ANALYSIS II, Homework 7

Due Wednesday 13.11.2013. Please hand in written answers for credit.

1. (a) Let A and B be two subsets of a normed space E. Suppose that A is closed in E and B is compact in E. Show that

$$A + B = \{x + y : x \in A,\ y \in B\}$$

is closed in E.

(b) Let $e_n = (0, ...0, 1, 0, ...) \in l^2$ for $n = 1, 2, ...$, and $A = \{e_n : n \in \mathbb{N}\}$ and $B = \{-e_n + \frac{1}{n}e_1 : n \in \mathbb{N}\}$. Show that A and B are closed and bounded sets in the space l^2 but $A + B$ is not closed in l^2.

2. Let $f \in C([0, 1], \mathbb{R})$. Suppose that for all $x \in [0, 1]$ we have that $|f(x)| \leq \int_0^x f(t)\ dt$. Show that $f(x) = 0$ for all $x \in [0, 1]$.

3. A metric space X is called separable, if there exists a countable set $\{x_1, x_2, ...\} \subset X$ such that $\overline{\{x_1, x_2, ...\}} = X$. Show that every precompact metric space is separable.

4. Show that the intersection of arbitrary many compacts sets in a metric space X is compact.

5. Let $f :]0, 1[\to]0, 1[$. True or false?

(a) If f is continuous and $(x_n)_n$ is a Cauchy sequence, then $(f(x_n))_n$ is a Cauchy sequence?

(b) If f maps every Cauchy sequence into a Cauchy sequence, then f is continuous?