ANALYSIS II, Homework 6

Due Wednesday 6.11.2013. Please hand in written answers for credit.

1. Let f be a continuously differentiable function on $[0, \infty)$, and suppose that

$$\int_0^\infty |f'(t)|^2 \, dt < \infty.$$

Show that f is uniformly continuous on $[0, \infty)$.

Hint: Cauchy-Schwarz inequality is useful.

2. Show that $\mathbb{R}^N$ endowed with the metric

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|f(n) - g(n)|}{1 + |f(n) - g(n)|}$$

is complete.

3. Let X and Y be metric spaces. Assume that $(x_n)_n$ is a Cauchy sequence in X and $f : X \to Y$ is uniformly continuous. Show that $(f(x_n))_n$ is a Cauchy sequence in Y.

4. Let X be the space

$$X = \{f \in C([0, 1], \mathbb{R}) : ||f||_\infty < 1\}.$$

True or false: $(X, || \cdot ||_\infty)$ is complete?

5. Let E be a normed space, F another normed space and $L(E, F)$ a Banach space. Here $L(E, F)$ is the space of all bounded linear operators from E into F endowed with the operator norm. Show that F is likewise a Banach space.