ANALYSIS II, Homework 8

Due Wednesday 20.11.2013. Please hand in written answers for credit.

1. Let $F(x)$ be a continously differentiable function defined on $[a, b]$ such that $F(a) < 0$ and $F(b) > 0$ and

$$0 < K_1 \leq F'(x) \leq K_2 < \infty \text{ for all } x \in [a, b].$$

Use the Banach fixed point theorem to find the unique root to the equation $F(x) = 0$.

Hint: Consider the function $f(x) = x - \lambda F(x)$ and choose λ carefully.

2. Consider the family $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$, where $f_n(x) = n \sin(\frac{x}{n})$ is defined on $I = \{x : 0 \leq x < \infty\}$. Investigate if

(a) $\mathcal{F}$ is equicontinuous,

(b) $\mathcal{F}$ is precompact in $(BC(I, \mathbb{R}), || \cdot ||_\infty)$.

3. Let $A \subset C([a, b], \mathbb{R})$ be a bounded set with respect to $|| \cdot ||_\infty$. Show that the set of all functions $F(x) = \int_a^x f(t)\, dt$ with $f \in A$ is precompact in $(C([a, b], \mathbb{R}), || \cdot ||_\infty)$.

4. Show that the equation

$$3x = cos^2(x) + 2$$

has exactly one real root.