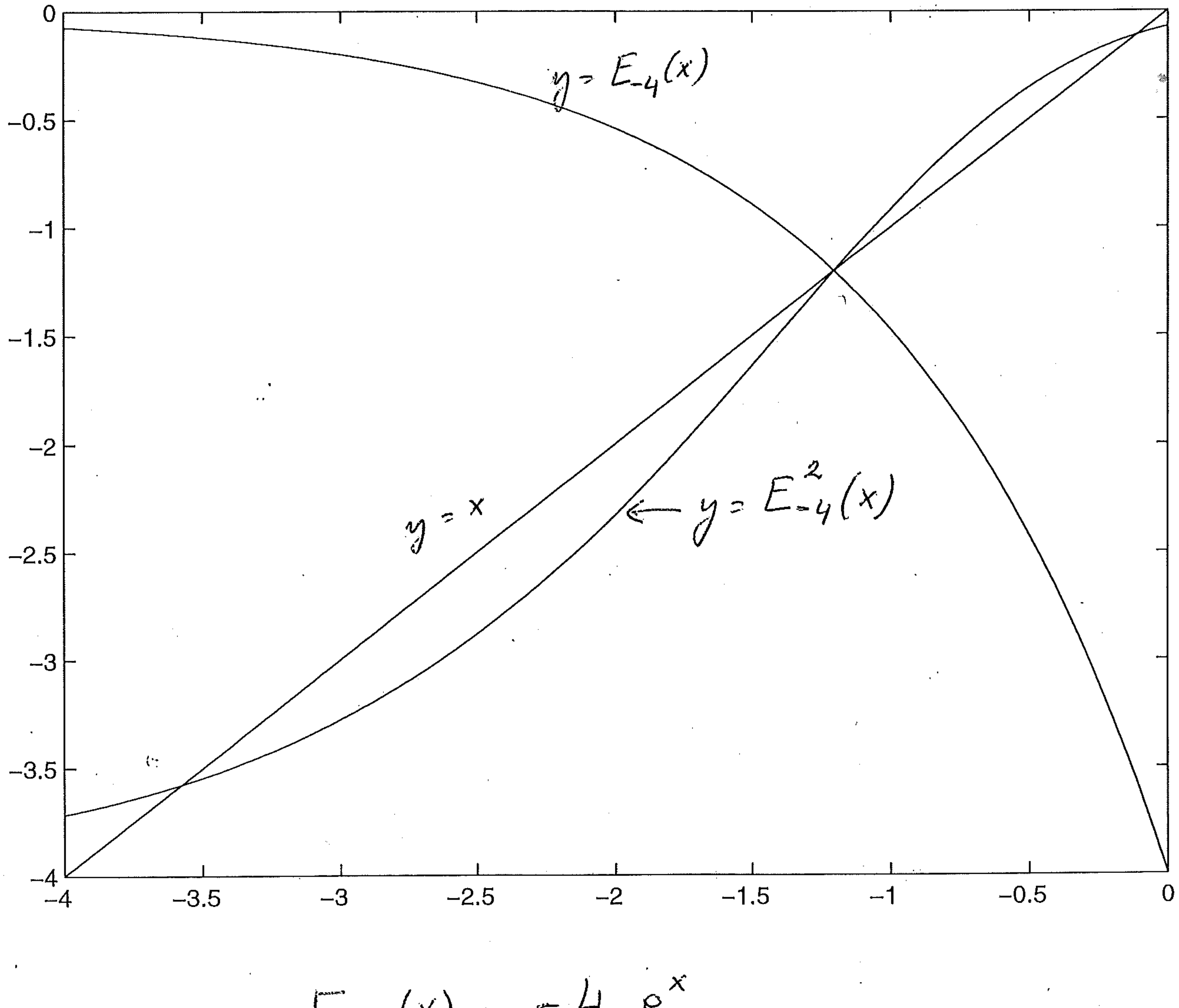

$$E_{-4}(x) = -4 \cdot e^{x}$$

Theorem 5.6. Let $\mu > 2+\sqrt{5}$. Then $\Lambda = I - \bigcup_{n=0}^{\infty} A_n$ is closed, totally disconnected and perfect.

Pf. $|F'(x)| > 1$ on $I_0 \cup I_1$ because

$$\mu - 2\mu x \geq \mu - 2\mu x_1 \quad \text{on } [0, x_1] = I_0 \quad \text{and}$$

$$\mu - 2\mu x \leq \mu - 2\mu x_2 < 0 \quad \text{on } [x_2, 1] = I_1$$

where x_1, x_2 are the roots of eq. $F_\mu(x) = 1$.

$$x_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\mu}} \quad (\mu \geq 4)$$

$$\mu - 2\mu x_1 = \sqrt{\mu^2 - 4\mu} \quad \text{which is} \quad \begin{matrix} \geq 1 & \text{if } \mu \geq 2+\sqrt{5} \\ > 1 & \text{if } \mu > 2+\sqrt{5} \end{matrix}$$

Call the value at x_1 $\lambda > 1$, i.e.

$$|F'(x)| \geq \lambda \quad \text{for all } x \in \Lambda.$$

Since $(F^2)'(x) = F'(\underbrace{F(x)}_{\in\Lambda}) \cdot F'(\underbrace{x}_{\in\Lambda})$ we have

$$|(F^2)'(x)| \geq \lambda^2, \quad x \in \Lambda$$

$\therefore$ (by induction)

$$|(F^n)'(x)| \geq \lambda^n, \quad x \in \Lambda.$$

Suppose Λ contains an interval $[x, y]$. Then, by the mean value theorem

$$F^n(x) - F^n(y) = (F^n)'(\xi)(x-y)$$

where $\xi \in (x, y) \subset \Lambda$.

14. $\therefore \quad |F^m(x) - F^m(y)| \geq \lambda^m |x-y|$, all $m \geq 1$.

But if m is so large that $\lambda^m |x-y| > 1$,

then this contradicts $x, y \in \Lambda$, especially $F^m(x), F^m(y)$
$\in [0,1]$.

Λ closed, because its complement is open.

$\Lambda \neq \emptyset$ since $x_1, x_2 \in \Lambda$.

Λ is perfect:

Assume p isolated element of Λ

$(\underset{x}{\cdot} \; \underset{p}{\cdot})$

$F^k(x) \notin [0,1]$

- If $x_k \to p$, $F^{m_k}(x_k) \notin [0,1]$, then $x_k \in A_{m_k}$.

Closest endpoint of an interval in A_{m_k}, call it y_k is at a distance to $x_k \to 0$. $\therefore$ $y_k \to p$. But interval endpoints are mapped into 0, thus $\in \Lambda$. We have $y_k \in \Lambda$, $y_k \to p$. p not isolated.

- If $\exists m$: F^m maps $(p-\varepsilon, p+\varepsilon) \setminus \{p\}$ out of $[0,1]$ and p inside $[0,1]$, then

$$F^m(p) = 0 \quad \text{and} \quad F^m(x) < 0, \quad x \in (p-\varepsilon, p+\varepsilon) \setminus \{p\}$$

or

$$F^m(p) = 1 \quad \text{and} \quad F^m(x) > 1, \quad \overline{x \neq p} \quad \text{"} \quad \text{——}.$$

15. Either way $(F^n)'(p) = 0$!

$$(F^n)'(p) = F'(F^{n-1}(p)) \cdot F'(F^{n-2}(p)) \cdots F'(F(p)) \cdot F'(p) = 0$$

F' is 0 at the point $\frac{1}{2}$ only. $\because$ $F^k(p) = \frac{1}{2}$ for some k, meaning $F^{k+1}(p) > 1$, $F^{k+2}(p) < 0$, $F^n(p) \to -\infty$, $n \to \infty$. Contradicts $p \in \Lambda$!

$\because$ $p \in \Lambda$ isolated $\Rightarrow$ contradiction

DEF. $\Gamma \subset \mathbb{R}$ is a *repelling* (*attracting*) *hyperbolic set* for f if

Γ is closed

$f(\Gamma) \subset \Gamma$ (invariant under f)

$|(f^n)'(x)| > 1 \quad (< 1)$

for all $x \in \Gamma$ and all $n \geq$ some N.

1.6. Symbolic Dynamics

Def. 6.1. $\Sigma_2 = \{ s = (s_0 s_1 s_2 \ldots) \mid s_i = 0 \text{ or } 1 \}$

[also $2^{\mathbb{N}} = \{ s = (s_0, s_1, s_2, \ldots) \mid s_i = 0 \text{ or } 1 \} = \{0, 1\}^{\mathbb{N}}$, infinite product of $\{0,1\}$ with itself]

sequence space

16. Σ_2 is a metric space, if we define
$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \qquad (<\infty).$$

<u>Proposition 6.2.</u> d is a metric

<u>Proposition 6.3.</u>

$s_i = t_i,\ i = 0,1,\dots,n \implies d(s,t) \le 2^{-n}$

$d(s,t) < 2^{-n} \implies s_i = t_i,\ i = 0,1,2,\dots,n.$

<u>Cor.</u> d is a complete metric

Pf. • $d(s,t) \geq 0$ evident

• $d(s,t) = 0 \iff s_i = t_i$ all $i = 0,1,\dots$
$\iff s = t$

• $d(s,t) = d(t,s)$ evident

$|r_i - s_i| + |s_i - t_i| \geq |r_i - t_i|$ all i

$\therefore$ • $d(r,s) + d(s,t) \geq d(r,t).$

$\therefore$ d metric

If $s_i = t_i,\ i = 0,1,\dots,n$ then
$$d(s,t) = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \le \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{2^{-n-1}}{\frac{1}{2}}$$
$$= 2^{-n}.$$

$d(s,t) < 2^{-n} \implies s_i = t_i,\ i = 0,1,2,\dots,n$

[because if not $d(s,t) \geq \frac{1}{2^n}$]

17. Let $s^{(n)}$ be a Cauchy-sequence:

$$d(s^{(n)}, s^{(m)}) \to 0 \quad \text{as} \quad n, m \to \infty$$

or $\forall k \, \exists n_k : \quad d(s^{(n)}, s^{(m)}) < 2^{-k}, \quad n, m \geq n_k$

$s^{(n)}$ constant ~~from~~ up to kth component for $n \geq n_k$

Let $t_i = s_i^{(n)}$ for $n \geq n_k$, $i = 0, 1, 2, \dots, k$

Then $t = \lim_{n \to \infty} s^{(n)}$, i.e. $d(t, s^{(n)}) \to 0$, $n \to \infty$.

Def. 6.4. $\sigma : \Sigma_2 \to \Sigma_2$ is defined by

$$\sigma(s_0 s_1 s_2, \dots) = (s_1 s_2 s_3 \dots)$$

(<u>shift map</u>)

<u>Prop. 6.5</u> σ is continuous from Σ_2 onto itself.

Pf. $\varepsilon > 0$, $s = s_0 s_1 s_2, \dots$ Take $2^{-n} < \varepsilon$, and $\delta = 2^{-(n+1)}$. If $t = t_0 t_1 t_2 \dots$ and $\underbrace{d(s,t) < \delta}_{s_i = t_i,\ i = 0, \dots, n+1}$

then $\underbrace{d(\sigma(s), \sigma(t)) \leq 2^{-n}}_{\text{because } \sigma(s)_i = \sigma(t)_i,\ i = 0, 1, \dots, n} < \varepsilon$. $\square$

Prop. 6.6.

1. $|\operatorname{Per}(\sigma_n)| = 2^n$
2. $\operatorname{Per}(\sigma)$ is dense in Σ_2
3. $\exists$ a dense orbit for σ in Σ_2

18. Pf. 1. $s = (s_0 s_1 \cdots s_{n-1} s_0 s_1 \cdots s_{n-1} s_0 s_1 \cdots s_{n-1} \cdots)$, a repeating sequence, has property $\sigma^n(s) = s$. Conversely, $\sigma^n(s) = s$, implies that s is repeating.

Eventually periodic $\Leftrightarrow$ eventually repeating

2. $\mathrm{Per}(\sigma)$ dense?

If $t \in \Sigma_2$, then we can take $s_i = t_i$, $i = 0, \cdots, n$, and s repeating. Then $d(s,t) \le 2^{-n}$ and $s \in \mathrm{Per}(\sigma)$.

3.

$s^* = (0\ 1\ 00\ 01\ 10\ 11\ 000\ 001\ 010\ 011\ 100\ 101\ 110\ 111\ \cdots)$ all possible combinations of 0 and 1 of length 1, 2, 3, ...

$\sigma^n(s^*)$, $n = 1, 2, \cdots$ comes arbitrarily close to any $t \in \Sigma_2$.

1.7. Topological Conjugacy

Idea: To identify systems which are isomorphic or "the same".

Def. 7.4. $f: A \to A$, $g: B \to B$ maps on topological spaces A and B.