ANALYSIS II, Homework 9

Due Wednesday 27.11.2013. Please hand in written answers for credit.

1. Let $c_0(\mathbb{N},\mathbb{C}) = \{a = (a_1, a_2, ...) : \lim_{n\to\infty} a_n = 0\}$. On this space we can define the norm

$$||a||_{c_0} = ||a||_{\max} = \max_{n\geq 1} |a_n|.$$

(i) Show that $c_0(\mathbb{N},\mathbb{C})$ is a closed subspace of $l^\infty(\mathbb{N},\mathbb{C})$ (hence complete).

(ii) Show that every point $(b_1, b_2, ...) \in l^1(\mathbb{N},\mathbb{C})$ induces a bounded linear functional on $c_0(\mathbb{N},\mathbb{C})$ through the formula

$$F(a) = \sum_{n=1}^{\infty} b_n a_n,\ \ a = (a_1, a_2, ...) \in c_0(\mathbb{N},\mathbb{C}).$$

(iii) Show that the operator norm of F is the usual l^1-norm, i.e.

$$\sup_{||a||_{c_0}\leq 1} |F(a)| = \sum_{n=1}^{\infty} |b_n|.$$

2. Let H be a Hilbert space and let $F : H \to \mathbb{C}$ be a linear functional. Show that F is continuous if the set $\{x \in H : F(x) = 0\}$ is closed in H.

3. Let $\mathcal{P}_3(\mathbb{R})$ be the vector space of all real-valued polynomials $p : \mathbb{R} \to \mathbb{R}$ such that $p(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$. Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$ and

$$\int_0^1 |2 + 3x - p(x)|^2\, dx$$

is as small as possible.

4. Show that the closed and bounded set $B = \{x \in l^2 : ||x||_2 \leq 1\}$ is not compact in l^2.

5. Let

$$C = \{(x_1, x_2, ...) \in l^2 : |x_n| \leq \frac{1}{n} \text{ for each } n = 1, 2, ...\}.$$

Show that C is a compact subset of l^2.