ANALYSIS II, Homework 2

Due Wednesday 25.9.2013. Please hand in written answers for credit.

1. If $||x+y|| = ||x|| + ||y||$ for two vectors x and y in a normed space, then show that

$$||\alpha x + \beta y|| = \alpha ||x|| + \beta ||y||$$

for all $\alpha, \beta \geq 0$.

2. Let A be an arbitrary subset of a vector space E and let $[A]$ be the set of all finite linear combinations in A, i.e. vectors x that can be written as

$$x = \sum_{i=1}^{n} \lambda_i x_i, \quad x_i \in A, \lambda_i \in \mathbb{K}, \; i = 1, ..., n.$$

Show that

(a) $[A]$ is a subspace of E.

(b) $[A]$ is then smallest subspace of E which contains A.

3. (a) Let X be a nonempty set. Show that the function $d : X \times X \to \mathbb{R}^+$, $d(x, y) = 1$, for $x \neq y$, $d(x, y) = 0$, if $x = y$ induces a metric on X. Moreover, is it true or false: the countable intersection of open sets in (X, d) is open?

(b) Let $X = \mathbb{N}$ and define

$$d(m, n) = |m^{-1} - n^{-1}|.$$

Is it true or false: (X, d) is a metric space?

4. Let $d(x, y)$ be a metric on a set X. Show that the function

$$e(x, y) = \frac{d(x.y)}{1 + d(x, y)}$$

is another metric on X.

5. Let *Lip* denote the space of all functions f from $[0, 1]$ into $\mathbb{R}$ for which

$$M_f = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|} < \infty.$$

For $f \in Lip$, let

$$||f||_{Lip} = |f(0)| + M_f.$$

Prove that $|| \cdot ||_{Lip}$ is a norm and $||f||_\infty \leq ||f||_{Lip}$ for all $f \in Lip$.