1.

$f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C} \to \mathbb{C}$ or $\mathbb{R}^2 \to \mathbb{R}^2$)

Consider

$x_{n+1} = f(x_n)$, $x_0 \in \mathbb{R}$, i.e.

$x_n = (f \circ f \circ \dots \circ f)(x_0) = f^n(x_0)$

Def. $O^+(x) = \{x, f(x), f^2(x), \dots, f^n(x), \dots\}$

forward orbit of x

$O^-(x) = \{x, f^{-1}(x), f^{-2}(x), \dots\}$

backward orbit (if f is invertible)

$O(x) = \{\dots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \dots\}$

(full) orbit = bana = rata

Def. x fixed point if $f(x) = x$ fixpunkt kiintopiste

x periodic point with period n if $f^n(x) = x$

prime period of x = $\min\{n > 0 \mid n$ period of $x\}$

$\mathrm{Per}_n(f) = \{x \mid x$ period for f, with period $n\}$

$\mathrm{Fix}(f) = \{x \mid x$ fixed pt for $f\}$

periodic orbit = $O^+(x)$ if x periodic

period = jakso

Def. x eventually periodic if x is not periodic but $f^i(x)$ is.

Def Let p have period n. x is forward asymptotic to p if

$$\lim_{k \to \infty} f^{kn}(x) = p$$

$W^s(p)$, the stable set of p, is

$\{x \mid x \text{ is forward asymptotic to } p\}$

Def. If f is invertible (e.g. a homeomorphism)

$W^u(p)$, the unstable set of p, is

$\{x \mid x \text{ is backward asymptotic to } p\}$

or

$$\{x \mid \lim_{k \to -\infty} f^{kn}(x) = p\}$$

Def. x is a critical point of f if $f'(x) = 0$.
x is a non-degenerate critical point, if, in addition, $f''(x) \neq 0$,
and degenerate if $f''(x) = 0$.

Ex. 3.12 Translations of the circle

$$T_\lambda(\theta) = \theta + 2\pi\lambda \qquad (\text{mod } 2\pi)$$

Theorem 3.13 (Jacobi) Each orbit of T_λ is dense in S^1 if λ is irrational.

[Weyl: Orbit is "uniformly distributed"]

Def. 4.1. p periodic w. prime period n.
The number $(f^n)'(p)$ is the multiplier of p.
If $|(f^n)'(p)| \neq 1$ the point p is hyperbolic.

Prop. 4.4. Let $f \in C^1$ and let p be a hyperbolic fixed point with $|f'(p)| < 1$. Then there is an open interval $U \ni p$ such that

$$\lim_{n \to \infty} f^n(x) = p$$

i.e. $U \subset W^s(p)$.

Cor. The convergence is exponentially fast.

Jacobi $\quad T = T_\lambda$

$T(\theta) = 2\pi\lambda + \theta$

$\lambda \in \mathbb{Q} \iff$ all points periodic (period q if $\lambda = \frac{p}{q}$)

$\lambda \notin \mathbb{Q} \iff$ each orbit dense in S'

Proof outline:

(1) $T^n(\theta)$ are all distinct, $n = 0, 1, 2, \cdots$

(2) $\{T^n(\theta)\}$ has limit point $\in S'$

(3) $\exists\, k = n - m$ ~~so~~ that $|T^n(\theta) - T^m(\theta)| < \varepsilon$ and

$|T^k(\theta) - \theta| < \varepsilon$

(4) $\theta, T^k(\theta), T^{2k}(\theta), \cdots$ partition S' in arcs shorter than ε.

Weyl's theorem on equidistribution

$\forall a, b \quad (a < b)$

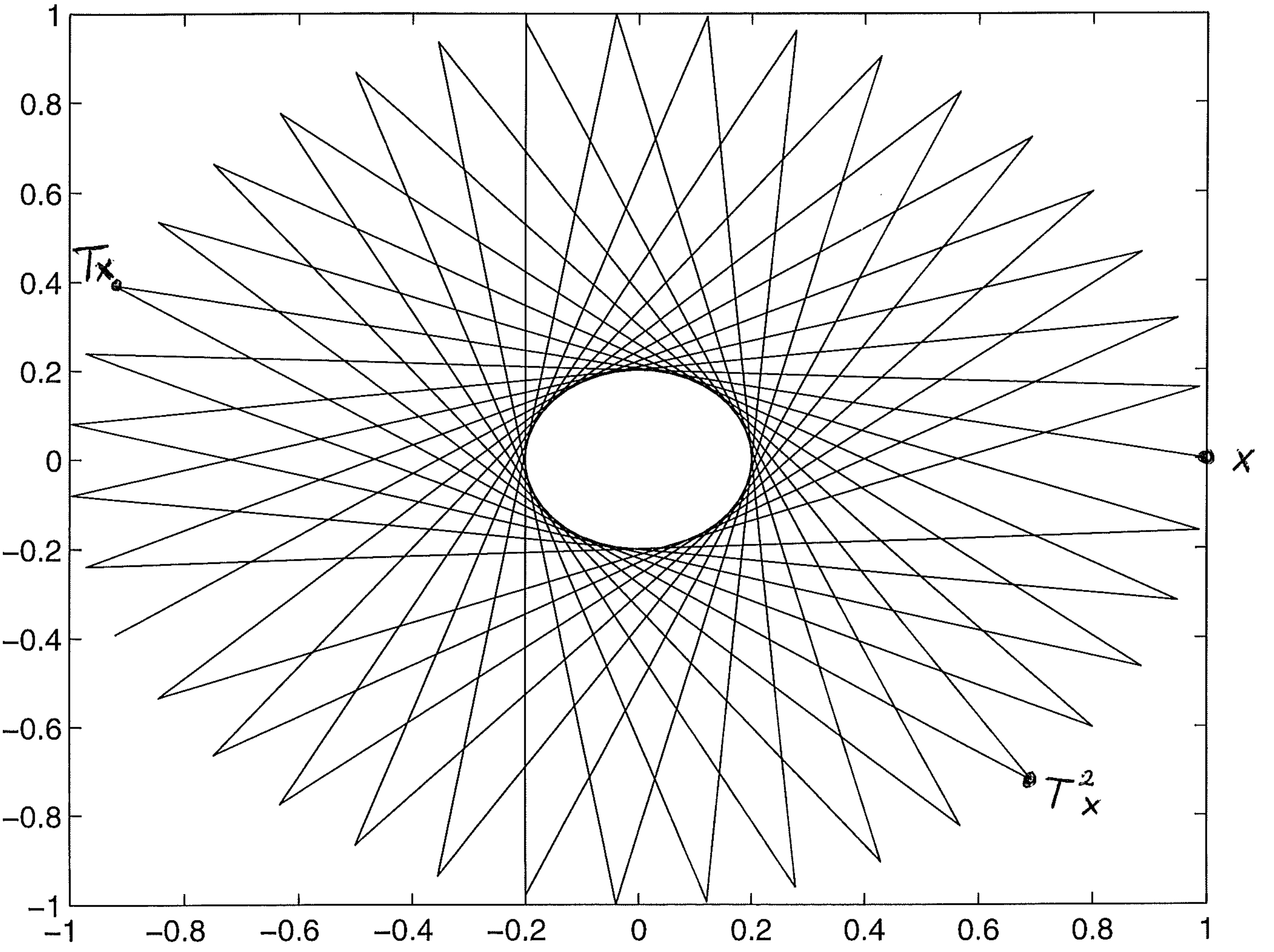
$$\frac{\#\{1 \le m \le N \mid T^m(\theta) \in (a,b)\}}{N} \longrightarrow \frac{b-a}{2\pi}$$

Cor. $\{ n\lambda - [n\lambda] \mid n = 1, 2, \ldots \}$ dense in $[0,1)$

if $\lambda \notin \mathbb{Q}$.

$$\frac{\#\{1 \le m \le N \mid m\lambda - [m\lambda] \in (a,b)\}}{N} \longrightarrow b - a$$

$T_{\frac{17}{39}}^{n}(0), \quad n = 0, \ldots, 38$



Def. 4.5. p hyperbolic periodic point
If $|(f^n)'(p)| < 1$ then p is called an
<u>attractor</u> (attracting periodic point, <u>sink</u>)

Note: $\exists\, U \ni p : \quad f^n(U) \subset U$
Also $U \subset W^s(p)$.
We call U a <u>local</u> <u>stable</u> <u>set</u> $W^s_{loc}(p)$.

Prop. 4.6. p hyperbolic fixed point
If $|f'(p)| > 1$ then $\exists\, U \ni p$, interval,
such that $x \in U \setminus \{p\} \Rightarrow \exists\, k > 0 : f^k(x) \notin U$.

N.B. $k = k(x)$!

Def 4.7. p is called a <u>repellor</u> (repelling fixed point, <u>source</u>).

Note. Similar def. and results for f^n.

Note. $U \setminus \{p\} \subset W^u(p)$
$U - \{p\}$ is a local unstable set $W^u_{loc}(p)$.

Non-hyperbolic points:
weakly attracting, weakly repelling
one-sided weakly attr., weakly rep.

Ex. 4.9.

$$Q_c(x) = x^2 + c$$

Fix p.? $x^2 + c = x \iff x^2 - x + c = 0$

has solution iff $\underline{c \leq \frac{1}{4}}$

$c = \frac{1}{4}$ Fixed point at $x = \frac{1}{2}$. Note: $Q_c(x) = (x - \frac{1}{2}) + (x - \frac{1}{2})^2 + \frac{1}{2}$ when $c = \frac{1}{4}$.

$c < \frac{1}{4}$ Fixed points at

$$x_1 = \frac{1}{2} + \sqrt{\frac{1}{4} - c}$$

$$x_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - c}$$

Multiplier: $1 + \sqrt{1-4c}$ at x_1

$1 - \sqrt{1-4c}$ at x_2

Also $Q_c(x) = (1 + \sqrt{1-4c})(x - x_1) + (x - x_1)^2 + x_1$

Hence we can write the dynamical system

$$y_{n+1} = Q_c(y_n)$$

in the form

$$y_{n+1} - x_1 = (1 + \sqrt{1-4c})(y_n - x_1) + (y_n - x_1)^2$$

or, with a change of coordinates

$$\bar{y}_{n+1} = (1 + \sqrt{1-4c})\, \bar{y}_n + \bar{y}_n^2$$

(Here $\bar{y}_\cdot = y_\cdot - x_1$)

Ex. $f_\beta(x) = \beta x + (1-\beta x)x \quad , \quad \beta > 0$

$$f_\beta(0) = 0$$
$$f_\beta(1) = 1$$
$$f'_\beta(x) = 1+\beta - 2\beta x$$
$$f'_\beta(0) = 1+\beta \quad > 1$$
$$f'_\beta(1) = 1-\beta$$
$$f_2(x) = 2x + (1-2x)x = 3x - 2x^2$$
$$= -(x-1) - 2(x-1)^2 + 1$$

Dyn. syst. for $\beta = 2$ of the form

$$x_{n+1} = -(x_n - 1) - 2(x_n-1)^2 + 1$$

or

$$\bar{x}_{n+1} = -\bar{x}_n - 2\bar{x}_n^2$$

where $\bar{x}_\cdot = x_\cdot - 1$.

LOGISTIC MAP (Ex. 4.10, Ch. 1.5)

$$F_\mu(x) = \mu x(1-x) \quad , \quad \mu > 1$$

$$F_\mu(0) = 0 \quad , \quad F'_\mu(0) = \mu > 1 .$$

$$F_\mu(p_\mu) = p_\mu \quad \text{where} \quad p_\mu = \frac{\mu-1}{\mu} \in (0,1)$$

$$F'_\mu(x) = \mu - 2\mu x \quad , \quad F''_\mu(x) = -2\mu$$

$$F'_\mu(p_\mu) = 2-\mu \qquad \therefore p_\mu \text{ attr. for } \mu \in (1,3).$$

Prop. 5.1. Let $F_\mu(x) = \mu x(1-x)$, $\mu > 1$.

1. $F_\mu(0) = 0$, $F_\mu\left(\frac{\mu-1}{\mu}\right) = \frac{\mu-1}{\mu}$, $F_\mu(1) = 0$

2. $0 < \frac{\mu-1}{\mu} < 1$ if $\mu > 1$.

$p_\mu := \frac{\mu-1}{\mu}$

Prop. 5.2. If $\mu > 1$, then

$F_\mu^n(x) \to -\infty$ if $x < 0$ or $x > 1$.

Pf. Alt. 1. For $x < 0$ we have $F_\mu(x) = \mu x - \mu x^2 < \mu x$,
$F_\mu^2(x) < \mu^2 x$ [because F_μ is increasing]
$\vdots$
$F_\mu^n(x) < \mu^n x \to -\infty$

Alt. 2. For $x < 0$ we have $F_\mu(x) < \mu x < x$.
Also $F_\mu'(x) = \mu - 2\mu x > 0$, i.e. F_μ increasing on $(-\infty, 0)$. Hence $F_\mu^2(x) < F_\mu(x) < x < 0$.
By induction
$$F_\mu^{n+1}(x) < F_\mu^n(x) < 0.$$
The seq. $F_\mu^n(x)$ is decreasing. If it has a limit p, then p is necessarily a fixed point: $F_\mu^n(x) \to p$, $F_\mu^{n+1}(x) \to F_\mu(p)$ means $F_\mu(p) = p$. But F_μ has no fixed point < 0. $\therefore \lim F_\mu^n(x) \to -\infty$.

For $x > 1$, we have $F_\mu(x) < 0$, so

$F_\mu^n(x) \to -\infty$ also in this case $\square$

<u>Prop. 5.3.</u> Let $1 < \mu < 3$.

1. F_μ has a repelling fixed point at 0, F_μ has an attracting fixed point at p_μ.

2. $\lim\limits_{n\to\infty} F_\mu^n(x) \to p_\mu$ for all $x \in (0,1)$.

<u>Pf.</u> of 2.

<u>$1 < \mu \leq 2$</u>

F_μ is increasing on $[0, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, 1]$.

Also $0 < x < \frac{\mu-1}{\mu} \left(\leq \frac{1}{2}\right) \implies x < F_\mu(x) < \frac{\mu-1}{\mu}$

$\mu x - \mu x^2 - x = 0$ at $x = 0$ and $x = \frac{\mu-1}{\mu}$ and positive between its zeroes.

Iterating:

$x < F_\mu(x) < F_\mu^2(x) < \frac{\mu-1}{\mu}$ and, by induction,

$F_\mu^n(x) < F_\mu^{n+1}(x) < \frac{\mu-1}{\mu}$.

The increasing sequence $F_\mu^n(x)$ converges to some $p \le \frac{\mu-1}{\mu}$. But, again, p must be a fixed point, since $\lim F_\mu^n(x) = \lim F_\mu^{n+1}(x) = p$, and p_μ is the only available fixed point.

$$\therefore \lim_{n\to\infty} F_\mu^n(x) = \frac{\mu-1}{\mu}.$$

If

$$\frac{\mu-1}{\mu} \le x \le \frac{1}{2} \quad \text{then} \quad \frac{\mu-1}{\mu} \le F_\mu(x) \le x$$

$$\le \frac{\mu}{4} \le \frac{1}{2}$$

see above: $\mu x - \mu x^2 - x \le 0$ outside the zeroes

We get

$$\frac{\mu-1}{\mu} \le F_\mu^2(x) \le F_\mu(x) \le x \le \frac{1}{2}$$

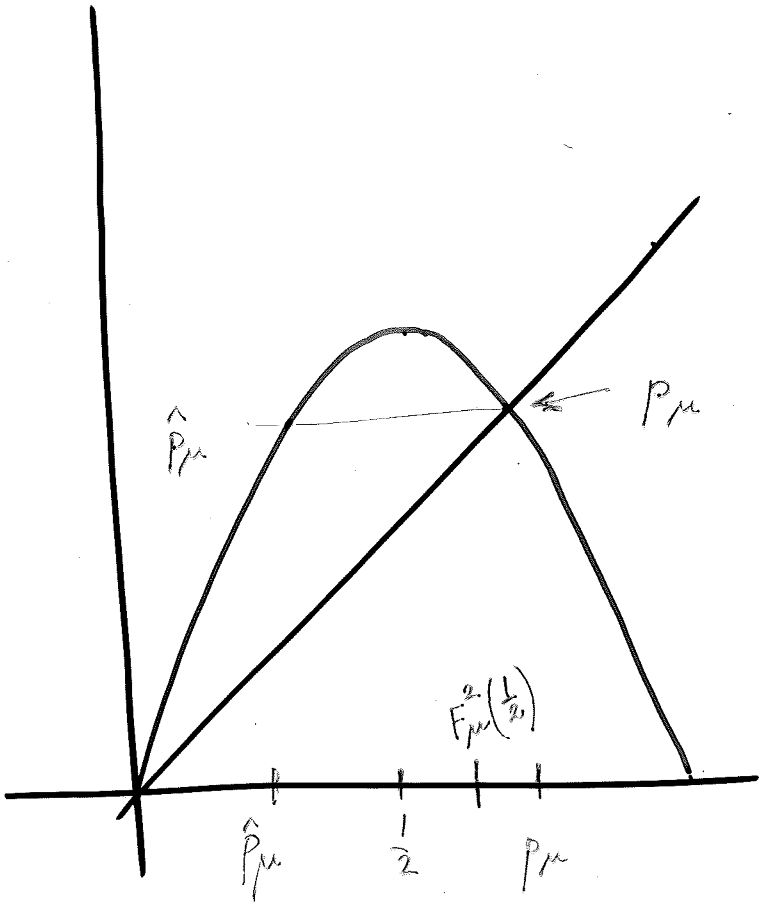
and

$$F_\mu^{n+1}(x) \le F_\mu^n(x) \text{ all } n.$$

The decreasing sequence $F_\mu^n(x)$ must have a limit $p \ge \frac{\mu-1}{\mu}$. Only fixed point available is $\frac{\mu-1}{\mu}$ itself. $\therefore F_\mu^n(x) \to \frac{\mu-1}{\mu}$, as $n \to \infty$

Finally, $\frac{1}{2} < x < 1 \Rightarrow F_\mu(x) < \frac{1}{2}$ so the previous cases apply. Thus $0 < x < 1 \Rightarrow \lim_{n\to\infty} F_\mu^n(x) = \frac{\mu-1}{\mu}$.

$2 < \mu \leq 3$



$p_\mu \equiv \dfrac{\mu-1}{\mu} > \dfrac12$

$\hat p_\mu = \dfrac{1}{\mu} < \dfrac12$.

$F_\mu(\hat p_\mu) = p_\mu$

$\dfrac{\mu}{4} > \dfrac{\mu-1}{\mu}$ because

$$\frac{\mu^2}{4} - \mu + 1 > 0$$
$$\|$$
$$\frac14(\mu^2 - 4\mu + 4)$$
$$\|$$
$$\frac14(\mu-2)^2$$

$0 < x < \hat p_\mu \implies x < F_\mu(x)$

so $x, F_\mu(x), \dots, F_\mu^n(x)$ increasing **until** $\in [\hat p_\mu, p_\mu]$

"can't stop because no fixed point in $(0, \hat p_\mu)$"

$p_\mu < x < 1 \implies 0 < F_\mu(x) < \hat p_\mu$, so

$F_\mu(x), F_\mu^2(x), \dots, F_\mu^n(x)$ increasing **until** $\in [\hat p_\mu, p_\mu]$

$\hat p_\mu \leq x \leq p_\mu \implies F_\mu(x) \in [p_\mu, \frac{\mu}{4}]$

Further $\boxed{F_\mu^2(x) \in (\frac12, p_\mu]}$ where F_μ is decreasing

and $F_\mu^2(x)$ is increasing. We have, then,

$\frac12 \leq x \leq p_\mu \implies \frac12 < F_\mu^2(\frac12) \leq F_\mu^2(x) \leq F_\mu^2(p_\mu) = p_\mu$.

Since $\boxed{x < F_\mu^2(x)}$ we get an increasing sequence

$$x,\ F_\mu^2(x),\ F_\mu^4(x),\ \dots \to p \le \frac{\mu-1}{\mu}.$$

As usual, there is no choice but $p = p_\mu$. Then the odd powers converge to p_μ, too.

Why is $F_\mu^2[\hat{p}_\mu, p_\mu] = F_\mu^2[\frac{1}{2}, p_\mu] \subset [\frac{1}{2}, p_\mu]$?

Look at

$$F_\mu^2(\tfrac{1}{2}) = F_\mu(\tfrac{\mu}{4}) = \tfrac{\mu}{4}\cdot\mu\,(1-\tfrac{\mu}{4}) = \frac{\mu^2}{4} - \frac{\mu^3}{16}$$

This expression is $\frac{1}{2}$ for $\mu = 2$, $\frac{9}{16}$ for $\mu = 3$, maximal at $\mu = \frac{8}{3}$ with value $\frac{16}{27}$, minimal at $\mu = 2$ (for $\mu \in [2, 3]$).

Why is $x < F_\mu^2(x)$ for $x \in [\frac{1}{2}, p_\mu)$.

Answer 1. If not we'd have $x = F_\mu^2(x)$ somewhere in the interval and p_μ is the only fixed point for F_μ^2. (Look at graph!)

Answer 2. Careful analysis of 4th degree expression $F_\mu^2(x) - x$ to show that there are no zeroes in $[\frac{1}{2}, p_\mu]$ other than right endpoint p_μ.

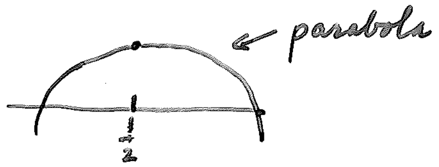
12 If $F_\mu^2(x) - x$ has a zero in $[\frac{1}{2}, p_\mu)$ then, necessarily, its second derivative must be 0 somewhere between $\frac{1}{2}$ and p_μ. Let us show that this is impossible!

$$(F_\mu^2(x) - x)'' = F_\mu''(F_\mu(x)) \cdot F_\mu'(x)^2 + F_\mu'(F_\mu(x)) \cdot F_\mu''(x)$$

$$= (-2\mu)\{(\mu - 2\mu x)^2 + \mu - 2\mu(\mu x - \mu x^2)\}$$

$$= (-2\mu)\{6\mu^2 x^2 - 6\mu^2 x + \mu^2 + \mu\}.$$

This parabola has its maximal value in $x = \frac{1}{2}$ (where the derivative $(-2\mu)(12\mu^2 x - 6\mu^2)$ is 0). The value is $\mu^2(\mu - 2) > 0$ (recall: $2 < \mu \leq 3$).



Thus we need to show that the expression is > 0 at p_μ, too.

For $x = p_\mu$ we get the value

$$(-2\mu)(\mu^2 - 5\mu + 6)$$

which is positive for $2 < \mu < 3$ (and 0 for $\mu = 2$ and $\mu = 3$)

$\therefore$ $2 < \mu \leq 3 \implies F_\mu^2(x) - x$ has no zeroes in $[\frac{1}{2}, p_\mu)$.

12a

F_μ with $\mu > 3$ has a 2-period in $(0,1)$:

Proof: The fixed points are 0 and $p_\mu = \frac{\mu-1}{\mu}$.

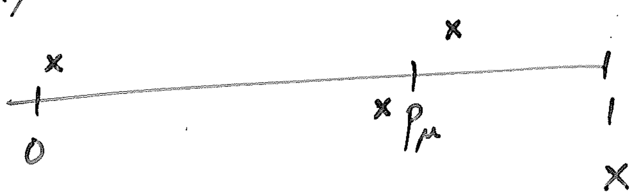
F_μ^2 has derivative $(2-\mu)^2 > 1$ in the fixed point:

$$(F_\mu^2)'(p_\mu) = F_\mu'(F_\mu(p_\mu)) \cdot F_\mu'(p_\mu)$$
$$= (F_\mu'(p_\mu))^2 = (2-\mu)^2.$$

Thus the eq. $F_\mu^2(x) - x$ has at least (actually exactly) 4 roots in $[0,1]$: 0, p_μ and ~~two~~ further roots.

The derivative of F_μ^2 being >1 in the fixed points we get

$$F_\mu^2(x) - x > 0 \quad \text{for } x > 0,\ x \approx 0+$$
$$F_\mu^2(x) - x < 0 \quad \text{for } x < p_\mu,\ x \approx p_\mu-$$
$$F_\mu^2(x) - x > 0 \quad \text{for } x > p_\mu,\ x \approx p_\mu+$$
$$F_\mu^2(1) - 1 = -1$$



Use the Intermediate Value Theorem!

Now take $\mu > 4$.
Write F instead of F_μ.
Consider $\mathcal{O}^+(x) = \{x, F(x), F^2(x), \dots\}$
When $\mu > 4$ "most" orbits go to $-\infty$, but some do not.
Put
$$\Lambda = \{x \in [0,1] \mid F^n(x) \not\to -\infty \text{ as } n \to \infty\}$$

$0, 1, p_\mu, \hat{p}_\mu \in \Lambda$

Let $A_0 = \{x \in (0,1) : F(x) > 1\}$. $A_0 \subset \Lambda^c$.
$x \in A_0$ has an orbit which escapes from $(0,1)$ in one step.
A_0 is a symmetric interval around $\frac{1}{2}$ of the form (a, b) where $F(a) = F(b) = 1$.
Write $I = [0,1]$, $I_0 = [0, a]$, $I_1 = [b, 1]$.

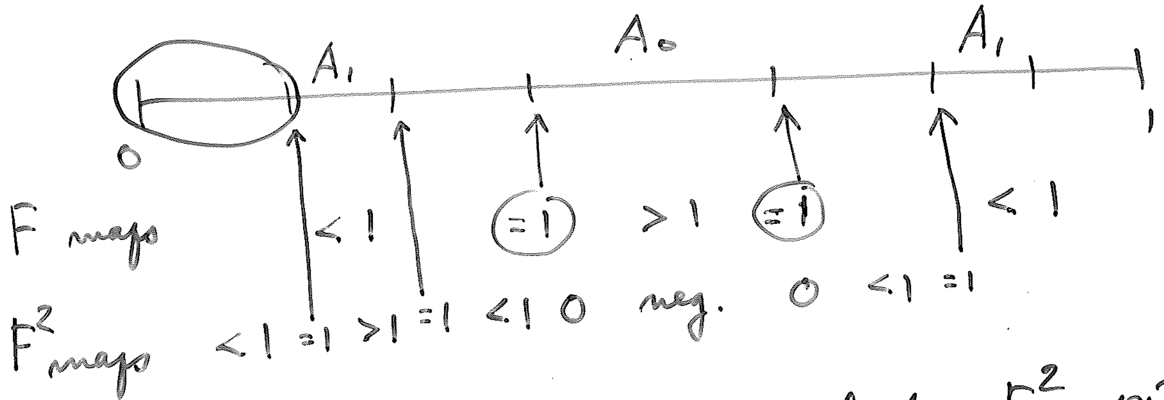
$I = I_0 \cup A_0 \cup I_1$.
Note that F is monotone on I_0 (I_1).

$F(I_0) = F(I_1) = I$.
Hence there is an interval in I_0 (I_1) mapped onto A_0.
Let $A_1 = \{x \in I_0 \cup I_1 \mid F(x) \in A_0\}$

12c A_1 consists of two open intervals, one is the middle interval of I_0, the other one is the middle interval of I_1.

F^2 maps these intervals <u>out of</u> $[0,1]$.

$$A_1 = \{ x \in I_0 \cup I_1 \mid F^2(x) > 1 \}$$



The red interval is mapped by F^2 onto $[0,1]$. It has "a middle interval" mapped by F^2 onto A_0. This is part of A_2

Def.
$$A_2 = \{ x \in I \setminus (A_0 \cup A_1) \mid F^2(x) \in A_0 \}$$
or $F^3(x) > 1$

A_2 consists of 4 open intervals

$I \setminus (A_0 \cup A_1 \cup A_2)$ consists of 8 closed intervals

<u>Note</u> The interval endpoints are eventually fixed $F^k(x) = 1$, $F^{k+1}(x) = 0$.

F^3 maps these 8 intervals onto $[0,1]$. The graph of F^3 intersects the main diagonal at least 8 times.

$\therefore$ There are at least 8 points of period 3.

Generalizing:

$$I \setminus (A_0 \cup A_1 \cup \dots \cup A_n)$$

contains all $x \in I$ such that $F^{n+1}(x) \in I$.

It consists of 2^{n+1} intervals. On each of these intervals F^{n+1} is strictly monotone. Each of these closed intervals is mapped onto $[0,1]$ by F^{n+1}.

$\therefore$ There are at least 2^{n+1} points in $Per_{n+1}(F)$.

$$\Lambda := I \setminus \bigcup_{k=0}^{\infty} A_k$$

is a non-empty closed set, a so-called <u>Cantor set</u>

Georg Cantor 1845 - 1918

closed, totally disconnected, perfect

contains no interval

contains no isolated points