How to Gamble If You Must (and if you really hate to gamble)

Robert Chen, Ilie Grigorescu, and Larry Shepp

University of Miami, Coral Gables, and Rutgers University

Abstract

You are in a sub-fair casino, with fortune $f_0 \in (0,1)$, and you want to turn it into a fortune of size one in discounted time. You may stake any amount, s_n, $0 < s_n \leq f_n$ and at any odds, $r_n > 0$, if your fortune is f_n at any time, $n \geq 0$. We assume every such gamble has an expected payoff with a *fixed value*, cs_n, with $-1 < c < 0$. The problem is to maximize the expected *discounted* payoff,

$V(f_0) = V(f_0; b, c) = \max_{\mathcal{S},\mathcal{R}} E_{f_0}^{\mathcal{S},\mathcal{R}} b^\tau \chi(f_\tau \geq 1)$

where the maximum is taken over all choices of stakes, $\mathcal{S}$, and odds, $\mathcal{R}$, and you get a positive payoff, namely b^τ, *only if* your fortune at time τ is at least unity. Here $0 < b < 1, -1 < c < 0$ are given parameters. The parameter b reflects that the gambler wants to stop gambling as soon as possible, while the parameter c reflects how subfair the casino is. The case $b = 1$ was solved earlier and it was shown that the optimal payoff is *not attained*. When $b < 1$, the solution is attained, and we solve explicitly for the unique optimal strategy $\mathcal{R}, \mathcal{S}$ and for $V = V(f, b, c)$. We show that there is an increasing sequence of numbers, $\phi_n \in [0,1)$, with ϕ_n tending to one, such that if $f_0 \in (\phi_{n-1}, \phi_n], n \geq 1$, then the optimal strategy is to make at most n bets; one stakes an amount s_0 at odds r_0, each given explicitly, at time 0. If you win any bet at any time, then your fortune is one and then you quit, but if you lose you then have a fortune, $f_1 \in (\phi_{n-2}, \phi_{n-1}]$, and then you then continue until you either win and get the reward $b^j, j \leq n$ and quit, or until you arrive at fortune 0 (after exactly n bets) and on such a path you get 0 reward. We show that ϕ_n tends to one very quickly: $1 - \phi_n \approx (-bc)^{\frac{n^2}{2}}$, and that $V(1^-, b, c) = \frac{b(1+c)}{1+bc}$. Set $\gamma^2 = \frac{-bc}{1-b(1+c)}$.

Then the ϕ_n and the values $v_n = V(\phi_n)$ at these break points, are $\phi_1 = 1-\gamma^2$, $v_0 = 0, v_1 = b(1+c)(1-\gamma^2)$, and for $n \geq 2$, ϕ_n, v_n are given recursively by:

$\phi_n = 1 + \frac{bc(1-\phi_{n-1})}{\frac{1-v_{n-1}}{1-v_{n-2}}\frac{1-\phi_{n-2}}{1-\phi_{n-1}} - b(1+c)}, v_n = b(1+c)(1 - \frac{1-\phi_n}{1-\phi_{n-1}}) + b(-c + (1+c)\frac{1-\phi_n}{1-\phi_{n-1}})v_{n-1}$.

Thus $\phi_2 = 1-\gamma^6$. For general $f_0 \in [0,1]$, between the break points, the optimal expected return is given explicitly, again, inductively. It is tempting to guess that for every n, ϕ_n is a power of γ since this holds for $n = 0, 1, 2$, but the asymptotics, $\phi \approx (-bc)^{\frac{n^2}{2}}$ shows this is false. In fact the recurrence shows that ϕ_3 is given by $\phi_3 = 1 + \frac{bc\gamma^{10}(1-b+\gamma^2)}{1-b-bc(1-b+\gamma^2)}$. Thus the ϕ's are explicit, but not simple.

We solve a more general problem as well, where there is a third parameter, $a > 0$, and the problem is to maximize

$V(f_0; a, b, c) = \max_{\mathcal{S},\mathcal{R}} E_{f_0}^{\mathcal{S},\mathcal{R}} b^\tau \chi(f_\tau \geq (1+a)^\tau)$.

Again the gambler wants to stop gambling (as reflected by the parameter b, but there is also a second kind of discounting, where fortunes decay in time, as reflected by the parameter a, so that one must achieve a growing fortune to obtain a positive reward. In the case $b = 1$, this problem was solved earlier. We show that if $a > 0, b \geq 0$, there are *at most a finite number of bets* no matter how close to one the initial fortune, f_0, is, as distinct to the case above where for $a = 0, b > 0$, the number of bets can be arbitrarily large if f_0 is close to one. Thus, in case $a > 0$, there are a finite number intervals defined by break points at which the optimal strategy makes at most n bets, $n \leq N(a,b,c) < \infty$. Of course, as $a \downarrow 0$, $N(a,b,c) \uparrow \infty$.

Discussion of prior results.

It is interesting to compare these results with the celebrated results for the Dubins-Savage and for the Vardi casinos. In the former, where there is only one fixed odds ratio r allowed and $b = 1$, the solution is highly non-unique. Although it is usually stated simply that "bold play" is optimal in the Dubins-Savage casino, which is true, it is somewhat misleading since there are many other optimal strategies as Dubins and Savage showed. In the Vardi casino, where again $b = 1$ but all $r > 0$ are allowed, there is no optimal solution though a sequence of strategies achieve the supremum value, given simply by $1-(1-f_0)^{1+c}$. Similar results have been shown to hold with $b = 1$ but $a > 0$. Thus if either discount factor, a, or b, is present, this gives rise to existence and uniqueness of the optimal solution. All of these problems and many similar ones belong to convexity theory, or infinite dimensional linear programming; it seems insightful to observe the consequences of discounting - it provides compactness and hence existence and uniqueness. The problem of this paper is one of many versions of the classical problem due to Dubins and Savage [1]; the new twist is that we assume there is a decline of utility or desire to continue gambling after every bet by a factor $0 < b < 1$. We bet on a binary outcome with odds r and we win with probability w and lose with probability $1 - w$. Thus our fortune moves from f to fortune $f + rs$ w.p. w and to fortune $f - s$ w.p. $1 - w$ (note that binary outcomes generate any possible gamble with a fixed expected return since these are the extreme points of a convex set). We must have $w = \frac{1+c}{1+r}$ so the expected return is cs, and we must have $-1 \leq c \leq r$ in order that w is a legitimate probability. The parameters $0 < b < 1$ and $-1 < c < r$ are assumed given. In the Dubins-Savage casino, the parameter r is fixed. In the present paper we assume that r can also be chosen to be any positive odds ratio at any time, as was suggested by Yehuda Vardi [12].

In Dubins and Savage's classic book [1] it is supposed the mafia will kill you if you do not repay your debt, normalized to be of size 1, to them. Your present fortune is $0 \leq f < 1$ and you are in a casino where certain gambles are available. You want to optimize the chance to reach one and remain alive. In Dubins's simplest case the casino allows only one *subfair* bet with odds r, i.e., you can stake any amount $s \leq f$ and reach the fortune

$f_1 = f + rs$ with probability $w < \frac{1}{1+r}$, or

$f_1 = f - s$ with probability $1 - w > \frac{r}{1+r}$.

It was proved by Dubins and Savage [1] that *bold play*, where you stake $s = \min(f, \frac{1-f}{r})$ at each bet until you either go broke or reach the goal is (non-uniquely) optimal. This is not as obvious as it may appear and with the variation where money decays by the factor $\frac{1}{1+a}$ and it is desired to finf $\max_\tau P(f_\tau = (1+a)^\tau)$, it is shown in [6] that bold play is no longer optimal and though it seems even more "obviously" optimal it is false. At certain initial fortunes it is necessary to play boldly but at other fortunes it is *provably suboptimal* to play boldly; the general optimal strategy is remains unknown. For the case when bets at any odds are permitted (Vardi casino), the problem was solved in [13], by methods similar to those used here, as discussed in the abstract. Related results were obtained in [8, 2, 3, 4, 7, 14, 9].

In more realistic casinos the casino allows more than one odds ratio, r, and then the optimal strategy is usually not known and hard to approximate much less determine without writing a large linear program. Yehuda Vardi raised the question in a conversation of whether bold play would also be optimal in a gambling house where *any* stake $s \leq f$ is allowed and *any odds* is allowed so long as the *expected return* is at most cs, where $c \in (-1, 0)$ is negative and given. Vardi's question has a neat answer [12]: the supremum over all betting strategies of the probability to reach $f = 1$, in Vardi's casino, although not attained by any strategy, is

$$P(f) = 1 - (1 - f)^{1+c}, 0 \leq f \leq 1.$$

The supremum is achieved as the limit of the probability attained by the strategy S_α as $\alpha \downarrow 0$, where S_α (boldly) stakes f when $f \leq \alpha$ and S_α (timidly) stakes $s = \frac{\alpha}{1-\alpha}(1 - f)$ when $\alpha \leq f < 1$. Note that $s \leq f$ as required. Whenever money is bet, all of it is bet (boldly) on the table with the right odds, r, to carry the fortune to unity in case of a win (similar to the Dubins-Savage case). [12] shows the strategy S_α obtains the winning probability

$$P_\alpha(f) = 1 + (1 - (1 + c)\alpha)^n (c - \frac{(1+c)(1-f)}{(1-\alpha)^n})$$

when $f \in I_n \equiv [(1 - (1 - \alpha)^n, 1 - (1 - \alpha)^{n+1}), n \geq 0$. The limit as $\alpha \downarrow 0$ is then seen to be $P(f) = 1 - (1 - f)^{1+c}$. John Lou [10] showed that the maximum of $V(f)$ over strategies for $\mathcal{R}, \mathcal{S}$ is very flat; having different odds available does not help the gambler very much. That is the value for the Vardi casino for a given c, not too negative, is equal to that of the Dubins-Savage casino for any fixed r to several decimal places. this makes one realize that the usual multiplicity of types of gambles in actual casinos have little value to the serious gambler, at least in terms of maximizing the chance to leave with a preset desired fortune.

In the simplest Dubins casino it is easy to find a formula for the value achieved by bold play. For example, for $r = 1$, if $f \in (0, 1)$ has the binary representation, $f = \sum_{j=1}^{\infty} \frac{1}{2^{n_j}}$, where $1 \leq n_1 < n_2 < \ldots$, then the reward obtained by bold play is:

$$V(f) = \sum_{j=1}^{\infty} (b(1 - \frac{1+c}{2}))^{j-1} (b(\frac{1+c}{2})^{n_j - n_{j-1}}$$

where $n_0 = 0$. One verifies (not at all trivially) that $V(f_n)$ is a supermartingale for any strategy of staking. It then follows from the standard arguments that V is an upper bound and since it is achieved by bold play it is the answer to the problem.

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Lawrence A Shepp 732-445-1143 FAX = 732-445-3428

Rutgers University Statistics Department, Hill Center, Room 580 Busch Campus, Piscataway, NJ 08854-8019

Home page: http://www.stat.rutgers.edu/ shepp e-mail: shepp@stat.rutgers.edu