Infinitely Divisible Processes

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Outline

1. Review of basic properties of ID laws
2. Large deviations and concentration inequalities
3. ID random measures and stochastic integration
4. Series representations of ID laws and processes. Simulation
5. ID processes – generalized Lévy-Itô representation
6. Stationary ID processes and ergodic properies

1. Review of basic properties of ID laws

1.1. Definition and first examples

Definition

A random vector X in $\mathbb{R}^d$ (or its distribution) is said to be infinitely divisible (ID) if for every $n \geq 1$ there exist iid random vectors $Y_{n,1}, \dots, Y_{n,n}$ in $\mathbb{R}^d$ (possibly on a different probability space) such that

$$X \stackrel{d}{=} Y_{n,1} + \cdots + Y_{n,n}.$$

Equivalently, a probability distribution μ on $\mathbb{R}^d$ is ID if for every $n \geq 1$ there exists a probability distribution μ_n on $\mathbb{R}^d$ such that

$$\mu = \underbrace{\mu_n * \cdots * \mu_n}_{n-times}$$

Easy to verify examples.

ID: Normal, Poisson, compound Poisson, geometric, negative binomial, exponential, gamma, Cauchy, ...

Proof: Check that for every n, $(\phi_X(t))^{1/n}$ *is a characteristic function, or otherwise.*

Not-ID: binomial, uniform.
Any random vector with bounded range is not ID, unless is constant.

1.2. Class $ID(\mathbb{R}^d)$ - basic properties

$ID(\mathbb{R}^d)$ denotes the class of all ID distributions in $\mathbb{R}^d$.

Proposition

(i) *If* $\mu_1, \mu_2 \in ID(\mathbb{R}^d)$, *then* $\mu_1 * \mu_2 \in \mathbb{R}^d$

(ii) *If* $\mu \in ID(\mathbb{R}^d)$, *then* $\hat{\mu}(u) \neq 0$ *for any* $u \in \mathbb{R}^d$

(iii) *If* $\mu_n \in ID(\mathbb{R}^d)$ *and* $\mu_n \to \mu$, *then* $\mu \in ID(\mathbb{R}^d)$

(iv) *If* $\mu \in ID(\mathbb{R}^d)$ *and* $V : \mathbb{R}^d \mapsto \mathbb{R}^k$ *is a linear transformation, then* $\mu \circ V^{-1} \in ID(\mathbb{R}^d)$

By (ii) there is a unique continuous function $C : \mathbb{R}^d \mapsto \mathbb{C}$ with $C(0) = 0$ such that

$$\hat{\mu}(u) = \exp(C(u)), \quad u \in \mathbb{R}.$$

The function $C(u)$ is called the cumulant function of μ and is denoted as

$$C(u) = C_\mu(u) = \log \hat{\mu}(u).$$

The famous Lévy-Khintchine formula gives a structural form of cumulants of ID distributions.

1.3. Lévy-Khintchine representation

Theorem (L-K representation)

Let X be an ID random vector X in $\mathbb{R}^d$. Then there exits a unique triplet (b, Σ, ν) such that

$$\log \mathbb{E} \exp\{i\langle u, X\rangle\} = i\langle b, u\rangle - \frac{1}{2}\langle u, \Sigma u\rangle + \int_{\mathbb{R}^d} (e^{i\langle u,x\rangle} - 1 - i\langle u, \tau(x)\rangle)\, \nu(dx) \quad (1)$$

where $u, b \in \mathbb{R}^d$, Σ is a nonnegative definite $d \times d$ matrix, and ν is a measure on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} \|x\|^2 \wedge 1\, \nu(dx) < \infty$. $\tau : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a fixed bounded measurable function such that $\lim_{x\to 0} \|x\|^{-2}(\tau(x) - x) = 0$.

Theorem (L-K representation, continue)

Σ is called a Gaussian covariance matrix, ν a Lévy measure, and τ a truncation function. Conversely, given (b, Σ, ν) and τ as above, there is an X satisfying (1).

We will write $X \sim \mathcal{ID}(b, \Sigma, \nu)$

The Lévy-Khintchine formula allows to write

$$X \stackrel{d}{=} G + Y$$

where $G \perp\!\!\!\perp Y$, G is Gaussian, $G \sim \mathcal{ID}(0, \Sigma, 0)$, and Y is of a Poissonian type, $Y \sim \mathcal{ID}(b, 0, \nu)$. This decomposition is very helpful.

Change of τ only affects b in the Lévy-Khintchine triplet and

$$b_{\tau_1} = b_{\tau_2} + \int_{\mathbb{R}^d} (\tau_2(x) - \tau_1(x))\,\nu(dx).$$

Remark

Different forms of the truncation $\tau(x)$ appear in the literature.

(i) $\dfrac{x}{\|x\|^2 + 1}$;

(ii) $x\mathbf{1}_{[0,1]}(\|x\|)$;

(iii) $\dfrac{x}{\max\{\|x\|, 1\}}$;

(iv) $\sin x$;

(v) $x\mathbf{1}_{[0,1]}(\|x\|) + x(2 - \|x\|)\mathbf{1}_{(1,2]}(\|x\|)$.

We will fix a truncation function proposed by Maruyama (1970). It is well suited for ID processes.

$$\tau(x) = [\![x]\!] = \left(\frac{x_1}{|x_1| \vee 1}, \ldots, \frac{x_d}{|x_d| \vee 1} \right),$$

$x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

1.4. Examples:

- Poisson distribution with parameter λ:

$$\hat{\mu}(u) = exp(\lambda(e^i u - 1)), \qquad \nu = \lambda\delta_1.$$

- Compound Poisson distribution (on $\mathbb{R}^d$):

$$\hat{\mu}(u) = exp(c(\hat{\rho}(u) - 1)), \qquad \nu = c\rho.$$

- Negative binomial with parameters $c > 0$ and $p \in (0, 1)$:

$$\hat{\mu}(u) = p^c(1 - qe^{iu})^{-c}, \quad \nu\{k\} = ck^{-1}q^k, \; k \in \mathbb{N}, \; q = 1 - p..$$

- Gamma distribution with parameters $\alpha, \beta > 0$:

$$\hat{\mu}(u) = (1 - i\alpha^{-1})^{-\beta}, \quad \nu(dx) = \beta x^{-1} e^{-\alpha x} \mathbf{1}_{(0,\infty)}(x)\, dx$$

ID distributions whose infinitely divisibility is difficult to prove:

- Student's t-distribution:

$$\mu(dx) = c(1 + x^2)^{-(\alpha+1)/2}\, dx, \quad \alpha \in (0, \infty)$$

- Pareto distribution:

$$\mu(dx) = c(1 + x)^{-\alpha-1}\mathbf{1}_{(0,\infty)}(x)\, dx, \quad \alpha \in (0, \infty)$$

ID distributions difficult to prove (continue):

- Log-normal distribution:

$$\mu(dx) = cx^{-1} e^{-\alpha(\log x)^2} \mathbf{1}_{(0,\infty)}(x)\, dx, \quad \alpha \in (0, \infty)$$

- F-distribution:

$$\mu(dx) = cx^{\beta-1}(1+x)^{-\alpha-\beta} \mathbf{1}_{(0,\infty)}(x)\, dx, \quad \alpha \in (0, \infty)$$

ID distributions difficult to prove (continue):

- Half-Cauchy distribution:

$$\mu(dx) = 2\pi(1+x^2)^{-1}\mathbf{1}_{(0,\infty)}(x)\,dx, \quad \alpha \in (0,\infty)$$

- Weibull distribution:

$$\mu(-\infty, x] = \begin{cases} 1 - e^{-x^{\alpha}} & x > 0 \\ 0 & x \leq 0, \end{cases}$$

$0 < \alpha \leq 1$.

ID distributions difficult to prove (continue):

- Logistic distribution:

$$\mu(-\infty, x] = (1 + e^{-x})^{-1}, \quad x \in \mathbb{R}$$

- Gumbel distribution:

$$\mu(-\infty, x] = e^{-e^{-x}}, \quad x \in \mathbb{R}$$

1.5. Lévy processes

Definition

A stochastic process $\{X_t\}_{t\geq 0}$ taking values in $\mathbb{R}^d$ is called a Lévy process if it has the following properties:

1. **Starts from 0:** $X_0 = 0$ a.s.;
2. **Independent increments:** For any $0 \leq t_1 < \cdots < t_n$, the random variables $X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent;
3. **Homogeneous increments:** For any $s < t$, $X_t - X_s \overset{d}{=} X_{t-s}$;
4. **Stochastic continuity:** For any $\epsilon > 0$, $\mathbb{P}(\|X_{t+s} - X_t\| > \epsilon) \to 0$ as $s \to 0$.
5. **Path property:** Sample paths $t \mapsto X_t$ are right continuous and have left limits a.s. Precisely,

$$\mathbb{P}\{\omega : \ t \to X_t(\omega) \text{ is right continuous with left limits}\} = 1.$$

Theorem

(i) $\{X_t\}_{t\geq 0}$ *is a Lévy process then* $X_t \in ID(\mathbb{R}^d)$ *for all* t. *Moreover, there exists a uniques Lévy-Khintchine triplet* (b, Σ, ν) *such that* $X_t \sim \mathcal{ID}(tb, t\Sigma, t\nu)$ *for all* t.

(ii) *Given a Lévy-Khintchine triplet* (b, Σ, ν) *there exists Lévy process* $\{X_t\}_{t\geq 0}$ *such that* $X_t \sim \mathcal{ID}(tb, t\Sigma, t\nu)$ *for all* t.

Examples

- Brownian motion in $\mathbb{R}^d$: $X_t \sim \mathcal{ID}(0, t\Sigma, 0)$
- Brownian motion with drift in $\mathbb{R}^d$: $X_t \sim \mathcal{ID}(tb, t\Sigma, 0)$
- Poisson process with parameter λ: $X_t \sim \mathcal{ID}(t\lambda, 0, t\lambda\delta_1)$
- Compound Poisson process in $\mathbb{R}^d$: $X_t \sim \mathcal{ID}(tb, 0, t\nu)$, where ν is a finite measure and b depends on ν.

2.6. Infinitesimal generator

$$P_t f(x) := \mathbb{E} f(x + X_t), \quad f \in L^\infty(\mathbb{R}^d).$$

forms a contraction semigroup.
For every $f \in C_0^2(\mathbb{R}^d) \cup C^3(\mathbb{R}^d)$ it's generator can be evaluated as

$$Af(x) = \frac{\partial P_t f(x)}{\partial t}\Big|_{t=0} = \langle b, \nabla f(x) \rangle + \frac{1}{2} \mathrm{Tr} \Sigma \nabla^2 f(x) + \int_{\mathbb{R}^d} (f(x+y) - f(x) - \langle \nabla f(x), [\![y]\!] \rangle) \nu(dy)$$

We reviewed

1. Definition and first examples
2. Class $ID(\mathbb{R}^d)$ - basic properties
3. Lévy-Khintchine representation
4. Examples
5. Lévy processes
6. Infinitesimal generator

2.1. Existence of g-moments of ID variables

A function $g : \mathbb{R}^d \mapsto \mathbb{R}_+$ is said to be log-subadditive if

$$g(x+y) \leq Kg(x)g(y) \quad \forall x, y \in \mathbb{R}^d.$$

g is locally bounded when $\sup_{\|x\| \leq r} g(x) < \infty$, $\forall\ r > 0$.

Theorem

Let $g : \mathbb{R}^d \mapsto \mathbb{R}_+$ be a log-subadditive locally bounded function, and let $X \sim \mathcal{ID}(b, \Sigma, \nu)$ be a random variable in $\mathbb{R}^d$. Then

$$\mathbb{E}g(X) < \infty \iff \int_{\{\|x\|>1\}} g(x)\, \nu(dx) < \infty.$$

In other words, finiteness of a g-moment of X is equivalent to the finiteness of the g-moment of $\nu_{|\{\|x\|>1\}}$.

For example, $g(x) = \|x\|^p$ $(p > 0)$, $g(x) = \exp(\|x\|^\beta)$ $(\beta \in (0,1])$ are log-subadditive. Log-subadditive functions have at most exponential growth, e.g.

$$g(x) \leq A \exp(B\|x\|), \quad x \in \mathbb{E},$$

for some positive constants A, B.

Hence $g(x) = \exp(\|x\| \log^+ \|x\|)$ is not log-subadditive.

Proof of the above theorem is in Sato's book, Chapter 5.25.

Analogous result holds for Banach spaces, see de Acosta (1980).

2.2. Estimation of moments

Let $X \sim \mathcal{ID}(0, \nu, b)$ be a mean zero random variable in $\mathbb{R}^d$. We have by differentiating of characteristic function

$$\mathbb{E}\|X\|^2 = \int_{\mathbb{R}^d} \|x\|^2 \, \nu(dx).$$

$$\mathbb{E}\|X\|^4 = \int_{\mathbb{R}^d} \|x\|^4 \, \nu(dx) + \Big(\int_{\mathbb{R}^d} \|x\|^2 \, \nu(dx)\Big)^2 + \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle x, y \rangle^2 \, \nu(dx)\nu(dy).$$

In a heavy tailed case we have $\mathbb{E}\|X\|^2 = \infty$ and usually $\mathbb{E}\|X\| < \infty$. How to estimate the latter quantity in terms of ν?

Theorem (M.B. Marcus, JR (2001))

Let $X \sim \mathcal{ID}(0, \nu, b)$ be a mean zero random variable in $\mathbb{R}^d$. Let $\ell = \ell(\nu)$ be a unique solution of the equation

$$\int_{\mathbb{R}^d} \|\ell^{-1}x\|^2 \wedge \|\ell^{-1}x\| \, \nu(dx) = 1.$$

Then

$$(0.25)\, \ell(\nu) \le \mathbb{E}\|X\| \le (2.125)\, \ell(\nu).$$

If ν is symmetric, the the upper bound constant can be decreased to 1.25.

Note: The above result was proved for random variables in Hilbert spaces.

Theorem (JR, M. Turner (2011))

Let $X \sim \mathcal{ID}(0, \nu, b)$ be a mean zero random variable in $\mathbb{R}^d$. Let $p \geq 1$ and $\ell = \ell(\nu)$ be a unique solution of the equation

$$\ell^{-2} \int_{\{\|x\| \leq \ell\}} \|x\|^2 \, \nu(dx) + \ell^{-p} \int_{\{\|x\| > \ell\}} \|x\|^p \, \nu(dx) = 1.$$

Then

$$(0.25) \, \ell(\nu) \leq (\mathbb{E}\|X\|)^{1/p} \leq K(p) \, \ell(\nu).$$

If $1 < p < 2$, then $K(p) \leq 2.3$.

Note: The above result was proved for random variables in Hilbert spaces.

2.3. Asymptotic behavior of the norm

Theorem (JR (1995))

Let $X \sim \mathcal{ID}(b, 0, \nu)$, *where* $\nu \neq 0$ *has bounded support in* $\mathbb{R}^d$. *Let*

$$R := \inf\{r > 0 : \ \nu\{x : \|x\| > r\} = 0\}$$

and

$$p := \nu\{x : \|x\| = R\}.$$

Then

$$\mathbb{E}\exp\left\{R^{-1}\|X\|\log^{+}(\alpha\|X\|)\right\} < \infty$$

for every $\alpha \in (0, (epR)^{-1})$. *(*$0^{-1} = \infty$.*)*

Methods: isoperimetric inequalities (see Talangrand (1989)), or hypercontractivity (see Kwapień-Szulga (1991)), combined with a technique similar to *découpage de Lévy* combined with certain methods of de Acosta.

Theorem (Large Deviations, JR (1995))

Let $X \sim \mathcal{ID}(b, \Sigma, \nu)$, *where* $\nu \neq 0$ *has bounded support in* $\mathbb{R}^d$. *Let* R *be as above. Then*

$$\lim_{t\to\infty} \frac{\log \mathbb{P}\{\|X\| > t\}}{t \log t} = -R^{-1}.$$

If ν *has unbounded support, then the above limit equals* 0.

Note: The above was proved for Banach space valued random variables.

Corollary

Let X *be as above. If* $\mathbb{E}\exp\{\|X\|^\beta\} < \infty$ *for some* $\beta > 1$, *the* X *is Gaussian; if this holds for* $\beta > 2$, *then* X *is nonrandom.*

2.4. Concentration inequality

Theorem (Houdré (2002))

Let $X \sim ID(0, \nu, b)$ be a random variable in $\mathbb{R}^d$ with Lévy measure ν of bounded support. Let R be as above and $V^2 = \int_{\mathbb{R}^d} \|x\|^2 \nu(dx)$. Then for every Lipschitz function $f : \mathbb{R}^d \mapsto \mathbb{R}$ with $\|f\|_{Lip} \leq 1$ and $t > 0$

$$\mathbb{P}(f(X) - \mathbb{E}f(X) \geq t) \leq \exp\left\{\frac{t}{R} - \left(\frac{t}{R} + \frac{V^2}{R^2}\right)\log\left(1 + \frac{Rt}{V^2}\right)\right\}$$

Corollary

Under above notation, for every $\theta < R^{-1}$

$$\mathbb{E}e^{\theta|f(X)|\log^+|f(X)|} < \infty.$$

2.5. Covariance Representation

Theorem (C. Houdré, V. Pérez-Abreu, D. Surgailis (1998))

Let $X \sim \mathcal{ID}(b, \Sigma, \nu)$ *be a vector in* $\mathbb{R}^d$ *such that* $\mathbb{E}\|X\|^2 < \infty$. *Let* $f, g : \mathbb{R}^d \mapsto \mathbb{R}$ *be Lipschitz. Then*

$$\mathrm{Cov}(f(X), g(X)) = \int_0^1 \mathbb{E}_s \int_{\mathbb{R}^d} (f(Y+x) - f(Y))(g(Z+x) - g(Z))\, \nu(dx) ds$$

where the expectation is with respect to probability measure $\mathbb{P}_s$ *on* $\mathbb{R}^{2d}$ *such that* $(Y, Z) \sim ID(0, \nu_s, b_s)$, *where* $b_s = (b, b)$ *and* $\nu_s = s\nu_1 + (1-s)\nu_0$, $s \in [0, 1]$. *Here* $\nu_0(du, dv) = \nu(du)\delta_0(dv) + \delta_0(du)\nu(dv)$ *is concentrated on the two main 'axes' of* $\mathbb{R}^{2d}$ *and* $\mu_1(du, dv)$ *is the push-forward of* ν *to the main diagonal of* $\mathbb{R}^{2d}$, $(u, u) \in \mathbb{R}^{2d}$.

Notice that

$\forall s \in [0, 1]$, under $\mathbb{P}_s$, Y and Z have the same distribution as X

$Y \perp\!\!\!\perp Z$ under $\mathbb{P}_0$, and

$Y = Z$ under $\mathbb{P}_1$.

2.6. Application of the covariance representation to concentration

Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be Lipschitz with $\|f\|_{Lip} \le 1$, and let $\mathbb{E}e^{t\|X\|} < \infty$, $t \in (0, t_0)$. Let $g(x) = e^{tf(x)}$. Assume for a moment that f is bounded, so that g is also Lipschitz, and that $\mathbb{E}f(X) = 0$.

Consider the Laplace transform $L(t)$ of $f(X)$, $L(t) = \mathbb{E}e^{tf(X)}$.

$$\frac{d}{dt}L(t) = \mathbb{E}f(X)e^{tf(X)} = \mathrm{Cov}(f(X), g(X))$$

By the covariance representation

$$\frac{d}{dt}L(t) = \int_0^1 \mathbb{E}_s \int_{\mathbb{R}^d} (f(Y+x) - f(Y))(e^{tf(Z+x)} - e^{tf(Z)})\, \nu(dx) ds$$
$$\leq \int_0^1 \mathbb{E}_s e^{tf(Z)} \int_{\mathbb{R}^d} |f(Y+x) - f(Y)|(e^{t|f(Z+x)-f(Z)|} - 1)\, \nu(dx) ds$$
$$\leq \int_0^1 \mathbb{E}_s e^{tf(Z)} \int_{\mathbb{R}^d} \|x\|(e^{t\|x\|} - 1)\, \nu(dx) ds = L(t)h(t),$$

where

$$h(t) := \int_{\mathbb{R}^d} \|x\|(e^{t\|x\|} - 1)\, \nu(dx).$$

Thus $\frac{L'(t)}{L(t)} \leq h(t)$, which yields

$$\mathbb{E}e^{tf(X)} = L(t) \leq \exp(\int_0^t h(s)\, ds), \quad t \in (0, t_0).$$

By the standard Cramér method in large deviations

$$\mathbb{P}(f(X) \geq a) \leq \exp(-\int_0^a h^{-1}(s)\, ds)$$

Now once can remove restrictions on f to get

$$\mathbb{P}(f(X) - \mathbb{E}f(X) \geq t) \leq \exp(-\int_0^t h^{-1}(s)\, ds).$$

If ν has bounded support, we can bound h easily to get the tail bound given on a previous slide (for finite dim spaces and extend to Banach spaces). It can be applied to other ID random vectors as well (see Houdré (2002)).

For current results on integrability of seminorms of chaos variables see Andreas Basse (2009).

We covered the following:

1. Existence of g-moments of ID variables
2. Estimation of moments
3. Asymptotic behavior of the norm
4. Concentration inequality
5. Covariance representation
6. Application of the covariance representation to concentration

3.1 Lévy processes and random measures

Given a real valued Lévy process $\{X_t\}_{t\geq 0}$, for any $A \in \mathcal{B}_b(\mathbb{R}_+)$ (bounded Borel sets) the stochastic integral

$$M(A) = \int_0^\infty \mathbf{1}_A(t)\, dX_t$$

can be defined.
The process $\{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+)\}$ has the following properties:

(i) $M(\emptyset) = 0$ a.s.

(ii) If $A_j \in \mathcal{B}_b(\mathbb{R}_+)$ are disjoint sets then $M(A_j)$ are independent, $j = 1, 2, \ldots,$ and if $\bigcup_j A_j \in \mathcal{B}_b(\mathbb{R}_+)$, then

$$M(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} M(A_j) \quad a.s.$$

(iii) For every $A \in \mathcal{B}_b(\mathbb{R}_+)$, $M(A)$ is ID with

$$M(A) \sim \mathcal{ID}(\lambda(A)b, \lambda(A)\sigma^2, \lambda(A)\nu)$$

where λ is the Lebesgue measure on $\mathbb{R}_+$ and (b, σ^2, ν) is the Lévy-Khintchine triplet of X_1.

Such process $\{M(A) : A \in \mathcal{B}_b(\mathbb{R}_+)\}$ is called an infinitely divisible random measure (IDRM) associated with the Lévy process $\{X_t\}_{t \geq 0}$.

Conversely, if M satisfies (i)–(iii), then

$$X_t = M(0, t], \quad t \geq 0$$

defines a Lévy process (in law).

IDEA:

IDRM's can be defined on more general spaces than $\mathbb{R}_+$, for example $\mathbb{R}_+^d$, which leads to interesting random fields such as Lévy sheets, etc.

3.2. IDRM's

Let S be a set and $\mathcal{S}_0$ be a σ-ring of subsets of S.

Definition

A stochastic process $M = \{M(A)\}_{A \in \mathcal{S}_0}$ is called an infinitely divisible random measure (IDRM) if

(i) $M(\emptyset) = 0$ a.s.

(ii) For every $\{A_i\} \subset \mathcal{S}_0$ pairwise disjoint, $\{M(A_i)\}$ forms a sequence of independent random variables and if $\bigcup_i A_i \in \mathcal{S}_0$, then

$$M(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} M(A_i) \quad a.s.$$

(iii) For every $A \in \mathcal{S}_0$, $M(A)$ has an ID distribution.

Let M be an IDRM on $(S, \mathcal{S}_0)$. Then for every $A \in \mathcal{S}_0$ the distribution of $M(A)$ is determined by its Lévy triplet

$$M(A) \sim (\beta(A), \gamma(A), \nu_0(A, \cdot)). \tag{2}$$

Condition (ii) and the uniqueness of Lévy triplets imply that for every $\{A_i\}_{i=1}^n \subset \mathcal{S}_0$ pairwise disjoint

$$\beta(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \beta(A_i), \qquad \gamma(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \gamma(A_i)$$

and

$$\nu_0(\bigcup_{i=1}^n A_i, \cdot) = \sum_{i=1}^n \nu_0(A_i, \cdot).$$

These relations extend to the countable additivity. Before ding that we need the following.

Since we prefer working with σ-finite measures from now on we assume the following condition on a σ-ring $\mathcal{S}_0$ of subsets of S. There exists an increasing sequence $\{S_n\} \subset \mathcal{S}_0$ such that

$$S = \bigcup_n S_n.$$

For instance, when $\mathcal{S}_0 = \mathcal{B}_b(\mathbb{R}_+)$, then $S_n = [0, n]$.

Put

$$\mathcal{S} = \sigma(\mathcal{S}_0).$$

Theorem

(A) *Let M be an IDRM on $(S, \mathcal{S}_0)$ as above. Then*

(i) $\beta : \mathcal{S}_0 \mapsto \mathbb{R}$ *is a signed measure,*

(iii) γ *extends to a* σ*-finite measure on* $\mathcal{S}$*,*

(iii) *there exists a* σ*-finite measure* ν *on* $\mathcal{S} \otimes \mathcal{B}(\mathbb{R})$ *such that* $\forall$ $A \in \mathcal{S}_0,\ B \in \mathcal{B}(\mathbb{R})$,

$$\nu(A \times B) = \nu_0(A, B).$$

(B) *Let* (β, γ, ν) *satisfy the conditions given in* **(A)**. *Then there exists a unique (in the sense of finite-dimensional distributions) IDRM* $M = \{M(A)\}_{A \in \mathcal{S}_0}$ *such that* (2) *holds.*

Theorem (continue)

(C) *Let* (b, γ, ν) *be as in* **(A)**. *Define a measure*

$$m(A) = |\beta|(A) + \gamma(A) + \int_A \int_{\mathbb{R}} \min\{x^2, 1\}\, \nu(ds, dx), \quad A \in \mathcal{S}_0,$$

where $|\beta| = \beta^+ - \beta^-$ *is the Jordan decomposition of measure* β *into positive and negative parts.*
m is called a <u>*control measure*</u> *of* M *because it has the property:* $m(A) = 0 \iff M(A') = 0$ *a.s. for all* $A' \subset A$.

Since all above measures are σ-finite, we can define measurable functions

$$b(s) := \frac{d\beta}{dm}(s),$$

$$\sigma^2(s) := \frac{d\gamma}{dm}(s),$$

and a measure kernel $\rho(s, dx)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$\nu(ds, dx) := \rho(s, dx)m(ds),$$

by disintegrating measure ν.

In this way we attach to every $s \in S$ a Lévy-Khintchine triplet $(b(s), \sigma^2(s), \rho(s, \cdot))$. Symbolically,

$$M(ds) \sim \mathcal{ID}(b(s), \sigma^2(s), \rho(s, \cdot))$$

An IDRM is said to be homogeneous when b, σ^2, ρ do not depend on s. Thus in the homogeneous case, in (2) we have for all A

$$\beta(A) = m(A)b$$

$$\gamma(A) = m(A)\sigma^2$$

$$\nu_0(A, \cdot) = m(A)\rho(\cdot)$$

for one Lévy-Khintchine triplet (b, σ^2, ρ).

In conclusion,

Theorem

$(\sigma^2(s), \rho(s,\cdot), b(s))$ *is a generating triplet of an infinitely divisible distribution* $\mu(s,\cdot)$ *on* $\mathbb{R}$ *such that*

$$|b(s)| + \sigma^2(s) + \int_{\mathbb{R}} \min\{x^2, 1\}\,\rho(s,dx) = 1$$

For every $B \in \mathcal{B}(\mathbb{R})$, $s \mapsto \mu(s,B)$ *is measurable; thus* μ *is a probability kernel on* $S \times \mathcal{B}(\mathbb{R})$. *Let*

$$C(s,u) = ib(s)u - \frac{1}{2}\sigma^2(s)u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iu[\![x]\!])\,\rho(s,dx)$$

be the cumulant function of $\mu(s,\cdot)$. *Then cumulant function* $C_{M(A)}$ *of* $\mathcal{L}\{M(A)\}$ *is of the form*

$$C_{M(A)}(u) = \int_A C(s,u)\,m(ds).$$

3.3. Stochastic integral - definition & existence

By a simple function on S we understand a finite linear combination of indicators of sets from $\mathcal{S}_0$, $f(s) = \sum_{j=1}^{n} a_j \mathbf{1}_{A_j}(s)$, $A_j \in \mathcal{S}_0$. For such function the integral is defined in an obvious way:

$$\int f \, dM = \sum_{j=1}^{n} a_j M(A_j).$$

(By a convention, we skip the region of integration when the region of integration is the whole space.)

In order to extend the integral beyond simple functions we need to introduce a distance, say d_M, such that if $d_M(f_n, f) \to 0$ then $\int f_n \, dM$ converges in probability to some random variable X. Then we define $\int f \, dM = X$.

For a random variable X, let $\|X\|_0 := \mathbb{E}\{|X| \wedge 1\}$. Clearly $\|X_n - X\|_0 \to 0$ if and only if $X_n \stackrel{P}{\to} X$. Define for a simple function $f : S \mapsto \mathbb{R}$,

$$\|f\|_M := \sup \| \int \phi f \, dM \|_0,$$

where the supremum is taken over all simple functions $\phi : S \mapsto \mathbb{R}$ with $|\phi| \leq 1$. Notice that since $s \mapsto \phi(s) f(s)$ is a simple function according to our definition, $\|f\|_M$ is well-defined.

It is easy to verify the following properties: for any simple functions f and g,

- $\|f\|_M = 0 \iff f = 0$ m-a.e.
- $\|f + g\|_M \leq \|f\|_M + \|g\|_M$
- $\|\theta f\|_M \leq \|f\|_M$ for any $|\theta| \leq 1$

These are properties of an F-norm on a vector space. Naturally, $d_M(f, g) := \|f - g\|_M$ is a metric on the vector space of simple functions.

Definition

We say that a function $f : S \mapsto \mathbb{R}$ is M-integrable if there exists a sequence $\{f_n\}$ of simple functions such that

(i) $f_n \to f$ m-a.e.

(ii) $\lim_{k,n\to\infty} \|f_n - f_k\|_M = 0.$

If (i)-(ii) hold, then we define

$$\int f\, dM = \lim_{n\to\infty} \int f_n\, dM,$$

where the limit is taken in probability.

Define the following functions:

$$B(s,x) = b(s)x + \int (\llbracket xy \rrbracket - x\llbracket y \rrbracket)\, \rho(s,dy).$$

$$V(s,x) = \int \llbracket xy \rrbracket^2 \, \rho(s,dy).$$

and

$$\Phi_M(s,x) := |B(s,x)| + \sigma^2(s)x^2 + V(s,x), \tag{3}$$

$s \in S, \quad x \in \mathbb{R}$.
If M is homogeneous, then above functions depend only on x.

Theorem

A measurable function $f : S \mapsto \mathbb{R}$ is M-integrable if and only if

$$\int \Phi_M(s, f(s))\, m(ds) < \infty$$

where Φ_M is given by (3).
The integral has an infinitely divisible distribution with the cumulant function

$$C_{\int f\, dM}(u) = \int C(s, uf(s))\, m(ds)$$

and the generating triplet

$$\int f\, dM \sim \mathcal{ID}\left(b_f, \sigma_f^2, \nu_f\right),$$

where b_f , σ_f^2, and ν_f are given by

Theorem (continue)

$$b_f = \int B(s, f(s))\, m(ds),$$

$$\sigma_f^2 = \int \sigma^2(s) f^2(s)\, m(ds),$$

and for every $B \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned} \nu_f(B) &= \nu(\{(s,x) : f(s)x \in B \setminus \{0\}\}) \\ &= \int \rho(s, \{x : x \in B/f(s)\})\, m(ds). \end{aligned}$$

Example (Poisson random measure)

Let M be a Poisson random measure on S.
$M \sim \mathcal{ID}(m(A), 0, m(A)\delta_1)$, so M is homogeneous generated by $(1, 0, \delta_1)$ and m. We compute

$$B(x) = x + \int (\llbracket xy \rrbracket - x \llbracket y \rrbracket)\, \delta_1(dy) = x + \llbracket x \rrbracket - x = \llbracket x \rrbracket$$

and

$$V(x) = \int \llbracket xy \rrbracket^2\, \delta_1(dy) = \llbracket x \rrbracket^2.$$

Thus

$$\Phi_M(x) = \llbracket x \rrbracket^2 + |\llbracket x \rrbracket|.$$

Notice that $|\llbracket x \rrbracket| \leq \Phi_N(s, x) \leq 2|\llbracket x \rrbracket|$ and $|\llbracket x \rrbracket| = |x| \wedge 1$. By the above Theorem

$$L(S, M) := \{f : S \mapsto \mathbb{R} : \int |f| \wedge 1\, dm < \infty\}.$$

Example (Poisson random measure, continue)

Since

$$C(u) = \int (e^{iux} - 1 - iu[\![x]\!])\,\delta_1(dx) + iu = e^{iu} - 1,$$

by (21) we have

$$C_{\int f\,dM}(u) = \int \left[e^{iuf(s)} - 1\right]\, m(ds),$$

for every $f \in L(S, M)$.

Example (Compensated Poisson random measure)

Let $\bar{M} = M - m$, where M is a PRM with mean measure m. $\bar{M}(A) \sim \mathcal{ID}(0, 0, m(A)\delta_1)$, so M is homogeneous generated by $(0, 0, \delta_1)$ and m. We compute

$$B(x) = \int (\llbracket xy \rrbracket - x\llbracket y \rrbracket)\, \delta_1(dy) = \llbracket x \rrbracket - x$$

and

$$V(s) = \int \llbracket xy \rrbracket^2\, \delta_1(dy) = \llbracket x \rrbracket^2.$$

Thus

$$\Phi_{\bar{M}}(x) = \llbracket x \rrbracket^2 + |x - \llbracket x \rrbracket| = x^2 \wedge |x|.$$

We get

$$L(S, \bar{M}) = \{f : S \mapsto \mathbb{R} : \int f^2 \wedge |f|\, dm < \infty\}.$$

Example (Compensated Poisson random measure, continue)

Since

$$C(u) = \int (e^{iux} - 1 - iu[\![x]\!])\, \delta_1(dx) = e^{iu} - 1 - iu,$$

by (21) we have

$$C_{\int f\, dM}(u) = \int \left[e^{iuf(s)} - 1 - iuf(s) \right] m(ds),$$

for every $f \in L(S, \bar{M})$.

Example (Symmetric α-stable random measure)

Let M be a symmetric α-stable random measure determined by $\mathbb{E}\exp(iuM(A)) = \exp(-m(A)|u|^\alpha)$. $M(A) \sim \mathcal{ID}(0, 0, m(A)\theta_\alpha)$, where $\theta_\alpha(dx) = c|x|^{-\alpha-1}dx$, where $c > 0$ is a constant. M is homogeneous generated by $(0, 0, \delta_1)$ and m. Since $B(x) = 0$, we get

$$\Phi_M(x) = V(x) = c\int [\![xy]\!]^2\,|y|^{-\alpha-1}dy = \frac{4c}{\alpha(2-\alpha)}|x|^\alpha.$$

Hence

$$L(S, M) = \{f : S \mapsto \mathbb{R} : \int |f|^\alpha\,dm < \infty\}.$$

Example (Symmetric α-stable random measure, continue)

Since

$$C(u) = \int (e^{iux} - 1 - iu[\![x]\!])\, \theta_\alpha(dx) = -|u|^\alpha,$$

we have

$$C_{\int f\, dM}(u) = -|u|^\alpha \int |f(s)|^\alpha\, m(ds),$$

for every $f \in L(S, M)$. Therefore, $\int f\, dM$ is a symmetric α-stable random variable with parameter $(\int |f|^\alpha\, dm)^{1/\alpha}$.

Example (Gamma random measure)

Let M be a gamma random measure with shape measure m. $M \sim \mathcal{ID}(m(A), 0, m(A)\eta)$, where $\eta(dx) = x^{-1}e^{-x}\,dx$, so M is homogeneous generated by $(k, 0, \eta)$ and m. Here

$$k = \int_0^\infty [\![x]\!] x^{-1} e^{-x}\,dx.$$

We compute

$$\begin{aligned} B(x) &= kx + \int_0^\infty ([\![xy]\!] - x[\![y]\!]) y^{-1} e^{-y}\,dy \\ &= \int_0^\infty [\![xy]\!] y^{-1} e^{-y}\,dy \end{aligned}$$

and

$$V(x) = \int [\![xy]\!]^2 \,\rho(s, dy) = \int_0^\infty [\![xy]\!]^2 y^{-1} e^{-y}\,dy.$$

$$\Phi_M(x) = |B(x)| + V(x).$$

Example (Gamma random measure, continue)

we infer that there are constants $c, C > 0$ such that

$$c\Psi(x) \leq \Phi_M(x) \leq C\Psi(x), \quad x \in \mathbb{R},$$

where

$$\Psi(x) = \begin{cases} |x| & |x| \leq 1, \\ \log(e|x|) & |x| > 1. \end{cases}$$

Hence

$$L(S, M) = \{f : S \mapsto \mathbb{R} : \int \Psi(f)\, dm < \infty\}.$$

Since

$$C(u) = -\frac{1}{2}\log(1 + u^2) + i \arctan(u),$$

we get the cumulant of $\int f\, dM$ in an explicit form as well.

3.5. Lévy bases

In modeling time plays a distinguished role. For this reason it is convenient to make a separate definition.

Definition

A Lévy basis is an IDRM M on $S = \mathbb{R} \times V$, where $\mathbb{R}$ (or its subset) is viewed as a time set and V as a space of marks. Typically one assumes that the control measure M on $S = \mathbb{R} \times V$ satisfies

$$m(\{t\} \times V) = 0 \quad \forall\ t \in \mathbb{R}..$$

One can then consider filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{R}}$ and stochastic integration of adapted processes with respect to M.

3.6. Metric properties of the space of integrable functions

$L(S, M)$ the space of functions $f : S \mapsto \mathbb{R}$ for which $\int f\, dM$ exists.
$L(S, M)$ is a linear vector space equipped with an F-norm

$$\|f\|_M := \sup \| \int \phi f\, dM\|_0,$$

where the supremum is taken over all simple functions $\phi : S \mapsto \mathbb{R}$ with $|\phi| \leq 1$. Under the metric

$$d_M(f, g) := \|f - g\|_M$$

the stochastic integral: $f \mapsto \int f\, dM$ is a continuous operation.
We want an more direct way to measure the distance.

Theorem

The space $L(S, M)$ is a complete linear metric space under the metric d_M. For any $f_n, f \in L(S, M)$

$$d_M(f_n, f) \to 0 \quad \Longleftrightarrow \quad \int \Phi_M(s, f_n(s) - f(s))\, m(ds) \to 0.$$

Under the metric d_M simple functions are dense in $L(S, M)$.

We covered:

1. Lévy processes and random measures
2. IDRM's
3. Stochastic integral - definition & existence
4. Examples
5. Lévy bases
6. Metric properties of the space of integrable functions

By a natural analogy to Gaussian processes we consider

Definition

Let T be an arbitrary nonempty set. A process $\mathbf{X} = \{X_t\}_{t \in T}$ is said to be an infinitely divisible stochastic process if for any $t_1, \ldots, t_n \in T$ the random vector

$$(X_{t_1}, \ldots, X_{t_n})$$

has an infinitely divisible distribution.

Examples:

1. Lévy processes.
   $\mathbf{X} = \{X_t\}_{t \geq 0}$ has independent and stationary increments. If $\mathbf{X}$ has only independent increments, then is called an additive process.

2. Linearly additive random fields.
   A random field $\mathbf{X} = \{X_t\}_{t \in \mathbb{R}^d}$ is called linearly additive if for every $a, b \in \mathbb{R}^d$, the process $\{X_{a+sb}\}_{s \in \mathbb{R}}$ has independent increments.
   Mori (1992) characterized all infinitely divisible, stochastically continuous, linearly additive random fields. (Chentsov type representations.)

3. Multiparameter Lévy processes.
   These are linearly additive random fields as in (2) such that for every $a, b \in \mathbb{R}^d$, $\{X_{a+sb}\}_{s\in\mathbb{R}}$ is a two-sided Lévy process.

4. Brownian and Lévy sheets.
   Let M be an IDRM with control Lebesgue measure on $\mathbb{R}_+^d$. The random field indexed by $t = (t_1, \ldots, t_d) \in \mathbb{R}_+^d$ defined by

$$X(t) = M([0, t_1] \times \cdots \times [0, t_d])$$

   is called a Lévy sheet (Brownian sheet when M is Gaussian).

5. Gaussian processes.

   Recall that a stochastic process $\mathbf{X} = \{X_t\}_{t\in T}$ is said to be Gaussian if $\forall\ t_1, \ldots, t_n \in T$ and $a_1, \ldots, a_n \in \mathbb{R}$

$$\sum_{j=1}^{n} a_j X_{t_j}$$

   has normal distribution on $\mathbb{R}$.

6. Symmetric α-stable (SαS) processes. Defined similarly as Gaussian processes except we require that $\sum_{j=1}^n a_j X_{t_j}$ has a SαS distribution.
7. Stationary ID processes.
For instance, Ornstein-Uhlenbeck processes

$$X_t = \int_{-\infty}^{t} e^{-\lambda(t-s)}\, dZ_s, \quad t \in \mathbb{R}.$$

Such processes are stationary solutions to Langevin equation

$$dX_t = -\lambda X_t + dZ_t,$$

$\{Z_t\}_{t\in\mathbb{R}}$ is a two-sided Lévy process with $\mathbb{E}\log^+|Z_1| < \infty$.
More generally, moving average processes

$$X_t = \int_{\mathbb{R}} f(t-s)\, dZ_s, \quad t \in \mathbb{R}.$$

(Conditions for existence of such integrals were given in the previous lecture.)

- Mixed moving average processes

$$X_t = \int_{\mathbb{R}\times V} f(t-s)\, M(ds\, dv), \quad t \in \mathbb{R}.$$

where M is IDRM on $\mathbb{R} \times V$ with control measure $\lambda \otimes \eta$. For example, mixed Ornstein-Uhlenbeck process

$$X_t = \int_{-\infty}^{t} \int_{\mathbb{R}_+} e^{-\lambda(t-s)}\, M(ds\, d\lambda), \quad t \in \mathbb{R}.$$

Harmonizable processes

$$X_t = \int_{\mathbb{R}^d} e^{itx}\, M(dx), \quad t \in \mathbb{R}$$

where M is an IDRM.

8. Stationary increment ID processes.
For instance,

- Linear fractional α-stable motion:

$$X_H(t) = \int_{\mathbb{R}} \left\{ a \left[(t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \right] + b \left[(t-s)_-^{H-1/\alpha} - (-s)_-^{H-1/\alpha} \right] \right\} dZ_s, \quad t \in \mathbb{R}.$$

M is a stationary α-stable random measure.

- Non-anticipative fractional Lévy process:

$$M_d(t) = \int_{\mathbb{R}} \left((t-s)_+^d - (-s)_+^d \right) dZ_s, \quad t \in \mathbb{R}.$$

- Well-balanced fractional Lévy process:

$$N_d(t) = \int_{\mathbb{R}} \left(|t-s|^d - |s|^d \right) dZ_s, \quad t \in \mathbb{R}.$$

9. Stationary increment moving average (SIMA) ID processes.

$$X_t = \int_{\mathbb{R}} (f(t-s) - f_0(-s))\, dZ_s, \quad t \in \mathbb{R}.$$

Cases $f_0 = f$ and $f_0 = 0$.
Mixed SIMA = SIMMA

$$X_t = \int_{\mathbb{R}} (f(t-s,v) - f_0(-s,v))\, M(ds\, dv), \quad t \in \mathbb{R}.$$

ID random vectors are classified by their Lévy-Khintchine triplets. Can one extend this classification to ID processes?

In his fundamental work Maruyama (1970) defined a Lévy measure of an infinitely divisible process on a σ-ring of subsets of $\mathbb{R}^T$. Such σ-ring has a complicated structure when the index set T is uncountable. Moreover, Maruyama's proof does not seem to be complete.

Below we present a simpler and natural construction of Lévy-Khintchine triplets for ID processes.

Let T be a nonempty set. For every $S \subset T$, consider the product measurable space

$$(\mathbb{R}^S, \mathcal{B}^S) = \prod_{t \in S} (R_t, \mathcal{B}_t)$$

where $(R_t, \mathcal{B}_t) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Let $p_{U,S} : \mathbb{R}^U \mapsto \mathbb{R}^S$ denote the canonical projection, $S \subset U \subset T$, and write p_S for $p_{T,S}$.

Let $\mathbf{0}_S$ denote the origin of $\mathbb{R}^S$, which is the set $\mathbf{0}_S = \prod_{t \in S} \{0_t\}$, $0_t = 0$.

Denote by $\mathcal{P}_f(T)$ the family of finite subsets of T.

When T is countable and ν is a Lévy measure on $\mathbb{R}^T$, the condition $\nu(\mathbf{0}_T) = 0$ guarantees the uniqueness of ν. Such condition does not make sense when T is uncountable because $\mathbf{0}_T \notin \mathcal{B}^T$. We circumvent this difficulty by extending the meaning of "ν does not charge the origin".

Definition

Let ν be a measure on $(\mathbb{R}^T, \mathcal{B}^T)$. We say that ν does not charge the origin if for every $A \in \mathcal{B}^T$ there exists a countable subset S of T such that

$$\nu(A) = \nu(A \setminus p_S^{-1}(\mathbf{0}_S)).$$

Definition (Path Lévy measure)

A measure ν on $(\mathbb{R}^T, \mathcal{B}^T)$ is said to be a Lévy measure if it does not charge the origin and for every $t \in T$

$$\int_{\mathbb{R}^T} \left(|x_t|^2 \wedge 1 \right) \, \nu(dx) < \infty.$$

Theorem

Let $\{\nu_F : F \in \mathcal{P}_f(T)\}$ *be a family of Lévy measures such that*

(i) *for every* $F \in \mathcal{P}_f(T)$, ν_F *is a Lévy measure on* $(\mathbb{R}^F, \mathcal{B}^F)$,

(ii) *for every* $F, G \in \mathcal{P}_f(T)$ *with* $F \subset G$,

$$\nu_F = \nu_G \circ p_{G,F}^{-1} \quad \text{on} \quad \mathcal{B}^F \cap (\mathbf{0}_F)^c.$$

Then there exists a unique measure ν *on* $(\mathbb{R}^T, \mathcal{B}^T)$ *such that* ν *does not charge the origin and for every* $F \in \mathcal{P}_f(T)$

$$\nu \circ p_F^{-1} = \nu_F \quad \text{on} \quad \mathcal{B}^F \cap (\mathbf{0}_F)^c.$$

ν *will be called a path Lévy measure.*

Remark

Path Lévy measure does not need be σ-finite but it is unique for a consistent system of finite dimensional Lévy measures.

The difficulty in the proof comes from the fact that such system does not form a projective family of measures because of the condition of no mass at the origin (needed for the uniqueness).

Our result does not require any integrability conditions, however, so it applies to general classes of measures that do not charge the origin.

Theorem

Let $\mathbf{X} = \{X_t\}_{t \in T}$ *be an infinitely divisible stochastic process. Then there exist a unique triplet* (b, Σ, ν) *consisting of*

(i) $b \in \mathbb{R}^T$,

(ii) *a nonnegative symmetric operator* $\Sigma : \mathbb{R}^{(T)} \mapsto \mathbb{R}^T$,

(iii) *a Lévy measure* ν *on* $\mathbb{R}^T$

such that for any $y \in \mathbb{R}^{(T)}$

$$\mathbb{E}e^{i \sum_{t \in T} y_t X_t} =$$

$$\exp\left\{ i\langle y, b\rangle - \frac{1}{2}\langle y, \Sigma y\rangle + \int_{\mathbb{R}^T} \left(e^{i\langle y, x\rangle} - 1 - i\langle y, [\![x]\!]\rangle \right) \nu(dx) \right\}.$$

Notation in the theorem:
For $x \in \mathbb{R}^T$ the truncation $[\![x]\!] \in \mathbb{R}^T$ is defined by

$$[\![x]\!]_t := \frac{x_t}{|x_t| \vee 1}, \quad t \in T.$$

$$\mathbb{R}^{(T)} = \{x \in \mathbb{R}^T : x_t = 0 \text{ for all but finitely many } t\}.$$

$$\langle y, x \rangle = \sum_{t \in T} y_t x_t, \quad y \in \mathbb{R}^{(T)},\ x \in \mathbb{R}^T.$$

Proposition

Let ν be a Lévy measure on $\mathbb{R}^T$. Then ν is σ-finite if and only if

$$\nu(p_{T_0}^{-1}(\mathbf{0}_{T_0})) = 0.$$

for some countable set $T_0 \subset T$.

Corollary

If an ID process $\mathbf{X} = \{X_t\}_{t \in T}$ is separable in probability, i.e., there is a countable set $T_0 \subset T$ such that $\forall\ t \in T\ \exists\ \{t_n\} \subset T_0$ such that $X_{t_n} \xrightarrow{P} X_t$, then the path Lévy measure ν of $\mathbf{X}$ is σ-finite.

As a consequence of the theorem we get,

$$\mathbf{X} \stackrel{d}{=} \mathbf{G} + \mathbf{Y},$$

where $\mathbf{G} = \{G_t\}_{t \in T}$ is a mean zero Gaussian process with the covariance operator Σ, $\mathbf{Y} = \{Y_t\}_{t \in T}$ is an infinitely divisible process of the Poissonian-type with the triplet $(0, \nu, b)$, and $\mathbf{G}$, $\mathbf{Y}$ are independent.
Σ is described by covariance function of $\mathbf{G}$ by

$$\langle y, \Sigma y \rangle = \sum_{s, \in T} y_s y_t \mathrm{Cov}(G_s, G_t), \quad y \in \mathbb{R}^{(T)}.$$

From now on will concentrate on infinitely divisible processes without Gaussian part.

Another view at Lévy and additive processes

1. Let $\mathbf{X} = \{X_t\}_{t\geq 0}$ be a Lévy process with

$$\mathbb{E}e^{iuX_t} = e^{t\psi(u)},$$

$$\psi(u) = \int_{-\infty}^{\infty} (e^{iuv} - 1 - iu[\![v]\!])\, \eta(dv).$$

Here $T = \mathbb{R}_+$. What is the Lévy measure ν of $\mathbf{X}$?

Let $F = \{0 \leq t_1 < \cdots < t_n\} \subset \mathbb{R}_+$ and let ν_F be the Lévy measure of $X_F = (X_{t_1}, \ldots, X_{t_n})$. Since the Lévy measure of $(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$ is concentrated on the axes and equals to

$$\sum_{k=1}^{n} (t_k - t_{k-1})\delta_0 \times \cdots \times \delta_0 \times \underbrace{\eta}_{k} \times \delta_0 \times \cdots \times \delta_0,$$

we get

$$\int_{\mathbb{R}^n} g(x)\,\nu_F(dx) = \sum_{k=1}^{n}(t_k - t_{k-1}) \int_{-\infty}^{\infty} g(\underbrace{0,\ldots,0}_{k-1}, v,\ldots,v)\,\eta(dv).$$

for every $g : \mathbb{R}^n \mapsto \mathbb{C}$ such that $|g(x)| \le C(\|x\|^2 \wedge 1)$.
Let $p_F : \mathbb{R}^{\mathbb{R}_+} \mapsto \mathbb{R}^F$ denote the projection. By the consistency,

$$\begin{aligned}\int_{\mathbb{R}^T} g\left(p_F(x)\right)\nu(dx) &= \int_{\mathbb{R}^n} g(x)\,\nu_F(dx)\\ &= \sum_{k=1}^{n}(t_k - t_{k-1}) \int_{-\infty}^{\infty} g(\underbrace{0,\ldots,0}_{(k-1)-\text{times}}, v,\ldots,v)\,\eta(dv)\\ &= \int_0^{\infty}\int_{-\infty}^{\infty} g\left(p_F(v\mathbf{1}_{[s,\infty)})\right)\eta(dv)ds.\end{aligned}$$

Hence Lévy measure ν of a Lévy process $\mathbf{X}$ is the image measure of $\eta \otimes \text{Leb}$ by

$$\mathbb{R} \times \mathbb{R}_+ \ni (v, s) \mapsto v\mathbf{1}_{[s,\infty)} \in \mathbb{R}^{\mathbb{R}_+}.$$

In particular, every such ν is concentrated on the set of one-step functions

$$S = \text{supp } \nu = \{v\mathbf{1}_{[s,\infty)} : v \in \mathbb{R},\ s \geq 0\}.$$

(Precisely, $\nu_*(\mathbb{R}^{\mathbb{R}_+} \setminus S) = 0$.)

For a Poisson process with parameter λ,

$$\text{supp } \nu = \{\mathbf{1}_{[s,\infty)} : s \geq 0\}$$

and ν is the image measure of $\eta \otimes \text{Leb}$ by the map $s \mapsto \mathbf{1}_{[s,\infty)}$.

2. Additive processes.

Proposition

Let $\mathbf{X} = \{X_t\}_{t \geq 0}$ *be a Poissonian-type infinitely divisible process with Lévy measure* ν*. Then* $\mathbf{X}$ *is an independent increment (additive) process if and only if* ν *is concentrated on the set*

$$S = \{v\mathbf{1}_{[s,\infty)} : v \in \mathbb{R},\ s \geq 0\}$$

i.e., $\nu_*(\mathbb{R}^T \setminus S) = 0$.

From the perspective of the support, Lévy (or additive) processes constitute a rather narrow class of infinitely divisible processes.

NOTE:

'Geometry' of the support of a Lévy measure determines to some extend sample path properties. For example, 'bad' properties (such as discontinuities) of functions in the support of the Lévy measure are inherited by sample paths of the process.

Series representation if time is OK.

Generalized Lévy-Itô representation

Proposition

Let ν be a Lévy measure on $\mathbb{R}^T$. Let E be a Borel space equipped with a σ-finite measure n and a measurable function $f : E \mapsto \mathbb{R}^T$ such that

$$n \circ f^{-1} = \nu.$$

Let N be a Poisson random measure on E with intensity measure n. Then the stochastic integral

$$X_t := \int_E f_t(x) \left(N(dx) - \frac{\nu(dx)}{1 \vee |f_t(x)|} \right), \quad t \in T$$

is well-defined and represents an infinitely divisible process with the triplet $(0, 0, \nu)$. Here $f_t(x)$ is the value of $f(x) \in \mathbb{R}^T$ at t.

Example

Let $E = \mathbb{R} \times \mathbb{R}_+$ and let $n = \eta \times \text{Leb}$ be a measure on E, where η is a Lévy measure on $\mathbb{R}$. Consider $f : E \mapsto \mathbb{R}^{\mathbb{R}_+}$,

$$f_t(x) = v\mathbf{1}_{[s,\infty)}(t), \quad x = (v, s) \in E.$$

We know that $\nu = n \circ f^{-1}$ is a Lévy measure of a Lévy process, which by the above Proposition has the form

$$\begin{aligned} X_t &= \int_{\mathbb{R}\times\mathbb{R}_+} v\mathbf{1}_{[s,\infty)}(t) \left(N(dv, ds) - \frac{\eta(dv)ds}{1 \vee |v|\mathbf{1}_{[s,\infty)}(t)} \right) \\ &= \int_{\mathbb{R}\times[0,t]} v \left(N(dv, ds) - \frac{\eta(dv)ds}{1 \vee |v|} \right) \\ &= \int_{[-1,1]\times[0,t]} v\,(N(dv, ds) - \eta(dv)ds) \\ &\quad + \int_{[-1,1]^c\times[0,t]} v\, N(dv, ds) + at. \end{aligned}$$

Generalized Lévy-Itô representation shows that every ID process without Gaussian part is a stochastic integral process with respect to a Poisson random measure.

Processes with special distributional properties, such as stable, selfdecomposable, etc. have special Lévy measures that can be factored. This leads to representations with respect to other than Poisson or Gaussian random measures.

1. Stochastic integral processes
2. Generating triplet of ID processes
3. Another view at Lévy and additive processes
4. Generalized Lévy-Itô representation
5. Stochastic integral representation of ID processes

5. Simulation of Lévy processes: an overview

Motivation Consider an SDE:

$$dX_t = f(X_{t-})\, dZ_t,$$

where $\{Z_t\}$ is a Lévy process. Interested in the numerical value of

$$\mathbb{E}g(X_T)$$

for some know function g and terminal value T. The Euler scheme:

$$X^{(n)}_{\frac{(k+1)}{n}T} = X^{(n)}_{\frac{k}{n}T} + f\left(X^{(n)}_{\frac{k}{n}T}\right)\left(Z_{\frac{(k+1)}{n}T} - Z_{\frac{(k+1)}{n}T}\right), \quad X^{(n)}_0 = X_0.$$

$$Eg(X_T) \approx Eg(X^{(n)}_T) \leftarrow \text{use MONTE CARLO technique.}$$

The latter step requires many simulations of a Lévy process.
We may consider other SDEs involving Lévy processes and different approximation schemes.
General problem: Given a functional $\Psi(Z_{(\cdot)})$ of a path $\{Z_t : t \in [0, T]\}$, find

$$\mathbb{E}g\left(\Psi(Z_{(\cdot)})\right).$$

Numerical method:

$$\mathbb{E}g\left(\Psi(Z_{(\cdot)})\right) \approx \mathbb{E}g\left(\Psi(Z_{(\cdot)}^{(n)})\right),$$

where the latter expectation is obtained by Monte Carlo technique, requires fast and efficient simulation of approximations $\{Z_t^{(n)} : t \in [0, T]\}$ to a Lévy process $\{Z_t : t \in [0, T]\}$.

1. Notation

$$Ee^{iuX(t)} = \exp(tC(u)),$$

where $C(u)$ is given by the Lévy-Khintchine formula

$$C(u) = iau - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iu\mathbf{1}_{\{|x|\leq 1\}}\right) \nu(dx), \qquad (4)$$

Jumps $J(t) = X(t) - X(t-)$ of a Lévy process $\{X(t)\}$:

The counting process

$$N((0,t] \times B) = \text{Card}\{s \leq t : J(s) \in B\}, \quad B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$$

is a Poisson point process with rate $dt\,\nu(dx)$.

Thus $\underline{\nu(dx) \text{ is the intensity of jumps of size } x}$.

ν is the Lévy measure of $X(1)$, satisfying $\nu(\{0\}) = 0$ and

$$\int \min\{|x|^2, 1\}\,\nu(dx) < \infty.$$

The Lévy-Itô representation:

$$X(t) = at + \sigma B(t) + \lim_{\epsilon \downarrow 0} \Big\{ \sum_{s \le t} J(s) \mathbf{1}_{\{|J(s)| > \epsilon\}} - t \int_{\epsilon < |x| \le 1} x \nu(dx) \Big\}, \tag{5}$$

where $\{B(t)\}$ is a standard Brownian motion independent of the process of jumps $\{J(t)\}_{t \ge 0}$. The convergence on the right hand side holds a.s. uniformly in t on each finite interval.

When $\nu(\mathbb{R}) < \infty$ we can write

$$X(t) = \mu t + \sigma B(t) + \sum_{j=1}^{N(t)} V_j,$$

where $\{B(t)\}$ is a standard Brownian motion, $\{N(t)\}$ is a Poisson process with rate $\lambda = \nu(\mathbb{R})$, V_j are i.i.d. with the distribution $\nu/\nu(\mathbb{R})$, $\{B(t)\}$, $\{N(t)\}$, and $\{V_j\}$ are independent of each other. Moreover, $\mu = a - \int_{|x|\le 1} x\nu(dx)$.

If

$$\int \min\{|x|, 1\}\,\nu(dx) < \infty, \tag{6}$$

then $\sum_{s\le t} |J(s)| < \infty$ a.s. and (5) reduces to

$$X(t) = \mu t + \sigma B(t) + \sum_{s\le t} J(s),$$

where μ is as above. Sample paths of $\{X(t)\}$ have bounded variation on each finite interval if and only if $\sigma = 0$ and (6) holds.

If additionally $\mu \ge 0$ and $\nu((-\infty, 0)) = 0$, then $\{X(t)\}$ has nondecreasing sample paths and is called a subordinator.

A Lévy process $\{X(t)\}$ is a Markov process with transition probabilities

$$P_{t,x}(\cdot) = \mathbb{P}(X(t) + x \in \cdot).$$

Explicit forms of $P_{t,x}$ or their densities $p_{t,x}$ are known in some cases but generally they are unknown.

The *generating triplet* (a, σ^2, ν) or the *cumulant* $C(u)$ of $X(1)$ are usually used as an identifiable parametrization of Lévy processes.

2. Simulation of 1-dimensional Lévy processes

Simulation of a Brownian motion and/or of a compound Poisson process can be found in many textbooks and will not be discussed here. Brownian and Poissonian-type components of a Lévy process can be generated independently of each other.

The main problem is to simulate a Poissonian-type component of a Lévy process having infinite Lévy measure.
In that case, by the Lévy-Itô representation, sample paths of $\{X(t)\}$ have infinitely many jumps in each finite interval.
Exact simulation of such process is obviously impossible.

2 (A). Random walk approximation

Suppose $\{X(t)\}$ is a Lévy process determined by the Lévy-Khintchine formula (4) with $\sigma = 0$ and an infinite Lévy measure ν. Fix the time domain $[0, T)$, $n \geq 1$, and put $h = T/n$. Generate the increments $\Delta_j^h X = X(jh) - X((j-1)h)$ as i.i.d. random variables with the distribution $P_h(\cdot) = \mathbb{P}(X(h) \in \cdot)$, $j = 1, \ldots, n-1$, and let

$$X^h(t) = \begin{cases} 0 & \text{if } 0 \leq t < h, \\ \Delta_1^h X + \cdots + \Delta_j^h X & \text{if } jh \leq t < (j+1)h. \end{cases}$$

Process $\{X^h(t)\}_{0 \leq t < T}$ is a random walk approximation to $\{X(t)\}_{0 \leq t < T}$.

Example (Standard gamma process)

This is a Lévy process such that $X(t) \sim \text{Gamma}(t,1)$. The density of $X(h)$ is given by

$$p_h(x) = \frac{1}{\Gamma(h)} x^{h-1} e^{-x}, \quad x > 0.$$

There exist many algorithms for generating $\Delta_j^h X$'s with this density; see the book of Devroye (1986) for a survey on this topic.

Johnk's algorithm ($h < 1$):

REPEAT *Generate i.i.d. Uniform(0,1) r.v.'s* U, V *Set* $Y = U^{1/h}$, $Z = V^{1/(1-h)}$ UNTIL $Y + Z \leq 1$;

Generate Exponential(1) r.v. W;

RETURN $\frac{YW}{Y+Z}$; *Set* $\Delta_j^h X = \frac{YW}{Y+Z}$.

Example (Stable process)

A stable Lévy process is determined by

$$Ee^{iuX(t)} = e^{t\psi(u)}$$

where the cumulant function $\psi(u)$ is of the form

$$\psi(u) = \begin{cases} -\theta^\alpha |u|^\alpha (1 - i\beta \operatorname{sgn} u \tan \frac{\pi\alpha}{2}) + i\mu u & \text{if } 0 < \alpha < 2, \ \alpha \neq 1 \\ -\theta |u| (1 + i\beta \frac{2}{\pi} \operatorname{sgn} u \log |u|) + i\mu u & \text{if } \alpha = 1, \end{cases}$$

where $0 < \alpha < 2$, $\theta > 0$, and $-1 \le \beta \le 1$. If $\beta = \mu = 0$, the

$$Ee^{iuX(t)} = e^{-t\theta|u|^\alpha}.$$

Densities $p_h(x)$ are known in only a few cases of α.

Example (Stable process: Chambers, Mallow and Struck algorithm)

Chambers, Mallow and Struck (1976) gave an algorithm for simulation of arbitrary one-dimensional stable r.v.
In the symmetric case, i.e. $\psi(u) = -\theta^\alpha |u|^\alpha$, this algorithm has a specially simple form

$$\Delta_j^h X = \theta h^{1/\alpha} \frac{\sin \alpha U_j}{(\cos U_j)^{1/\alpha}} \left(\frac{\cos((1-\alpha)U_j)}{V_j} \right)^{(1-\alpha)/\alpha},$$

where U_j are i.i.d. Uniform$(-\pi/2, \pi/2)$ r.v.'s independent of i.i.d. Exponential(1) r.v.'s V_j.

Example (Damien, Laud & Smith algorithm (1995))

This algorithm provides an approximation to an infinitely divisible random variable. Let ν be a Lévy measure and define

$$\theta(dx) = x^2/(1+x^2)\,\nu(dy)$$

and $c = \int_{-\infty}^{\infty} \theta(dx)$. Fix $m \geq 1$ and let (U_i, V_i), $i = 1, \ldots, m$, be i.i.d. pairs such that U has distribution $\theta(dx)/c$ and given $U = u$, $V \sim Poisson(\lambda_m(u))$, where $\lambda_m(u) = c(1+u^2)/(mu^2)$. Then, as $m \to \infty$,

$$\sum_{i=1}^{m} \left(U_i V_i - \frac{c}{mU_i} \right) \tag{7}$$

converges in distribution to an infinitely divisible random variable with the generating triplet $(\int (x\mathbf{1}_{\{|x|\leq 1} - \frac{x}{1+x^2})\nu(dx), 0, \nu)$. The centers $\frac{c}{mU_i}$ in (7) are not needed when (6) holds.

Example (Damien, Laud & Smith algorithm, cont.)

Using this algorithm for an approximate simulation of a Lévy process $\{X(t)\}$, one needs to fix n, the number of partitions of the time domain $[0, T]$ and m, the number of of terms in (7) to approximately generate the increment $\Delta_j^h X$. Two types of errors are committed, one from using a discrete skeleton and another one from each approximation at a grid point.

Comments on the random walk approximation method.

A discrete skeleton $\{X(jh) : j = 0, 1, \dots\}$ is a random walk. Therefore, if a simulation method of $X(h)$ is available, then one simulates $X(t)$ approximately when $t \notin h\mathbb{Z}$.

A disadvantage of such method is that one cannot precisely identify the location and magnitude of the large jumps. Large jumps are important, especially in the heavy tailed case, because they determine various functionals of a Lévy process. Another complication may be that a simulation of $X(h)$ may be numerically tedious (e.g., when its density involves special functions).

Series representations provide uniform along sample paths approximation of Lévy processes. Let ν be a Lévy measure. Define its <u>tail integral</u> by

$$U(x) = \begin{cases} \nu((x,\infty)), & x \geq 0, \\ -\nu((-\infty,x]), & x < 0 \end{cases}$$

and the inverse of U by

$$U^{-1}(u) = \begin{cases} \inf\{x > 0 : U(x) \leq u\}, & u \geq 0, \\ \inf\{x < 0 : U(x) \leq u\} \wedge 0, & u < 0. \end{cases}$$

Theorem

Let $\{\gamma_j\}$ be a Poisson point process on $\mathbb{R}$ with unit rate. Fix $T > 0$ and let $\{\tau_j\}$ be i.i.d. Uniform$(0, T)$. Then

$$X(t) = \sum_{j=1}^{\infty} \left[U^{-1}(\gamma_j/T)\mathbf{1}_{\tau_j \le t} - a_j t \right]$$

converges uniformly to a Lévy process $\{X(t) : t \in [0, T]\}$ with the characteristic triplet $(0, 0, \nu)$. Here

$$a_j = \int_{j-1 \le |u| \le j} U^{-1}(u)\mathbf{1}_{|U^{-1}(u)| \le 1} \, du.$$

Remark: $\gamma_j = \epsilon_j \Gamma_j/2$, where $\{\Gamma_j\}$ is a random walk with Eponential(1)-steps independent of the Bernoulli($\frac{1}{2}$) sequence of signs $\{\epsilon_j\}$.

Example (Stable process (LePage 1980))

$\nu(dx) = c_1 \alpha x^{-\alpha-1} \mathbf{1}_{x>0} dx + c_{-1} \alpha |x|^{-\alpha-1} \mathbf{1}_{x<0} dx.$

$$U(x) = \begin{cases} c_1 x^{-\alpha}, & x \geq 0, \\ -c_{-1}|x|^{-\alpha}, & x < 0 \end{cases}$$

and

$$U^{-1}(u) = \begin{cases} (u/c_1)^{-1/\alpha}, & u \geq 0, \\ -(|u|/c_{-1})^{-1/\alpha}, & u < 0. \end{cases}$$

Hence

$$X(t) = \sum_{j=1}^{\infty} \Big[\epsilon_j \Big(\frac{|\gamma_j|}{T c_{\epsilon_j}} \Big)^{-1/\alpha} \mathbf{1}_{\tau_j \leq t} - a_j t \Big],$$

where $\epsilon_j = \operatorname{sgn}(\gamma_j)$.

Example (Standard gamma process, series representation)

$\nu(dx) = x^{-1}e^{-x}\,dx$. $U(x) = \int_x^\infty t^{-1}e^{-t}\,dt = \mathrm{Ei}(x)$, the exponential integral function. Neither Ei nor Ei^{-1} have explicit forms. Nevertheless, one can represent a gamma process as

$$X(t) = \sum_{j=1}^{\infty} \mathrm{Ei}^{-1}(\gamma_j/T)\mathbf{1}_{\tau_j \le t}.$$

There are important Lévy processes whose Lévy measure tail functions do not have explicit inverses. In such situation, using the series formula can be numerically expensive.

Theorem (Generalized series representations)

Let $\{\Gamma_j\}$ be a random walk with Eponential(1)-steps, $\{V_j\}$ an i.i.d. sequence of random elements in some measurable space E, and $\{\tau_j\}$ and i.i.d. Uniform$(0, T)$. Assume that all sequences $\{\Gamma_j\}$, $\{V_j\}$, and $\{\tau_j\}$ are independent of each other. Let $H : \mathbb{R}_+ \times E \mapsto \mathbb{R}$ be such that

$$T^{-1} \int_0^\infty P(H(u, V) \in \cdot)\, du = \nu(\cdot).$$

Then

$$X(t) = \sum_{j=1}^\infty \left[H(\Gamma_j, V_j) \mathbf{1}_{\{U_j \le t\}} - t b_j \right], \quad 0 \le t \le T. \tag{8}$$

converges uniformly to a Lévy process $\{X(t) : t \in [0, T]\}$ with the characteristic triplet $(0, 0, \nu)$. Here $b_j = \int_{j-1}^j \mathbb{E} H(s, V_1) \mathbf{1}_{|H(s,V_1)| \le 1}$.

The trick is to find (guess) a function H and a marking variable V so that (9) holds.

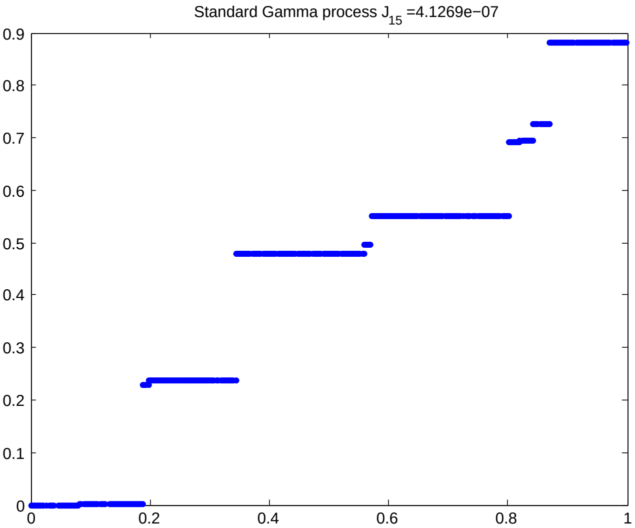
Example (Standard gamma process, series representation revisited)

Bondesson (1982) gave the following formula for a gamma process.

$$X(t) = \sum_{j=1}^{\infty} e^{-\Gamma_j/T} V_j \mathbf{1}_{\tau_j \le t},$$

where $\{V_j\}$ is i.i.d. Exponential(1) independent of $\{\Gamma_j, \tau_j\}$.

Here $H(u, v) = e^{-u/T} v$, $V \sim$ Exponential(1).


Standard Gamma process J_{15} =4.1269e−07
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1

Example (Tempered stable process algorithm (J.R. 2004))

We consider a symmetric Lévy process with
$\nu(dx) = \kappa_1 x^{-\alpha-1} e^{-\lambda_1 x} \mathbf{1}_{x>0}\, dx + \kappa_{-1} |x|^{-\alpha-1} e^{-\lambda_{-1}|x|} \mathbf{1}_{x<0}\, dx$.
The tail function, for $x > 0$,

$$U(x) = \nu((x,\infty)) = \kappa_1 \int_x^\infty t^{-\alpha-1} e^{-\lambda_1 t}\, dt = \kappa_1 \lambda_1^\alpha \mathrm{Ei}(\alpha+1, \lambda_1 x),$$

$U(x) = \kappa_{-1} \lambda_{-1}^\alpha \mathrm{Ei}(\alpha+1, \lambda_{-1}|x|)$ when $x < 0$.

$$Ei(p, x) = \int_x^\infty t^{-p} e^{-t}\, dt.$$

In order to avoid inverting $Ei(p, \cdot)$ (many times) in the usual series expansion we need to find an appropriate H and marking V.

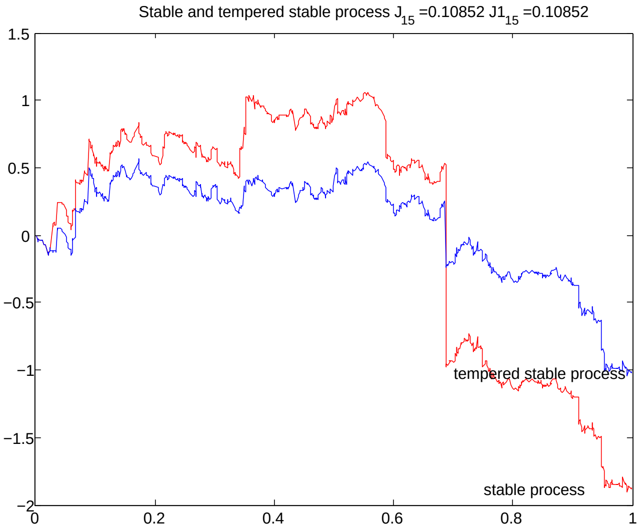
Example (Tempered stable process algorithm, cont.)

(Symmetric case: $\kappa_1 = \kappa_{-1} = \kappa$, $\lambda_1 = \lambda_{-1} = \lambda$.) Let $\{\epsilon_j\}$, $\{\eta_j\}$, and $\{\xi_j\}$ be sequences of i.i.d. random variables such that $\mathbb{P}(\epsilon_j = \pm 1) = 1/2$, $\eta_j \sim$ Exponential(λ), and $\xi_j \sim$ Uniform$(0,1)$. All random sequences $\{\Gamma_j\}$, $\{U_j\}$, $\{\epsilon_j\}$, $\{\eta_j\}$, and $\{\xi_j\}$ are assumed to be independent of each other. Then

$$X(t) = \sum_{j=1}^{\infty} \epsilon_j \left(\left(\frac{\alpha \Gamma_j}{2\kappa T} \right)^{-1/\alpha} \wedge \eta_j \xi_j^{1/\alpha} \right) \mathbf{1}_{\{U_j \le t\}}, \quad 0 \le t \le T$$

represents a symmetric tempered α-stable process with Lévy measure as above.

Here $V = (\epsilon, \eta, \xi)$, $H(u, V) = \epsilon((\frac{\alpha u}{2\kappa T})^{-1/\alpha} \wedge \eta\xi)$.


Stable and tempered stable process J15 =0.10852 J115 =0.10852
1.5
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tempered stable process
stable process

Comments on series representations

Series representations provide uniform along sample paths approximation of Lévy processes. They are often easy to simulate. Usually the largest jumps of a Lévy process are included in the first few terms of the series. A disadvantage of this method is that some series may converge slowly. Therefore, one may need a huge number of terms to reach a desired accuracy of the approximation. On the other hand, with ever increasing computational speed, a slow convergence may be not an issue of practical importance for some applications.

$\{X(t)\}$ a Lévy process with the generating triplet $(a, 0, \nu)$ can be decomposed into a sum of two independent Lévy processes

$$X(t) = X^{\epsilon}(t) + X_{\epsilon}(t), \tag{9}$$

where $\{X^{\epsilon}(t)\}$ is a compound Poisson process with a drift and the distribution of jumps proportional to $\nu^{\epsilon} = \nu_{|\{|x|>\epsilon\}}$ and the process $\{X_{\epsilon}(t)\}$ has mean zero and the Lévy measure $\nu_{\epsilon} = \nu_{|\{|x|\leq\epsilon\}}$; $\epsilon > 0$ is fixed. We refer to $\{X_{\epsilon}(t)\}$ as a small-jump part of $\{X(t)\}$.

When the intensity of small jumps is high (as for a stable process with $\alpha > 1$) it was proposed to replace X_ϵ by a Brownian motion with variance $\sigma_\epsilon^2 = \int_{(-\epsilon,\epsilon)} x\,\nu(dx)$:

$$X(t) = X^\epsilon(t) + X_\epsilon(t) \approx X^\epsilon(t) + \sigma_\epsilon W(t),$$

where $\{W(t)\}$ is a standard Brownian motion independent of $\{X^\epsilon(t)\}$.

Asmussen and J.R. (2001) rigorously discussed this approximation and showed that as $\epsilon \to 0$

$$\sigma_\epsilon^{-1} X_\epsilon \xrightarrow{d} W \quad \Longleftrightarrow \quad \sigma_{c\sigma_\epsilon \wedge \epsilon} \sim \sigma_\epsilon \quad (\forall c > 0).$$

One can select the level of cut ϵ experimentally.

To this end one needs a consistent procedure for generating $\{X^\epsilon(t)\}$ as $\epsilon \downarrow 0$. This is possible when a shot noise series representation is available. We take

$$X^\epsilon(t) = \sum_{j:\,\Gamma_j \le \epsilon^{-1}} \left[H(\Gamma_j, V_j)\mathbf{1}_{\{U_j \le t\}} - t a_j \right].$$

Some work on Berry-Esseen bounds and Edgeworth approximations was done by Asmussen and J.R. (2001) but precise estimates on errors of such approximations are missing.

Comments on Poisson and Gaussian approximations.

If one discards small jumps or if one replaces them by their mean value, then the resulting process is a compound Poisson process with a drift. This is a Poisson approximation of a Lévy process. The large jumps are precisely simulated.
However, when the intensity of small jumps is high, discarding them may produce a substantial error. In such case one can often replace the small jump part by a Brownian motion with small variance. This approximation complements the method based on series representations because is applicable when the series converges slowly.

4. Simulation of multidimensional Lévy processes

$$\mathbf{E}e^{i\langle y,X(t)\rangle} = \exp\left\{t\left[i\langle a,y\rangle + \int_{\mathbb{R}^d}(e^{i\langle y,x\rangle} - 1 - i\langle y,x\rangle\mathbf{1}_{\|x\|\le 1})\,\nu(dx)\right]\right\}.$$

Contrary to the one-dimensional case, close formulas for simulation of increments of multidimensional Lévy processes are rarely available.

We need approximate methods:

- Generalized shot noise series representations of Lévy processes
- Poisson and Gaussian approximations

$$X(t) = \sum_{j=1}^{\infty} \left[H(\Gamma_j, V_j) \mathbf{1}_{\{U_j \le t\}} - t a_j \right], \quad 0 \le t \le T.$$

Here $\{\Gamma_j\}$ is the sequence of arrival times in a Poisson process of rate one, $\{V_j\}$ is a sequence of iid random elements taking values in a Borel space S and having the common distribution Q, $\{\tau_j\}$ an iid sequence of uniform on $[0, T]$ random variables, and the sequences $\{\Gamma_j\}$, $\{V_j\}$, and $\{\tau_j\}$ are independent of each other. Furthermore, $\{a_j\}$ is a sequence of vectors in $\mathbb{R}^d$ and $H : \mathbb{R}_+ \times S \mapsto \mathbb{R}^d$ is a measurable map such that the function $r \mapsto \|H(r,s)\|$ is nonincreasing for each $s \in S$ and

$$\nu(B) = \int_0^{\infty} Q(\{s \in S : \ H(r,s) \in B \setminus \{0\}\}) \, dr, \qquad B \in \mathcal{B}(\mathbb{R}^d).$$

There are many ways of representing a Lévy process $\{X(t)\}$ in this form.

Series representation based on a polar decomposition of ν was proposed by LePage (1980).

$$\nu(A) = \int_{S^{d-1}} \int_0^\infty \mathbf{1}_A(xv)\, \rho(dx, v)\sigma(dv), \quad A \in \mathcal{B}(\mathbb{R}_0^d)$$

where σ is a probability measure on the unit sphere S^{d-1} of $\mathbb{R}^d$ and $\{\rho(\cdot, v)\}_{v \in S^{d-1}}$ is a measurable family of Lévy measures on $(0, \infty)$. Put

$$\rho^{-1}(u, v) := \inf\{x > 0 : u \geq \rho((x, \infty), v)\}.$$

Let $\{V_i\}$ be an i.i.d. sequence in S^{d-1} with the common distribution σ such that $\{V_i\}$ is independent of $\{\Gamma_i, \tau_i\}$.

$$X(t) = \sum_{j=1}^{\infty} \left[\rho^{-1}(\Gamma_j / T, V_j) V_j \mathbf{1}_{\{U_j \leq t\}} - t a_j\right], \quad 0 \leq t \leq T.$$

Another method of generating series representations is by Lévy copulas, introduced by Tankov (PhD thesis, Ecole Polytechnique, France, 2004.) (See also Cont, and Tankov, P. (2004, book) and Kallsen and Tankov 2006).

For the sake of simplicity, we give the definition for $d = 2$.
A function $F : (-\infty, \infty]^2 \mapsto (-\infty, \infty]$ is called a Lévy copula if

(i) $F(x, y) < \infty$ if $(x, y) \neq (\infty, \infty)$

(ii) $F(x, y) = 0$ if $\min\{|x|, |y|\} = 0$

(iii) F is 2-increasing, i.e., $F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1) \geq 0$ whenever $x_1 \leq x_2$ and $y_1 \leq y_2$

(iv) $F^{(i)}(x) = x$ for $i = 1, 2$. $(F^{(1)}(x) = F(x, \infty) - F(x, -\infty))$.

μ_F a measure determined by Lévy copula F by
$\mu_F((x_1, x_2] \times (y_1, y_2]) = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$

Let ν_1, ν_2 be marginal Lévy measures of ν on $\mathbb{R}^2$. Let U_1, U_2 be tail integrals of ν_1, ν_2, respectively. (see page 17)

For any Lévy measure ν there is a unique Lévy copula F such that

$$(u, v) \mapsto (U_1^{-1}(u), U_2^{-1}(v))$$

transports μ_F onto ν. Conversely, given one-dimensional Lévy measures ν_1, ν_2 a Lévy copula produces a Lévy measure ν on $\mathbb{R}^2$.

$$X(t) = \sum_{j=1}^{\infty} \left[\left(U_1^{-1}(\Gamma_j / T), U_2^{-1}(V_j) \right) \mathbf{1}_{\{U_j \le t\}} - t a_j \right], \quad 0 \le t \le T.$$

V_j is independent of $\{\Gamma_i\}_{i \neq j}$ and its conditional distribution given Γ_j is determined by Lévy copula F.

There is no universal formula for producing 'good' series representations of Lévy processes. It's better to see (11) as a general patterns and sometimes guess a representation.

(with Serge Cohen, 2007)

$\{X_\epsilon(t)\}$ 'small-jump' part of $\{X(t)\}$

$$\mathbf{E}e^{i\langle y, X_\epsilon(t)\rangle} = \exp\left\{t\int_{\mathbb{R}^d}[e^{i\langle y,x\rangle} - 1 - i\langle y, x\rangle]\nu_\epsilon(dx)\right\},$$

$$\Sigma_\epsilon = \int_{\mathbb{R}^d} xx^\top \nu_\epsilon(dx).$$

We assume that the matrix Σ_ϵ is nonsingular for sufficiently small $\epsilon > 0$.

$\{W(t)\}_{t\geq 0}$ a standard Brownian motion in $\mathbb{R}^d$. Then

$$\Sigma_\epsilon^{-1/2} X_\epsilon \xrightarrow{d} W \quad \Longleftrightarrow \quad \int_{\langle \Sigma_\epsilon^{-1} x, x\rangle > c} \langle \Sigma_\epsilon^{-1} x, x\rangle \, \nu_\epsilon(dx) \to 0 \quad \forall c > 0.$$

This result generalized the one-dim result of Asmussen and J.R. (2001). We now have

$$X(t) = X^\epsilon(t) + X_\epsilon(t) \approx X^\epsilon(t) + \Sigma_\epsilon^{1/2} W(t).$$

Direct computation of Σ_ϵ is often impossible. In (S. Cohen and J.R. 2007) we find computationally tractable matrices A_ϵ asymptotically equivalent to Σ_ϵ for processes of interest.

In particular, we provide Poisson-Gaussian approximation for stable and tempered stable Lévy processes in $\mathbb{R}^d$.

Verification that the Gaussian approximation holds is often difficult (opposite to $d = 1$).

Application to tempered stable processes

Recall that the Lévy measure of a tempered α–stable process $\mathbf{X}$ in $\mathbb{R}^d$ is of the form

$$\nu(dr, du) = \alpha r^{-\alpha-1} q(r,u)\, dr\, \lambda(du),$$

in polar coordinates, where $\alpha \in (0,2)$, λ is a finite measure on S^{d-1}, and $q : (0,\infty) \times S^{d-1} \mapsto (0,\infty)$ is a Borel function such that, for each $u \in S^{d-1}$, $q(\cdot, u)$ is completely monotone with $q(0+, u) = 1$ and $q(\infty, u) = 0$.

We show that

$$\epsilon^{\alpha-2}\Sigma_\epsilon = \epsilon^{\alpha-2}\int_{\mathbb{R}^d} xx^\top\,\nu_\epsilon(dx) \to \Lambda,$$

as $\epsilon \to 0$, where

$$\Lambda = \frac{\alpha}{2-\alpha}\int_{S^{d-1}} uu^\top\,\lambda(du).$$

Finally,

$$X(t) \stackrel{d}{=} a_\epsilon(t) + \epsilon^{1-\alpha/2}\Lambda^{1/2}W_t + N_\epsilon(t) + Y_\epsilon(t),$$

where

$$N_\epsilon(t) = \sum_{\{j\colon \Gamma_j \le \epsilon^{-\alpha} T\|\lambda\|\}} \mathbf{1}(\tau_j \le t)\left(\left(\frac{\Gamma_j}{T\|\lambda\|}\right)^{-1/\alpha} \wedge e_j u_j^{1/\alpha}\|v_j\|^{-1}\right)\frac{v_j}{\|v_j\|},$$

and

$$\epsilon^{\alpha/2-1} \sup_{t\in[0,T]} \|Y_\epsilon(t)\| \xrightarrow{\mathbb{P}} 0 \quad \text{as } \epsilon \to 0.$$

6. Open questions

- Precise estimation of the error in approximate methods.
- How to choose cutting level ϵ?
- Robustness. Continuity with respect to different functionals of Lévy processes such as solutions to stochastic differential equations.