

Regular Spectral Factorizations

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1 Description of the Problem

Problem 1 *Let \mathcal{X} be an invertible $n \times n$ matrix-valued function in the open right half plane, which belongs to H^∞ together with its inverse. Find conditions on the boundary function $\Pi(i\omega) = \mathcal{X}(i\omega)^* \mathcal{X}(i\omega)$ ($\omega \in \mathbf{R}$) which imply that the limit $E = \lim_{\lambda \rightarrow +\infty} \mathcal{X}(\lambda)$ exists, and compute this limit. Here λ tends to infinity along the positive real axis.*

By mapping the right half-plane into the unit disk we obtain the following equivalent formulation of our problem:

Problem 2 *Let \mathcal{X} be an invertible $n \times n$ matrix-valued function in the open unit disk, which belongs to H^∞ together with its inverse. Find conditions on the boundary function $\Pi(z) = \mathcal{X}(z)^* \mathcal{X}(z)$ ($|z| = 1$) which imply that the limit $E = \lim_{\eta \rightarrow 1^-} \mathcal{X}(\eta)$ exists, and compute this limit. Here η tends to 1 from the left along the positive real axis.*

The function \mathcal{X} is called a *spectral factor* of the ‘‘Popov function’’ Π , and this spectral factor is *regular* if the limit E exists. Observe that $\Pi(i\omega) = |\mathcal{X}(i\omega)|^2$ in the important scalar case. There is also an infinite-dimensional version of the same problem, where the values of \mathcal{X} lie in the space of bounded linear operators on a separable Hilbert space U . In this case, we obtain three different versions of the problem, depending on the topology we use to compute the limit E , i.e., as a uniform operator limit, or as a strong limit, or as a weak limit. In this case we would also like to know if E is invertible.

2 History of the Problem and Motivations

This open problem arose fairly recently out of some new results on infinite-dimensional regulator problems and the corresponding Riccati equations for the very general class of infinite-dimensional linear system known as (weakly) *regular well-posed linear systems*, see [5, 8, 9, 10, 14, 15, 17]. The existence of solutions to Riccati equations is the key to solving many control problems and it is important to prove existence for as large a class of systems as possible. The approach used here is to reduce the regulator problem to the associated spectral factorization problem, appeal to known results for the existence of a spectral factor \mathcal{X} , and then to use the beautiful properties of well-posed linear systems to construct a realization of this spectral factor from a given realization of the original system. If the spectral factor is regular, the appropriate algebraic Riccati equations can be derived, and the optimal solution can be constructed. The conclusions are perfect generalizations of the corresponding finite-dimensional conclusions (see [3]), apart from the following gaps:

- Although, under the standard assumptions, the existence of a spectral factor is guaranteed, it need not be regular, i.e., the limit E need not exist. This regularity property is essential in the derivation of the Riccati equation.
- The resulting Riccati equation contains the operator $(E^*E)^{-1}$. Hence we need to compute E^*E , and E^*E must be invertible.

3 Known Results

Different sets of conditions are known that imply that the spectral factor is regular and that the limit E can be computed. They are all in one way or another related to the smoothness of the Popov function Π , which in terms of the original system typically can be written in the form

$$\Pi(i\omega) = R + NG(i\omega) + (NG(i\omega))^* + G(i\omega)^*QG(i\omega),$$

where $G(s) = C(sI - A)^{-1}B$ is the system transfer function and R , Q , and N are various weighting operators. The corresponding Riccati equation Π and the equation for the optimal feedback operator K are in this case

$$\begin{aligned} A^*\Pi + \Pi A + Q &= (B^*\Pi + N)^*(E^*E)^{-1}(B^*\Pi + N), \\ K &= -(E^*E)^{-1}(B^*\Pi + N). \end{aligned}$$

We have (partially) positive answers in the following cases (here we assume for simplicity that A generates an exponentially stable semigroup; see [1, 4, 6, 13, 17] for more details):

- The original system has a bounded control operator B and an admissible observation operator C , e.g., it is of Pritchard-Salamon type;
- The dimension of the output space is finite, and the original system has an admissible control operator B and a bounded observation operator C ;
- A generates an exponentially stable analytic semigroup, and $C(-A)^{-\gamma}B$ is bounded for some $\gamma < 1$;
- The dimension of the input and output spaces are finite, and the impulse response of the original system is an L^1 -function;
- The dimensions of the input and output spaces are finite, and $G(s)$ is a rational function of e^{-Ts} , for some $T > 0$.

Some combinations of these cases are also possible. It is known that the spectral factor need not be regular; for an example see [17]. Some other illustrating examples are given in [7, 16]. What is lacking are verifiable conditions under which the spectral factor is regular. In particular, it is not known to what extent systems modelled as a boundary control problem for the wave equation in several space dimensions have a regular spectral factor.

An abstract formula for the limit E is given in [7, 8] and, as one would expect, in the classical case and all the first 4 special cases itemized above the limit satisfies $E^*E = R$. What we would like to be able to do is to compute it in all cases.

4 Related Results

The same problem appears in a more general setting where one looks for sign-indefinite J -spectral factors of a given boundary function, and derives the corresponding algebraic Riccati equations. These become important in the solution of H^∞ type problems; see [4, 11, 12].

In another generalization we keep the assumption that \mathcal{X} is invertible in the open right half plane and “outer”, but we do not require the inverse to be in H^∞ . This case is related to the so called singular regulator problem where $R = 0$, and clearly we expect E to be zero. Some sufficient conditions for this to be true are given in [2].

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