Fractional Poisson motion and related models

Ingemar Kaj
Uppsala University
ikaj@math.uu.se

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Lévy fractional Brownian motion

Gaussian random field $B_H(t)$, $t \in R^d$, $0 < H < 1$

$$\mathrm{Cov}(B_H(s), B_H(t)) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H})$$

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- Well-balanced representation

$$B_H(t) = C_H \int_{R^d} (|t-x|^{H-d/2} - |x|^{H-d/2})\, M(dx)$$

$M(dx)$ Gaussian measure with control measure dx
$M(A) \sim N(0, |A|)$, $\quad M(A \cap B) = 0$, $A \cap B = \emptyset$

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- Harmonizable representation

$$B_H(t) = C_H \int_{R^d} \frac{e^{-it\cdot x} - 1}{|x|^{d/2+H}} (M_1(dx) + iM_2(dx))$$

Random balls representation

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$(2, H)$-Takenaka field $B_H(t) = M_H(V_t)$, where

$$\begin{aligned} V_t &= \{\text{all spheres separating } 0 \text{ and } t\} \\ &= \{(x,r) : |x| \le r\} \triangle \{(x,r) : |x-t| \le r\} \end{aligned}$$

[Samorodnitsky, Taqqu 1994]

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$$B_H(t) = C_H \int_{R \times R_+} \int_{B(x,r)} 1_{[0,t]}(y)\, dy\, M_H(dx,dr), \quad d = 1$$

$$B_H(t) = C_H \int_{R^d \times R_+} \int_{B(x,r)} \Big(\frac{1}{|y-t|^{d-1}} - \frac{1}{|y|^{d-1}} \Big)\, dy\, M_H(dx,dr)$$

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Viewpoint of gereralized random field

Consider the Gaussian, isotropic, generalized random field W_β, such that for a suitable family $\mathcal{M}_\beta$ of signed measures,

$$W_\beta(\mu) = C_\beta \int_{R^d \times R_+} \mu(B(x,r))\, M_\beta(dx, dr), \quad \mu \in \mathcal{M}_\beta$$

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W_β is self-similar with $H = (d-\beta)/2$ [Dobrushin ∼1975]:

$$W_\beta(\mu_c) \overset{fdd}{=} c^H W_\beta(\mu), \quad \mu_c(A) = \mu(c^{-1}A), \quad A \subset R^d$$

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[IK, Leskelä, Norros, Schmidt, 2007] (LRD, heavy-tailed models)
$d < \beta < 2d$, $-d/2 < H < 0$

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$1/2 < H < 1$?

Extension to any H

For any $H \neq Z$ if $d \geq 2$ and $H \neq \frac{1}{2}Z$ if $d = 1$, Dobrushin (1979) obtains a Gaussian, self-similar, istropic random field B_H such that

$$\mathrm{Cov}(B_H(\phi), B_H(\psi)) = C_H \int_{R^d} \widehat{\phi}(\xi)\overline{\widehat{\psi}}(\xi)|\xi|^{-d-2H}\, d\xi, \quad \phi, \psi \in \mathcal{S}$$

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Theorem (Biermé, Estrade, IK 2010)

Put $m = [H + 1/2]$ and $\beta_H = d - 2(H - m)$. Then $d - 1 < \beta_H < d$ or $d < \beta_H \leq d + 1$, and

$$B_H(\phi) \overset{fdd}{=} W_{\beta_H}((-\Delta)^{-m/2}\phi), \quad \phi \in \mathcal{S}_\infty$$

For $H > -d/2$,

$$\mathrm{Cov}(B_H(\phi), B_H(\psi)) = C_H \int_{R^d \times R^d} |y - y'|^{2H} \phi(y)\psi(y')\, dydy'$$

Two examples

Recall

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$$m = [H + 1/2], \quad \beta_H = d - 2(H - m), \quad B_H(\phi) \overset{fdd}{=} W_{\beta_H}((-\Delta)^{-m/2}\phi)$$

- $0 < H < 1/2$. Then $m = 0$, $\beta_H = d - 2H$. Also, $\mu = \delta_t - \delta_0 \in \mathcal{M}$ is an admissible measure. Thus

$$B_H(t) := W_{\beta_H}(\delta_t - \delta_0) = \int_{R^d \times R_+} (\delta_t - \delta_0)(B(x, r))\, M_{\beta_H}(dx, dr)$$

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- $1/2 < H < 1$. Now $m = 1$ and $\beta_H = d - 2H + 2$. Also, taking $\phi_t(y)\, dy \approx \delta_t(dy) - \delta_0(dy)$,

$$(-\Delta)^{-1/2}(\phi_t) = \left(|t - y|^{-(d-1)} - |y|^{-(d-1)}\right) dy$$

$$B_H(t) := \int_{R^d \times R_+} \int_{B(x,r)} \left(\frac{1}{|t - y|^{d-1}} - \frac{1}{|y|^{d-1}}\right) dy\, M_{\beta_H}(dx, dr)$$

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Replace $M_\beta(dx, dr)$ by compensated Poisson point measure

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As above, for $d - 1 < \beta < d$, $d < \beta < 2d$

$$\mu \to J_\beta(\mu) = C_\beta \int_{R^d \times R_+} \mu(B(x, r))\, \widetilde{N}_\beta(dx, dr)$$

and, for non-integer (non-half-integer) H,

$$P_H(\phi) = J_{\beta_H}((-\Delta)^{-m/2}\phi)$$

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Also,

$$\mathrm{Cov}(P_H(\phi), P_H(\psi)) = \mathrm{Cov}(B_H(\phi), B_H(\psi))$$

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In particular, for $t \in R^d$,

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$$P_H(t) := \int_{R \times R_+} |B(x,r) \cap [0,t]|\, \widetilde{N}_{\beta_H}(dx, dr), \quad d = 1$$

$$P_H(t) := \int_{R^d \times R_+} \int_{B(x,r)} \left(\frac{1}{|t-y|^{d-1}} - \frac{1}{|y|^{d-1}} \right) dy\, \widetilde{N}_{\beta_H}(dx, dr)$$

Fractional Poisson motion, $d = 1$

$1/2 < H < 1$ Put

$$\int f(x)\, P_H(dx) = \int_{R \times R_+} \int_{B(x,r)} f(y)\, dy\, \widetilde{N}_{\beta_H}(dx, dr)$$

Then

$$\log E \exp\left\{\theta \int f(x)\, P_H(dx)\right\} = \sum_{k=2}^{\infty} \frac{\theta^k}{k!} \int_{R^k} \frac{f(y_1)\ldots f(y_k)}{(y_{(k)} - y_{(1)})^{2(1-H)}}\, dy_1 \ldots dy_k$$

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This is (a version of) Mandelbrot's Fractal sum of pulses (1995):

$$M_t = \int_{R\times R_+} (1_{\{0<x-r<t\}} - 1_{\{0<x+r<t\}})\, \widetilde{N}_{\beta_H}(dx, dr),$$

which is shown to be approximated by FBM, see also [Marouby, 2011].
Indeed,

$$\{c^{-H} P_H(ct)\} \Rightarrow \{B_H(t)\}, \quad c \to \infty$$

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Avraram and Glynn (2011) consider Scheduled traffic:

Scheduled arrival times: $j \in Z$

Actual arrival times: $j + U_j$, where $P(U > x) \sim x^{-\beta}$

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Take $H = (1 - \beta)/2 \in (0, 1/2)$ and put

$$Z_t = \#\text{actual arrivals in } [0, t], \quad X_t^{(c)} = \frac{Z_{ct} - [ct]}{c^H}$$

Then

$$\{X^{(c)}(t)\} \Rightarrow \{B_H(t)\}$$

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"Proof"

$$Z_t - t \approx \int_{R \times R_+} (1_{\{0<x+u<t\}} - 1_{\{0<x<t\}})\, \widetilde{N}_{\beta_H}(dx, du) = P_H(t)$$

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