ANALYSIS II, Homework 3

Due Wednesday 2.10.2013. Please hand in written answers for credit.

1. Let

$$A = \{(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\} \subset \mathbb{R}^2.$$

Find the closure $\overline{A}$ of A.

2. For $f \in C^1([0,1], \mathbb{R})$, let

$$||f||_1 = |f(0)| + ||f'||_\infty,$$

$$||f||_2 = \max\{|\int_0^1 f(t)\ dt|, ||f'||_\infty\},$$

$$||f||_3 = (\int_0^1 |f(t)|^2\ dt + \int_0^1 |f'(t)|^2\ dt)^{\frac{1}{2}}.$$

Determine whether every $||\cdot||_i,\ i = 1, 2, 3$, is a norm.

3. The differential operator $D : C^1([0,1], \mathbb{K}) \to C([0,1], \mathbb{K})$ is defined by $Df = f'$. Show that D is linear. Is D bounded? Use the sup-norm both in $C^1([0,1], \mathbb{K})$ and $C([0,1], \mathbb{K})$.

4. Suppose that d is a metric that is induced by a norm in a vector space E according to $d(x,y) = ||x-y||$. Show that this type of metric has the two additional properties (x,y,z $\in E, \lambda \in \mathbb{C}$) :

$$(a)\quad d(\lambda x, \lambda y) = |\lambda|\ d(x,y),$$

$$(b)\quad d(x+z, y+z) = d(x,y).$$

Conversely, suppose that a metric $d(\cdot,\cdot)$ has the two properties listed above. Show that we can define a norm on E by $||x|| := d(x,0)$.

5. Consider the set of all n-tuples $a = (a_1, a_2, ..., a_n)$ of real numbers with distance

$$\rho_p(a,b) = (\sum_{k=1}^{n} |a_k - b_k|^p)^{\frac{1}{p}},$$

where p is a fixed number ≥ 1. Show that ρ_p is a metric in $\mathbb{R}^n$.