ANALYSIS II, Homework 4

Due Wednesday 9.10.2013. Please hand in written answers for credit.

1. Let $E = C([0,1],\mathbb{R})$ be equipped with its usual $||\cdot||_\infty$-norm. Prove or disprove the following assertions:

(a) $A = \{f \in X : f([0,1]) = [0,1]\}$ is closed in E,

(b) $B = \{f \in X : f \text{ is injective}\}$ is closed in E,

(c) $C = \{f \in X : f(\frac{1}{2}) = 0\}$ is closed in E.

2. On $C^1([0,1],\mathbb{R})$ consider the norm

$$||f|| = ||f'||_\infty + ||f||_\infty.$$

Let $g, h \in C([0,1],\mathbb{R})$ be fixed and $C([0,1],\mathbb{R})$ equipped with the $||\cdot||_\infty$-norm. Define the operator $T : C^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$

$$(Tf)(t) = g(t)f'(t) + h(t)f(t).$$

Show that T is linear and bounded.

3. Let $0 < p < \infty$, and define

$$l^p = \{x = (x_1, x_2, ...) : x_n \in \mathbb{C}, ||x||_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}} < \infty\}.$$

(a) Show that l^p is a complex vector space,

(b) Let $1 \le p < \infty$. Show that $(l^p, ||\cdot||_p)$ is a normed space,

(c) Consider the shift operators $T_r, T_l : l^2 \to l^2$ defined by

$$T_r(x) = (0, x_1, x_2, ...) \text{ and } T_l(x) = (x_2, .x_3, ...).$$

Calculate the norms of T_r and T_l.