Passive and Conservative Continuous Time Impedance and Scattering Systems. Part I: Well-Posed Systems

Olof J. Staffans
Åbo Akademi University
Department of Mathematics
FIN-20500 Åbo, Finland
http://www.abo.fi/~staffans/

June 12, 2002

Abstract

Let U be a Hilbert space. By a $\mathcal{L}(U)$-*valued positive analytic function* on the open right half-plane we mean an analytic function which satisfies the condition $\widehat{\mathfrak{D}} + \widehat{\mathfrak{D}}^* \geq 0$. This function need not be *proper*, i.e., it need not be bounded on any right half-plane. We study the question under what conditions such a function can be realized as the transfer function of an *impedance passive system*. By this we mean a continuous time state space system whose control and observation operators are not more unbounded than the (main) semigroup generator of the system, and in addition, there is a certain energy inequality relating the absorbed energy and the internal energy. The system is (impedance) *energy preserving* if this energy inequality is an equality, and it is *conservative* if both the system and its dual are energy preserving. A typical example of an impedance conservative system is a system of hyperbolic type with collocated sensors and actuators. We give several equivalent sets of conditions which characterize when a system is impedance passive, energy preserving, or conservative. We prove that a impedance passive system is well-posed if and only if it is proper. We furthermore show that the so called *diagonal transform* (which is a particular rescaled feedback/feedforward transform) maps a proper *impedance* passive (or energy preserving or conservative) system into a (well-posed) *scattering* passive (or energy preserving or conservative) system. This implies that, just as in the finite-dimensional case, if we apply negative output feedback to a proper impedance passive system, then the resulting system is (energy) stable. Finally, we show that every proper positive analytic function on the right half-plane a (essentially unique) well-posed impedance conservative realization, and it also has a minimal impedance passive realization.

Keywords

Dissipative, energy preserving, proper, collocated sensors and actuators, positive real, Caratheodory-Nevanlinna function, Titchmarsh-Weyl function, bounded real lemma, Kalman-Yakubovich-Popov lemma, diagonal transform.

1 Introduction

Let U be a Hilbert space. By a *$\mathcal{L}(U)$-valued positive analytic function* on $\mathbf{C}^+$ (= the open right half-plane) we mean an analytic function which satisfies the condition $\widehat{\mathfrak{D}} + \widehat{\mathfrak{D}}^* \geq 0$ (many other alternative names are also used for this class of functions, such as (impedance) *passive functions*, *Caratheodory-Nevanlinna functions*, *Weyl functions*, or *Titchmarsh-Weyl functions*; see, e.g., [1] and [3] for more detailed discussions of the history of this class of functions). This function need not be *proper*, i.e., it need not be bounded on any right half-plane. For example, the scalar functions $\widehat{\mathfrak{D}}(s) = 1/s$ and $\widehat{\mathfrak{D}}(s) = 1$ are proper (the former is even *strictly* proper since $\widehat{\mathfrak{D}}(\infty) = 0$), whereas $\widehat{\mathfrak{D}}(s) = s$ is not proper (all of these are positive analytic). In this article we introduce a class of continuous time impedance passive systems whose transfer functions are (not necessarily proper) positive analytic. Our class of systems contains all earlier state space realizations of positive analytic functions that we know of, but it is still not complete in the sense that not every positive analytic function has a realization in our class, one of the main exceptions being the function $\widehat{\mathfrak{D}}(s) = s$ mentioned above. (Actually, as we shall show in [18], all the exceptions are of this type.) For example, systems with collocated sensors and actuators belong to the class studied here.

As is well-known, every $\mathcal{L}(U)$-valued function $\widehat{\mathfrak{D}}$ which is analytic and bounded on some right half-plane (i.e., every proper transfer function) has a *well-posed realization*. By this we mean a well-posed linear system Σ whose transfer function is equal to the given function $\widehat{\mathfrak{D}}$. This system Σ has a state space (a Hilbert space) X, an input signal $u \in L^2_{\text{loc}}(\mathbf{R}^+; U)$, a state trajectory $x \in C(\mathbf{R}^+; X)$, and an output signal $y \in L^2_{\text{loc}}(\mathbf{R}^+; U)$ (here $\mathbf{R}^+ = [0, \infty)$). In the absence of an input signal (i.e., for $u = 0$), the evolution of the state x is described by a strongly continuous semigroup. That the transfer function of Σ is $\widehat{\mathfrak{D}}$ means that if the initial state is zero and if the input u is Laplace transformable, then the output y is also Laplace transformable and, on some right half-plane, the Laplace transform $\hat{y}$ of y is given by $\hat{y} = \widehat{\mathfrak{D}}\hat{u}$; here $\hat{u}$ is the Laplace transform of u. In Section 2 we give the formal definition of a well-posed linear system, and there we also describe the basic properties of such systems.

Not every positive analytic function is proper, so to develop a more general theory we need a class of systems which are not necessarily well-posed. The class of systems that we introduce in Section 2 is maybe not the most general one, but it has some nice properties which makes it possible to develop a meaningful theory for this class. We allow both the control and the observation operator to be as unbounded as the generator of the semigroup describing the autonomous behavior of the system. This is roughly twice as much unboundedness as may be present in a well-posed system.

The physical interpretation of a positive analytic function is that it is *energy absorbing* (in an impedance setting). This class of transfer functions appears in certain situations where the input u and the output y are related to each other in a specific way. For example, we could have a pair of wires connected to an electrical circuit, and let u be the voltage between the wires and y the current carried by the wires (or the other way around). In this and many other similar situations, the energy absorbed by the system in the time period $[0, t]$ is proportional to the integral $2\int_0^t \Re\langle u(s), y(s)\rangle\, ds$. It is well-known that if the initial state is zero (so that the Laplace transforms of the input and output satisfy $\hat{y} = \widehat{\mathfrak{D}}\hat{u}$ in some right-half-plane), then this energy is nonnegative for all possible input signals u if and only if $\widehat{\mathfrak{D}}$ is a positive analytic function.

Let us next explain what we mean by an *impedance passive system*. For simplicity we here stick to the well-posed case. The transfer function of an impedance passive system must be a positive analytic function, but this is not enough. A well-posed system Σ is an impedance passive system if for all initial states $x_0 \in X$, all input signals $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+; U)$, and all $t \geq 0$, the state $x(t)$ at time t and the output signal y satisfy

$$|x(t)|^2 \leq |x_0|^2 + 2\int_0^t \Re\langle u(s), y(s)\rangle\, ds. \tag{1}$$

Here $|x(t)|^2$ represents the *energy stored in the state* at time $t \geq 0$. An impedance passive system has the property that if at some time the state $x(t)$ is zero, then at this time moment the system can only absorb energy and not emit any energy (the time derivative of the absorbed energy function is positive). If a system Σ is impedance passive, then so is the dual system Σ^d (this system is defined in Section 2; its transfer function is $\widehat{\mathfrak{D}}^d(z) = \widehat{\mathfrak{D}}(\overline{z})^*$). A system Σ is *impedance energy preserving* if the preceding inequality holds in the form of an equality:

$$|x(t)|^2 = |x_0|^2 + 2\int_0^t \Re\langle u(s), y(s)\rangle\, ds, \tag{2}$$

and it is *impedance conservative* if both the original system Σ and the dual system Σ^d are impedance energy preserving. In some sense an impedance conservative realization describes a given positive analytic function in an 'optimal' way: all the energy absorbed or emitted by the system is stored in the state or withdrawn from the state, and the same statement is true also for the dual system. (There is no guarantee that all of the state energy can ever be withdrawn, as some of it may be trapped in the state forever.)

We begin in Section 2 with a presentation of the class of systems that we use to realize positive analytic functions. In the same section we define what we mean by a *well-posed system*. We continue in Section 3 by recalling the notions of *scattering* passive, energy preserving, and conservative systems, as presented in, e.g., [9], [21], and [28]. (The same classes of systems appear in [2] in a different notation.) These classes of systems are closely related to the corresponding classes of impedance systems introduced above. The only difference is that the expression for the absorbed energy is replaced by $\int_0^t |u(s)|^2\, ds - \int_0^t |y(s)|^2\, ds$, so that (1) becomes

$$|x(t)|^2 + \int_0^t |y(s)|^2\, ds \le |x_0|^2 + \int_0^t |u(s)|^2\, ds, \tag{3}$$

and (2) becomes

$$|x(t)|^2 + \int_0^t |y(s)|^2\, ds = |x_0|^2 + \int_0^t |u(s)|^2\, ds. \tag{4}$$

These systems are always well-posed, and they play an important role in our study of impedance passive, energy preserving, and conservative systems.

In Section 4 we are finally ready to give formal definitions of impedance passive, energy preserving, and conservative systems. We also give a number of equivalent conditions for a system to have one of these properties. For example, if the system is described by a (possibly infinite-dimensional) system of differential equations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{5}$$

where $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(U;X)$, $C \in \mathcal{L}(X;U)$, and $D \in \mathcal{L}(U)$, then one of our conditions (see formula (28)) says that this system is impedance passive if and only if

$$\begin{bmatrix} A + A^* & B \\ B^* & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & C^* \\ C & D + D^* \end{bmatrix}. \tag{6}$$

It is impedance energy preserving if and only if this inequality holds as an equality, and it is impedance conservative if furthermore the corresponding dual identity holds.

There is a simple transform, sometimes called the *diagonal transform*, which maps an impedance passive (or energy preserving or conservative) system into a scattering passive (or energy preserving or conservative) system. This transform is well-known in the finite-dimensional state space case, and also in a very general input/output setting (see [33, Section 8.15]) (it maps a positive analytic function into a contractive analytic function). In Section 5 we show that the same transform works in the infinite-dimensional state space setting as well if we apply it to a well-posed impedance passive system. In the same section we prove the following basic result: an impedance passive system is well-posed if and only if the transfer function of the system is bounded on some vertical line in the right half-plane. Furthermore, we show that every proper positive analytic function on $\mathbf{C}^+$ has a well-posed impedance conservative realization (which is essentially unique under a suitable minimality requirement), and it also has a minimal well-posed impedance passive realization. In the exponentially stable finite-dimensional case the last statement is a consequence of the impedance version of the *Kalman-Yakubovich-Popov lemma*, also known as the *positive (real) lemma*. According to that lemma, a matrix-valued proper rational transfer function $\widehat{\mathfrak{D}}$ with an exponentially stable minimal realization of the type (5) (with finite-dimensional X and U) is positive if and only if there exist matrices $P > 0$, Q, and W such that

$$\begin{bmatrix} PA + A^*P & PB \\ B^*P & 0 \end{bmatrix} = \begin{bmatrix} 0 & C^* \\ C & D + D^* \end{bmatrix} - \begin{bmatrix} Q^* \\ W^* \end{bmatrix} \begin{bmatrix} Q & W \end{bmatrix}; \tag{7}$$

see, e.g., [34, Theorems 13.25 and 13.26]. This identity has a simple energy interpretation: if we add another output $z(t) = Qx(t) + Wu(t)$ to the system in (5), then the solution x of (5) satisfies the energy balance equation

$$\langle x(t), Px(t)\rangle + \int_0^t |z(s)|^2\, ds = \langle x_0, Px_0\rangle + 2\int_0^t \Re\langle u(s), y(s)\rangle\, ds. \qquad (8)$$

If we replace the norm in the state space by the new norm $|x|_P = \sqrt{\langle x, Px\rangle}$, then the above identity becomes

$$|x(t)|_P^2 + \int_0^t |z(s)|^2\, ds = |x_0|_P^2 + 2\int_0^t \Re\langle u(s), y(s)\rangle\, ds, \qquad (9)$$

and this shows that, with this norm and with the added output z, the system (5) can be regarded as an mixed impedance/scattering energy preserving system. (The operator P disappears from (7) when we compute the adjoints with respect to the inner product $[x_1, x_2] = \langle x_1, Px_2\rangle$ induced by the new norm.) Dropping the extra output z we get a minimal impedance passive realization of $\widehat{\mathfrak{D}}$. See [32, Sections 5–7] for more details.

In our final Section 6 we give a feedback interpretation of the diagonal transform: it says that if we apply negative feedback to a proper impedance passive system, then the resulting closed-loop system is energy stable.

Many of the results presented above are also true for impedance passive systems which are not proper, hence not well-posed. In particular, it is still true for these non-well-posed systems that the diagonal transform is well-defined, and that it maps an impedance passive (or energy preserving or conservative) system into a (well-posed) scattering passive (or energy preserving or conservative) system. It is also true that a very large class of non-proper positive analytic functions on $\mathbf{C}^+$ (those that do not contain a pure differentiating action) have realizations in the class of impedance passive systems that we introduce here. We shall return to this in [18].

2 Infinite-Dimensional Linear Systems

Many infinite-dimensional linear time-invariant continuous-time systems can be described by the equations (5) on a triple of Hilbert spaces, namely, the input space U, the state space X, and the output space Y. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operator A is supposed to be the generator of a strongly continuous semigroup $t \mapsto \mathfrak{A}^t$. The generating operators A, B and C are usually unbounded, but D is bounded.

By modifying this set of equations slightly we get the class of systems which will be used in this work. In the sequel, we think about the block matrix $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as *one single (unbounded) operator* from $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, and write (5) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0. \tag{10}$$

The operator S completely determines the system. Thus, we may identify the system with such an operator, which we call the *node* of the system.

There are certain conditions that we need to impose on S in order to get a meaningful theory. First of all, S must be closed and densely defined as an operator from $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ into $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$. Let us denote the domain of S by $\mathcal{D}(S)$. Then S can be split into $S = \left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right]$, where S_1 maps $\mathcal{D}(S)$ into X and S_2 maps $\mathcal{D}(S)$ into Y. By analogy to the finite-dimensional case, let us denote $A\&B := S_1$ and $C\&D := S_2$, so that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ (the reader who finds this notation confusing may throughout replace $A\&B$ by S_1 and $C\&D$ by S_2). It is not true, in general, that $A\&B$ and $C\&D$ (defined on $\mathcal{D}(S)$) can be decomposed into $A\&B = \begin{bmatrix} A & B \end{bmatrix}$ and $C\&D = \begin{bmatrix} C & D \end{bmatrix}$; this is possible only in the case where $\mathcal{D}(S)$ can be written as the product of one subspace of X times another subspace of U. However, we shall require that an extended version of $A\&B$ can be decomposed as indicated above, so that *$A\&B$ is the restriction to $\mathcal{D}(S)$ of $\begin{bmatrix} A & B \end{bmatrix}$* for suitably defined operators A and B.

The decomposition of $A\&B$ is based on the familiar 'rigged Hilbert space structure' (sometimes referred to as a 'Gelfand triple').[1] Let A be a closed (unbounded) densely defined operator on the Hilbert space X with a nonempty resolvent set. We denote its domain $\mathcal{D}(A)$ by X_1. This is a Hilbert space with the norm $|x|_{X_1} := |(\alpha - A)x|_X$, where α is an arbitrary number in $\alpha \in \rho(A)$ (different numbers α give different but equivalent norms). We also construct a larger Hilbert space X_{-1}, which is the completion of X under the norm $|x|_{X_{-1}} := |(\alpha - A)^{-1}x|_X$. Then $X_1 \subset X \subset X_{-1}$ with continuous and dense injections. The operator A has a unique extension to an operator in $\mathcal{L}(X; X_{-1})$ which we denote by $A_{|X}$ (thereby indicating that the domain of this operator is all of X). The operators A and $A_{|X}$ are similar to each other and they have the same spectrum. Thus, for all $\alpha \in \rho(A)$, the operator $\alpha - A_{|X}$ maps X one-to-one onto X_{-1}. Its inverse $(\alpha - A_{|X})^{-1}$ is the unique extension to X_{-1} of the operator $(\alpha - A)^{-1}$.

We shall also need the dual versions of the spaces X_1 and X_{-1}. If we repeat the construction described above with A replaced by the (unbounded) adjoint A^* of A, then we get two more spaces, that we denote by X_1^d (the

[1] See, e.g., [9] or [20] or almost any other of the papers listed in the reference list for details.

analogue of X_1) and X^d_{-1} (the analogue of X_{-1}). Then $X^d_1 \subset X \subset X^d_{-1}$ with continuous and dense injections. If we identify the dual of X with X itself, then X^d_1 becomes the dual of X_{-1} and X^d_{-1} becomes the dual of X_1.[2] We denote the extension of A^* to an operator in $\mathcal{L}(X; X^d_{-1})$ by $A^*_{|X}$. This operator can be interpreted as the (bounded) adjoint of the operator A, regarded as an operator in $\mathcal{L}(X_1; X)$.

Definition 2.1. We call S a *system node* on the three Hilbert spaces (U, X, Y) if it satisfies condition (S) below:[3]

(S) $S := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \supset \mathcal{D}(S) \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ is a closed linear operator. Here $A\&B$ is the restriction to $\mathcal{D}(S)$ of $\left[A_{|X} \quad B\right]$, where A is the *generator of a C_0 semigroup* on X (the notations $A_{|X} \in \mathcal{L}(X; X_{-1})$ and X_{-1} were introduced in the text above). The operator B is an arbitrary operator in $\mathcal{L}(U; X_{-1})$, and $C\&D$ is an arbitrary linear operator from $\mathcal{D}(S)$ to Y. In addition, we require that

$$\mathcal{D}(S) = \left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \;\middle|\; A_{|X}x + Bu \in X\right\}.$$

It follows from the above definition that $A\&B \colon \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \supset \mathcal{D}(A\&B) \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, with $\mathcal{D}(A\&B) = \mathcal{D}(S)$, is a closed operator. Thus, $\mathcal{D}(S)$ becomes a Hilbert space with the graph norm of the operator $A\&B$. Furthermore, it is not difficult to show that the assumption that S is closed is equivalent to the assumption that *$C\&D$ is continuous from $\mathcal{D}(S)$ (with the graph norm of $A\&B$) to Y.*

We shall use the following names of the different parts of the system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. The operator A is the *main operator* or the *semigroup generator*, B is the *control operator*, $C\&D$ is the *combined observation/feedthrough operator*, and the operator C defined by

$$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in X_1,$$

[2] Often X_{-1} is *defined* to be the dual of X^d_1 when we identify the dual of X with X itself.

[3] This definition is equivalent to the corresponding definition used by Smuljan in [13] in 1986. Unfortunately, that paper (written in Russian) has not been properly known and recognized in the English literature, and many of its results have been (independently) rediscovered, among others by this author. The main part of [13] is devoted to system nodes which are well-posed (see our Definition 2.6). System nodes appear also in the work by Salamon [11, 12] in a less implicit way, again primarily in the well-posed case. Our notation $C\&D\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ corresponds to Smuljan's notation $N\langle x, u\rangle$ and Salamon's notation $\big(x - (\alpha - A)^{-1}Bu\big) + \widehat{\mathfrak{D}}(\alpha)u$. Compare this to formula (13) below.

is the *observation operator* of S.

An easy algebraic computation (see, e.g., [20, Section 4.7] for details) shows that for each $\alpha \in \rho(A) = \rho(A_{|X})$, the operator $\begin{bmatrix} 1 & (\alpha - A_{|X})^{-1}B \\ 0 & 1 \end{bmatrix}$ is an boundedly invertible mapping between $\begin{bmatrix} X \\ U \end{bmatrix} \to \begin{bmatrix} X \\ U \end{bmatrix}$ and $\begin{bmatrix} X_1 \\ U \end{bmatrix} \to \mathcal{D}(S)$. Since $\begin{bmatrix} X_1 \\ U \end{bmatrix}$ is dense in $\begin{bmatrix} X \\ U \end{bmatrix}$, this implies that $\mathcal{D}(S)$ is dense in $\begin{bmatrix} X \\ U \end{bmatrix}$. Furthermore, since the second column $\begin{bmatrix} (\alpha - A_{|X})^{-1}B \\ 1 \end{bmatrix}$ of this operator maps U into $\mathcal{D}(S)$, we can define the *transfer function* of S by

$$\widehat{\mathfrak{D}}(s) := C\&D \begin{bmatrix} (s - A_{|X})^{-1}B \\ 1 \end{bmatrix}, \qquad s \in \rho(A), \tag{11}$$

which is a $\mathcal{L}(U;Y)$-valued analytic function on $\rho(A)$. By the resolvent formula, for any two α, $\beta \in \rho(A)$,

$$\begin{aligned} \widehat{\mathfrak{D}}(\alpha) - \widehat{\mathfrak{D}}(\beta) &= C\big[(\alpha - A_{|X})^{-1} - (\beta - A_{|X})^{-1}\big]B \\ &= (\beta - \alpha)C(\alpha - A)^{-1}(\beta - A_{|X})^{-1}B. \end{aligned} \tag{12}$$

It is possible to alternatively define a system node by specifying the main operator A, the control operator B, the observation operator C, and the transfer function $\widehat{\mathfrak{D}}$ evaluated at some point $\alpha \in \rho(A)$.

Lemma 2.2. *Let A be the generator of a C_0 semigroup on a Hilbert space X, and let X_1, X_{-1} and $A_{|X}$ be the spaces and the operator induced by A, as explained in the text preceding Definition 2.1. Let $B \in \mathcal{L}(U;X_{-1})$, let $C \in \mathcal{L}(X_1;Y)$, and let $D \in \mathcal{L}(U;Y)$, where U and Y are two more Hilbert spaces. Let $A\&B$ be the restriction of $\begin{bmatrix} A_{|X} & B \end{bmatrix}$ to $\mathcal{D}(A\&B) = \big\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} X \\ U \end{bmatrix} \mid A_{|X}x + Bu \in X \big\}$. Finally, let $\alpha \in \rho(A)$, and define*

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C\big(x - (\alpha - A_{|X})^{-1}Bu\big) + Du, \qquad \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(A\&B).$$

Then $S := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \mathcal{D}(S) := \mathcal{D}(A\&B) \to \begin{bmatrix} X \\ Y \end{bmatrix}$ is a system node on (U, X, Y). The control operator of this system node is B, the observation operator is C, and the transfer function satisfies $\widehat{\mathfrak{D}}(\alpha) = D$.

Proof. Most of this is obvious. The only thing which needs to be checked is that the operator $C\&D$ defined above is continuous from $\mathcal{D}(S) = \mathcal{D}(A\&B)$ (with the graph norm of $A\&B$) to Y. However, this follows from the fact that

$$x - (\alpha - A_{|X})^{-1}Bu = (\alpha - A_{|X})^{-1}\big(\alpha x - (A_{|X}x + Bu)\big). \qquad \square$$

Thus, if we replace D by $\widehat{\mathfrak{D}}(\alpha)$ above, then we have written $C\&D$ in terms of A, B, C, and $\widehat{\mathfrak{D}}(\alpha)$:

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = \bigl(x - (\alpha - A_{|X})^{-1} Bu\bigr) + \widehat{\mathfrak{D}}(\alpha)u. \tag{13}$$

In particular, the right-hand side does not depend on how we choose $\alpha \in \rho(A)$.

As shown in [13, Theorem 1.2] (and also in [2] and [9]), if S is a system node on (U, X, Y), then the (unbounded) adjoint S^* of S is a system node on (Y, X, U). We shall refer to this system node as the *dual system node*, and we sometimes denote it by S^d. If we let A be the main operator of S, and let $B \in \mathcal{L}(U; X_{-1})$ and $C \in \mathcal{L}(X_1; Y)$ be the control and observation operators of S, then the main operator of S^d is $A^d = A^*$ (by this we mean the unbounded adjoint of A; see the paragraph before Definition 2.1), the control operator of S^* is $B^d = C^* \in \mathcal{L}(Y; X^d_{-1})$, and the observation operator is $C^d = B^* \in \mathcal{L}(X^d_1; U)$. Furthermore, if $\widehat{\mathfrak{D}}$ is the transfer function of S, then the transfer function $\widehat{\mathfrak{D}}^d$ of S^d is given by $\widehat{\mathfrak{D}}^d(s) = \widehat{\mathfrak{D}}(\overline{s})^*$ for $s \in \rho(A^*)$.

Every system node induces a 'dynamical system' of a certain type:

Lemma 2.3. *Let S be a system node on (U, X, Y). Then, for each $x_0 \in X$ and $u \in W^{2,1}_{\mathrm{loc}}(\mathbf{R}^+; U)$ with $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$, the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0, \tag{14}$$

has a unique solution (x, y) satisfying $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{D}(S)$ for all $t \geq 0$, $x \in C^1(\mathbf{R}^+; X)$, and $y \in C(\mathbf{R}^+; Y)$.

This lemma is proved in [9] (and also in [20]).[4]

By taking *Laplace transforms* in (14) we find that if u is Laplace transformable with transform $\hat{u}$, then the output y is also Laplace transformable with transform

$$\begin{aligned} \hat{x}(s) &= (s - A)^{-1} x_0 + (s - A_{|X})^{-1} B \hat{u}(s), \\ \hat{y}(s) &= C(s - A)^{-1} x_0 + \widehat{\mathfrak{D}}(s) \hat{u}(s), \end{aligned} \tag{15}$$

for $\Re s$ large enough. Thus, our definition of the transfer function is equivalent to the standard definition in the classical case.

[4] Well-posed versions of this lemma (see Definition 2.6) are (implicitly) found in [11] and [13] (and also in [21]). In the well-posed case we need less smoothness of u: it suffices to take $u \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^+; U)$. In addition y will be smoother: $y \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^+; Y)$.

Definition 2.4. By the *linear system* Σ generated by a system node S we understand the family Σ of maps defined by

$$\Sigma_0^t \begin{bmatrix} x_0 \\ \pi_{[0,t]}u \end{bmatrix} := \begin{bmatrix} x(t) \\ \pi_{[0,t]}y \end{bmatrix},$$

parametrized by $t \geq 0$, where x_0, $x(t)$, u, and y are as in Lemma 2.3 and $\pi_{[0,t]}u$ and $\pi_{[0,t]}y$ are the restrictions of u and y to $[0,t]$. We call x the *state trajectory* and y the *output function* of Σ with initial state x_0 and input function u.

In one of our proofs we shall use a technique which we refer to as 'exponential shifting:'

Lemma 2.5. *If $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a system node on (U, X, Y), then so is $S_\alpha = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] - \left[\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for every $\alpha \in \mathbf{C}$. The domains of these two nodes are the same: $\mathcal{D}(S_\alpha) = \mathcal{D}(S)$. If x is the state trajectory and y is the output function of the system Σ generated by S with initial state x_0 and input function u (as described in Lemma 2.3), then the functions $x_\alpha(t) = \mathrm{e}^{-\alpha t}x(t)$ and $y_\alpha(t) = \mathrm{e}^{-\alpha t}y(t)$ are the state trajectory and output function of the system Σ_α generated by S_α with initial state x_0 and input function $u_\alpha(t) = \mathrm{e}^{-\alpha t}u(t)$.*

We leave the easy proof to the reader. The same transform is also applicable to the more general (distribution) solutions which will be defined in a moment. Observe that by choosing $\Re\alpha$ large enough we can make the semigroup of the system Σ_α exponentially stable. (Therefore, in many cases we may assume without loss of generality that the system has an exponentially stable semigroup.)

So far we have *defined Σ_0^t only for the class of smooth data* given in Lemma 2.3. It is possible to extend this definition by allowing the state to take values in the larger space X_{-1} instead of in X, and by allowing y to be a distribution.

Let us first take a look at the state, which is supposed to be a solution of the equation $\dot{x}(t) = A_{|X}x(t) + Bu(t)$ for $t \geq 0$, with initial value $x(0) = x_0$. However, since $B \in \mathcal{L}(U; X_{-1})$, if $x_0 \in X$ and if $u \in L^1_{\mathrm{loc}}(\mathbf{R}^+; U)$, then this equation has a unique strong solution $x \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^+; X_{-1})$ (see, e.g., [20, Section 3.8]; the operator $A_{|X}$ is the generator of the C_0 semigroup that we get by extending the semigroup generated by A to X_{-1}). Thus, the notion of the state trajectory causes no problem if we are willing to accept a trajectory with values in X_{-1}.

To get a generalized definition of the output y under the same premises we can do as follows (see [20, Section 4.7] for details). Let $x_0 \in X$, $u \in$

$L^1_{\rm loc}(\mathbf{R}^+;U)$, and let $x \in W^{1,1}_{\rm loc}(\mathbf{R}^+;X_{-1})$ be the corresponding state trajectory. Define $\left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right]$ by

$$\begin{bmatrix} x_2(t) \\ u_2(t) \end{bmatrix} = \int_0^t (t-s) \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \, ds, \qquad t \geq 0$$

(this is the second order integral of $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$). Then $\begin{bmatrix} x_2(t) \\ u_2(t) \end{bmatrix} \in \mathcal{D}(S)$ for all $t \geq 0$, and we may define the output y by

$$y(t) = \left(C\&D \begin{bmatrix} x_2(s) \\ u_2(s) \end{bmatrix} \right)'', \qquad t \geq 0, \tag{16}$$

where we interpret the second order derivative in the distribution sense.[5]

Another possibility to extend Σ_0^t to a larger class of data is based on an additional *well-posedness assumption.*

Definition 2.6. A system node S is *well-posed* if, for some $t > 0$, there is a finite constant $K(t)$ such that the solution (x, y) in Lemma 2.3 satisfies

$$|x(t)|^2 + \|y\|^2_{L^2(0,t)} \leq K(t) \big(|x_0|^2 + \|u\|^2_{L^2(0,t)} \big). \tag{\textbf{WP}}$$

It is *energy stable* if there is some $K < \infty$ so that, for all $t \in \mathbf{R}^+$, the solution (x, y) in Lemma 2.3 satisfies

$$|x(t)|^2 + \|y\|^2_{L^2(0,t)} \leq K \big(|x_0|^2 + \|u\|^2_{L^2(0,t)} \big). \tag{\textbf{ES}}$$

It is not difficult to show that if (WP) holds for *one* $t > 0$, then it holds for *all* $t \geq 0$.

If a system node S is well-posed, then the corresponding system Σ can be *extended by continuity* to a family of operators

$$\Sigma_0^t := \left[\begin{array}{c|c} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \hline \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{array} \right]$$

from $\begin{bmatrix} X \\ L^2([0,t];U) \end{bmatrix}$ to $\begin{bmatrix} X \\ L^2([0,t];Y) \end{bmatrix}$. (We still denote the extended family by Σ.)

For more details, explanations and examples we refer the reader to [1], [2], [4], [5, 6] [7], [10], [11, 12], [13], [14, 15, 16, 17, 20], [21, 22], [23, 24, 25, 26, 27], [28], [29], and [30] (and the references therein).

[5] In the well-posed case, if $u \in L^2_{\rm loc}(\mathbf{R}^+;U)$, then it suffices to integrate $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ once, then apply $C\&D$, and finally differentiate once in the distribution sense.

3 Scattering Passive and Conservative Systems

The following definition is a slightly modified version of the definitions in the two classical papers [31, 32] by Willems (although we use a slightly different terminology: our *passive* is the *same as Willems' dissipative*).[6]

Definition 3.1. Let J be a bounded self-adjoint operator on $\left[\begin{smallmatrix} Y \\ U \end{smallmatrix}\right]$. A system node S on (U, X, Y) is *J-passive* if, for all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies

$$|x(t)|^2 - |x_0|^2 \leq \int_0^t \left\langle \left[\begin{smallmatrix} y(s) \\ u(s) \end{smallmatrix}\right], J \left[\begin{smallmatrix} y(s) \\ u(s) \end{smallmatrix}\right] \right\rangle ds. \qquad \textbf{(JP)}$$

It is *J-energy preserving* if the above inequality holds in the form of an equality: for all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies

$$|x(t)|^2 - |x_0|^2 = \int_0^t \left\langle \left[\begin{smallmatrix} y(s) \\ u(s) \end{smallmatrix}\right], J \left[\begin{smallmatrix} y(s) \\ u(s) \end{smallmatrix}\right] \right\rangle ds. \qquad \textbf{(JE)}$$

Physically, passivity means that *there are no internal energy sources.* An energy preserving system has neither any internal energy sources nor any sinks.

Different choices of J give different passivity notions. The case $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ is known as *scattering*. The case where $U = Y = \left[\begin{smallmatrix} V \\ V \end{smallmatrix}\right]$ and $J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is known as *impedance* (*admittance*, *immittance*, *resistance*, *conductance*). The case where $U = Y = \left[\begin{smallmatrix} V \\ W \end{smallmatrix}\right]$, and $J = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$ is known as *transmission* (*chain scattering*). In this article we focus on the *scattering* ($J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$) and *impedance* ($J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$) settings.

Definition 3.2. A system node S is *scattering passive*[7] if, for all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies

$$|x(t)|^2 - |x_0|^2 \leq \|u\|^2_{L^2(0,t)} - \|y\|^2_{L^2(0,t)}. \qquad \textbf{(SP)}$$

It is *scattering energy preserving* if the above inequality holds in the form of an equality: for all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies

$$|x(t)|^2 - |x_0|^2 = \|u\|^2_{L^2(0,t)} - \|y\|^2_{L^2(0,t)}. \qquad \textbf{(SE)}$$

[6] Another difference is that we have replaced Willems' more general *storage function* $S(x)$ by the quadratic function $|x|^2_X$. Our setting becomes the scattering version of the setting which Willems uses in the second part [32] if we simply take the norm in the state space to be $|x|^2 = \sqrt{S(x)}$ (this is possible whenever the storage function is quadratic and strictly positive).

[7] In [9], [28], [21, 22], [29], etc., these systems are called *dissipative.*

Finally, it is *scattering conservative* if both S and S^* are scattering energy preserving.

Thus, *every scattering passive system is well-posed*: the passivity inequality **(SP)** implies the well-posedness inequality **(WP)**.

A scattering passive system can be characterized in several different ways:

Theorem 3.3. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, Y). Then the following conditions are equivalent:*

(i) *Σ is scattering passive.*

(ii) *For all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 \leq |u(t)|_U^2 - |y(t)|_Y^2. \tag{17}$$

(iii) *For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$2\Re\left\langle A\&B\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], x_0\right\rangle_X \leq |u_0|_U^2 - \left|C\&D\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\right|_Y^2. \tag{18}$$

(iv) *For some (or equivalently, for all) $\alpha \in \rho(A)$ we have*[8]

$$\begin{bmatrix} A + A^*_{|X} & (\alpha + A^*_{|X})(\alpha - A_{|X})^{-1}B \\ B^*(\overline{\alpha} - A^*)^{-1}(\overline{\alpha} + A) & B^*(\overline{\alpha} - A^*)^{-1}(2\Re\alpha)(\alpha - A_{|X})^{-1}B \end{bmatrix} \\ + \begin{bmatrix} C^*C & C^*\widehat{\mathfrak{D}}(\alpha) \\ \widehat{\mathfrak{D}}(\alpha)^*C & \widehat{\mathfrak{D}}(\alpha)^*\widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \tag{19}$$

which is an operator inequality in $\mathcal{L}\left(\left[\begin{smallmatrix} X_1 \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} X_{-1}^d \\ U \end{smallmatrix}\right]\right)$.

(v) *For some $\alpha \in \rho(A) \cap \mathbf{C}^+$ (or equivalently, for all $\alpha \in \mathbf{C}^+$), the operator*

$$\begin{bmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} = \begin{bmatrix} (\overline{\alpha} + A)(\alpha - A)^{-1} & \sqrt{2\Re\alpha}\,(\alpha - A)^{-1}B \\ \sqrt{2\Re\alpha}\,C(\alpha - A)^{-1} & \widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \tag{20}$$

is a contraction. (Here $\mathbf{C}^+$ is the open right half-plane.)

This is [21, Theorem 7.4]. The main part of this theorem is also found in [2] (see, in particular, Definition 4.1, Proposition 4.1, Subsection 4.5, and Theorem 5.2 of [2]).

A similar result is valid for scattering energy preserving systems:

[8] See the paragraph before Definition 2.1 for the definition of $A^*_{|X}$ and X^d_{-1}.

Theorem 3.4. *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a system node on* (U, X, Y)*. Then the following conditions are equivalent:*

(i) Σ *is scattering energy preserving.*

(ii) *For all* $t > 0$*, the solution* (x, y) *in Lemma 2.3 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 = |u(t)|_U^2 - |y(t)|_Y^2. \tag{21}$$

(iii) *For all* $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,

$$2\Re\left\langle A\&B \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], x_0 \right\rangle_X = |u_0|_U^2 - \left|C\&D \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\right|_Y^2. \tag{22}$$

(iv) *For some (or equivalently, for all)* $\alpha \in \rho(A)$ *we have*

$$\begin{aligned} &\begin{bmatrix} A + A^*_{|X} & (\alpha + A^*_{|X})(\alpha - A_{|X})^{-1}B \\ B^*(\overline{\alpha} - A^*)^{-1}(\overline{\alpha} + A) & B^*(\overline{\alpha} - A^*)^{-1}(2\Re\alpha)(\alpha - A_{|X})^{-1}B \end{bmatrix} \\ &\qquad + \begin{bmatrix} C^*C & C^*\widehat{\mathfrak{D}}(\alpha) \\ \widehat{\mathfrak{D}}(\alpha)^*C & \widehat{\mathfrak{D}}(\alpha)^*\widehat{\mathfrak{D}}(\alpha) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \end{aligned} \tag{23}$$

which is an operator identity in $\mathcal{L}\left(\left[\begin{smallmatrix} X_1 \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} X_{-1}^d \\ U \end{smallmatrix}\right]\right)$.

(v) *For some* $\alpha \in \rho(A) \cap \mathbf{C}^+$ *(or equivalently, for all* $\alpha \in \mathbf{C}^+$*), the operator* $\left[\begin{smallmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{smallmatrix}\right]$ *defined in* (20) *is isometric.*

This theorem is proved in [9]. Most of this theorem is also found in [2].

By applying Theorem 3.4 both to the original system node S and to the dual system node S^* we get a set of systems which characterize *scattering conservative system nodes*. Some equivalent but *simpler conditions* are given in [9].

A finite-dimensional system is *scattering conservative if and only if it is energy preserving and the input and output spaces have the same dimension.* Some related (but more complicated) results are true also in infinite-dimensions. See [2], [9], and [21, 22] for details.

4 Impedance Passive and Conservative Systems

As we mentioned above, we get into the impedance setting by taking $J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ in Definition 3.1.

Definition 4.1. A system node S on (U, X, U) (note that $Y = U$) is *impedance passive* if, for all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies

$$|x(t)|_X^2 - |x_0|_X^2 \le 2 \int_0^t \Re\langle y(t), u(t)\rangle_U dt. \tag{IP}$$

It is *impedance energy preserving* if the above inequality holds in the form of an equality: for all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies

$$|x(t)|_X^2 - |x_0|_X^2 = 2 \int_0^t \Re\langle y(t), u(t)\rangle_U dt. \tag{IE}$$

Finally, S is *impedance conservative* if both S and the dual system node S^* are impedance energy preserving.

Note that in this case *well-posedness is neither guaranteed, nor always relevant.*

Theorem 4.2. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, U). Then the following conditions are equivalent:*

(i) *S is impedance passive.*

(ii) *For all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} |x(t)|_X^2 \le 2\Re\langle y(t), u(t)\rangle_U. \tag{24}$$

(iii) *For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$\Re\big\langle A\&B \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], x_0 \big\rangle_X \le \Re\langle C\&D \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], u_0\rangle_U. \tag{25}$$

(iv) *For some (or equivalently, for all) $\alpha \in \rho(A)$ we have*

$$\begin{aligned} &\begin{bmatrix} A + A^*_{|X} & (\alpha + A^*_{|X})(\alpha - A_{|X})^{-1}B \\ B^*(\overline{\alpha} - A^*)^{-1}(\overline{\alpha} + A) & B^*(\overline{\alpha} - A^*)^{-1}(2\Re\alpha)(\alpha - A_{|X})^{-1}B \end{bmatrix} \\ &\qquad \le \begin{bmatrix} 0 & C^* \\ C & \widehat{\mathfrak{D}}(\alpha)^* + \widehat{\mathfrak{D}}(\alpha) \end{bmatrix}, \end{aligned} \tag{26}$$

which is an operator inequality in $\mathcal{L}\Big(\left[\begin{smallmatrix} X_1 \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} X^d_{-1} \\ U \end{smallmatrix}\right]\Big)$.

(v) *For some* $\alpha \in \rho(A) \cap \mathbf{C}^+$ *(or equivalently, for all* $\alpha \in \mathbf{C}^+$*), the operator* $\left[\begin{smallmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{smallmatrix}\right]$ *defined in* (20) *satisfies*

$$\begin{bmatrix} \mathbf{A}(\alpha)^*\mathbf{A}(\alpha) & \mathbf{A}(\alpha)^*\mathbf{B}(\alpha) \\ \mathbf{B}(\alpha)^*\mathbf{A}(\alpha) & \mathbf{B}(\alpha)^*\mathbf{B}(\alpha) \end{bmatrix} \le \begin{bmatrix} 1 & \mathbf{C}(\alpha)^* \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(\alpha)^* \end{bmatrix}. \tag{27}$$

(vi) *The system node* $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is a dissipative operator in* $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$*, i.e., for all* $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$*,*

$$\Re \left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \le 0. \tag{28}$$

Note that if $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a system node, then so is $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$, and that the domains of these two nodes are the same (it depends only on $A\&B$, and not on $C\&D$).

Proof. The proof of parts (i)–(v) of this theorem is (almost) identical to the proof of Theorem 3.3 (= [21, Theorem 7.4]), so we leave it to the reader. Clearly, (vi) is just another way of writing condition (iii). □

As the following lemma shows, condition (vi) given above is actually true in a stronger sense.

Lemma 4.3. *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be an impedance passive system node on* (U, X, U)*. Then* $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right] - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *has a bounded inverse for all* α*,* $\beta \in \mathbf{C}^+$*. In particular,* $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is maximal dissipative.*

Proof. It follows from (iii) that the operator A is dissipative, and by the basic assumption on the system node S, A generates a C_0 semigroup. This implies that the semigroup generated by A is a *contraction* semigroup, and hence A is maximal dissipative (i.e., every $\alpha \in \mathbf{C}^+$ belongs to the resolvent set of A). As discussed at length in [9], for all $\alpha \in \rho(A)$, the operator $\left[\begin{smallmatrix} (\alpha - A)^{-1} & (\alpha - A_{|X})^{-1}B \\ 0 & 1 \end{smallmatrix}\right]$ maps $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ one-to-one onto $\mathcal{D}(S)$ (and both this operator and it inverse are continuous). A short algebraic computation shows that, for all $\alpha \in \rho(A)$ and $\beta \in \mathbf{C}$,

$$\begin{aligned} &\left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \right) \begin{bmatrix} (\alpha - A)^{-1} & (\alpha - A_{|X})^{-1}B \\ 0 & 1 \end{bmatrix} \\ &\quad = \begin{bmatrix} 1 & 0 \\ C(\alpha - A)^{-1} & \beta + \widehat{\mathfrak{D}}(\alpha) \end{bmatrix}. \end{aligned}$$

By part (iv) of Theorem 4.2, the operator $-\widehat{\mathfrak{D}}(\alpha)$ is dissipative. It is also bounded, hence it is maximal dissipative, i.e., $\beta + \widehat{\mathfrak{D}}(\alpha)$ is invertible for all $\beta \in \mathbf{C}^+$. However, this combined with the preceding factorization shows that $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right] - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ has a bounded inverse for all $\alpha, \beta \in \mathbf{C}^+$. At the same time we get an explicit expression for the inverse, namely

$$\begin{aligned}
&\left(\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \right)^{-1} \\
&\quad = \begin{bmatrix} (\alpha - A)^{-1} & (\alpha - A_{|X})^{-1}B \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C(\alpha - A)^{-1} & \beta + \widehat{\mathfrak{D}}(\alpha) \end{bmatrix}^{-1} \\
&\quad = \begin{bmatrix} (\alpha - A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} (\alpha - A_{|X})^{-1}B \\ 1 \end{bmatrix} [\beta + \widehat{\mathfrak{D}}(\alpha)]^{-1} \begin{bmatrix} -C(\alpha - A)^{-1} & 1 \end{bmatrix}.
\end{aligned} \tag{29}$$

Taking $\alpha = \beta$ we find that $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is maximal dissipative. ☐

Corollary 4.4. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, U). Then the following conditions are equivalent, and they are also equivalent to conditions (ii)–(vi) in Theorem 4.2:*

(i) *S is impedance passive.*

(vii) *The operator $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is the generator of a contraction semigroup on $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$.*

(viii) *For some $\alpha \in \rho(A) \cap \mathbf{C}^+$ (or equivalently, for all $\alpha \in \mathbf{C}^+$), the operator $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is invertible, and*

$$\begin{bmatrix} \mathbb{A}(\alpha) & \mathbb{B}(\alpha) \\ \mathbb{C}(\alpha) & \mathbb{D}(\alpha) \end{bmatrix} = \left(\overline{\alpha} + \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \right) \left(\alpha - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \right)^{-1} \tag{30}$$

is a contraction.

Proof. By Theorem 4.2 and Lemma 4.3, S is impedance passive if and only if the operator $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is maximal dissipative, or equivalently, if and only if this operator generates a contraction semigroup on $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. This proves the equivalence of (i) and (vii). The contraction semigroup mentioned above may be regarded as the one corresponding to a scattering passive system with input space $\{0\}$, state space $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$, and output space $\{0\}$ (i.e., the system has no input or output, just a state). The equivalence of (vii) and (viii) now follows from the equivalence of (i) and (v) in Theorem 3.3 applied to this special system. ☐

Corollary 4.5. *A system node S is impedance passive if and only if the dual system node S^* is impedance passive.*

Proof. If $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is impedance passive, then both A and $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] S$ are maximal dissipative on X respectively $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ (see Corollary 4.4). This implies that A^* and $S^* \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ are maximal dissipative. The latter condition is equivalent to the maximal dissipativity of $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] S^*$. By Corollary 4.4, this implies that also S^* is impedance passive. $\square$

Let us next take a closer look at the energy preserving case.

Theorem 4.6. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, U). Then the following conditions are equivalent:*

(i) *Σ is impedance energy preserving.*

(ii) *For all $t > 0$, the solution (x, y) in Lemma 2.3 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 = 2\Re\langle y(t), u(t)\rangle_U. \tag{31}$$

(iii) *For all $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,*

$$\Re\left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} = 0. \tag{32}$$

(iv) *For some (or equivalently, for all) $\alpha \in \rho(A)$ we have*

$$\begin{aligned} &\begin{bmatrix} A + A^*_{|X} & (\alpha + A^*_{|X})(\alpha - A_{|X})^{-1}B \\ B^*_{|X_1} & B^*(\overline{\alpha} - A^*)^{-1}(2\Re\alpha)(\alpha - A_{|X})^{-1}B \end{bmatrix} \\ &\qquad = \begin{bmatrix} 0 & C^* \\ C & \widehat{\mathfrak{D}}(\alpha)^* + \widehat{\mathfrak{D}}(\alpha) \end{bmatrix}, \end{aligned} \tag{33}$$

which is an operator identity in $\mathcal{L}\left(\left[\begin{smallmatrix} X_1 \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} X_{-1}^d \\ U \end{smallmatrix}\right]\right)$.

(v) *For some $\alpha \in \rho(A) \cap \mathbf{C}^+$ (or equivalently, for all $\alpha \in \mathbf{C}^+$), the operator $\left[\begin{smallmatrix} \mathbf{A}(\alpha) & \mathbf{B}(\alpha) \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) \end{smallmatrix}\right]$ defined in (20) satisfies*

$$\begin{bmatrix} \mathbf{A}(\alpha)^*\mathbf{A}(\alpha) & \mathbf{A}(\alpha)^*\mathbf{B}(\alpha) \\ \mathbf{B}(\alpha)^*\mathbf{A}(\alpha) & \mathbf{B}(\alpha)^*\mathbf{B}(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{C}(\alpha)^* \\ \mathbf{C}(\alpha) & \widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(\alpha)^* \end{bmatrix}. \tag{34}$$

(vi) *The system node* $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is skew-symmetric, i.e.,* $\mathcal{D}(S) = \mathcal{D}(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]) \subset \mathcal{D}(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]^*)$, *and*

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathcal{D}(S). \tag{35}$$

(vii) *For some* $\alpha \in \rho(A) \cap \mathbf{C}^+$ *(or equivalently, for all* $\alpha \in \mathbf{C}^+$*), the operator* $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is invertible, and the operator* $\left[\begin{smallmatrix} \mathbb{A}(\alpha) & \mathbb{B}(\alpha) \\ \mathbb{C}(\alpha) & \mathbb{D}(\alpha) \end{smallmatrix}\right]$ *defined in* (30) *is an isometry.*

Proof. We again leave most of the proof to the reader (it is very similar to the proof of Theorem 4.2). Note that $S^* \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is a system node with domain $\mathcal{D}(S^* \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]) = \mathcal{D}(S^*) \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ (it is the adjoint of the system node $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] S = \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$). The only slight difference is that we claim that $C = B^*_{|X}$ instead of the following formula which one first arrives at, namely

$$Cx = B^*(\overline{\alpha} - A^*)^{-1}(\overline{\alpha} + A)x, \qquad x \in X_1.$$

However, as $A^* = -A$ on X_1, the above formula can be rewritten as

$$Cx = B^*(\overline{\alpha} - A^*)^{-1}(\overline{\alpha} - A^*)x = B^*x, \qquad x \in X_1. \quad \square$$

An analogous but even simpler result is true for conservative impedance systems:

Theorem 4.7. *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a system node on* (U, X, U)*. Then the following conditions are equivalent:*

(i) Σ *is impedance conservative.*

(ii) *For all* $t > 0$*, the solution* (x, y) *in Lemma 2.3 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}|x(t)|_X^2 = 2\Re\langle y(t), u(t)\rangle_U, \tag{36}$$

and the same identity is true for the adjoint system.

(iii) *The system node* $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is skew-adjoint, i.e.,*

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* = - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}. \tag{37}$$

(iv) *$A^* = -A$, $B^* = C$, and $\widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(-\overline{\alpha})^* = 0$ for some (or equivalently, for all) $\alpha \in \rho(A)$ (in particular, this identity is true for all α with $\Re\alpha \neq 0$).*

(v) *For some $\alpha \in \rho(A) \cap \mathbf{C}^+$ (or equivalently, for all $\alpha \in \mathbf{C}^+$), the operator $\alpha - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is invertible, and the operator $\left[\begin{smallmatrix} \mathbb{A}(\alpha) & \mathbb{B}(\alpha) \\ \mathbb{C}(\alpha) & \mathbb{D}(\alpha) \end{smallmatrix}\right]$ defined in (30) is unitary.*

Proof. Most of this follows directly from Theorem 4.6. The only fact which requires a separate proof is that (iv) holds if and only if condition (iv) in Theorem 4.6 holds both for the original system and for the dual system.

Suppose that (iv) holds. Then it is obvious that three out of the four identities in (33) hold (the exceptional one being the one in the lower right corner). This last identity is proved as follows: it follows from (iv) and (12) that

$$\begin{aligned}
\widehat{\mathfrak{D}}(\alpha) &= -\widehat{\mathfrak{D}}(-\overline{\alpha})^* \\
&= -\widehat{\mathfrak{D}}(\alpha)^* + (-\alpha - \overline{\alpha})B^*(\overline{\alpha} - A^*)^{-1}(-\alpha - A^*)^{-1}B \\
&= -\widehat{\mathfrak{D}}(\alpha)^* + 2\Re\alpha B^*(\overline{\alpha} - A^*)^{-1}(\alpha - A)^{-1}B.
\end{aligned}$$

The corresponding adjoint identity is proved in the same way (note that (iv) is invariant under duality).

The proof of the converse direction is essentially the same: if (33) holds both for the original system and the dual system, then $A^* = -A$, $B^* = C$, and the bottom right corner of (33) together with the above computation shows that $\widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(-\overline{\alpha})^* = 0$. □

Example 4.8. Let A be the generator of a contraction semigroup on X. Define $S = \left[\begin{smallmatrix} A_{|X} & A_{|X} \\ -A_{|X} & -A_{|X} \end{smallmatrix}\right]$ with $\mathcal{D}(S) = \left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right] \;\middle|\; x + u \in \mathcal{D}(A)\right\}$. Then S is an impedance passive system node on (X, X, X) (use part (vi) of Theorem 4.2 and note that $\left[\begin{smallmatrix} A & A \\ A & A \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] A \begin{bmatrix} 1 & 1 \end{bmatrix}$ can be interpreted as the dissipative operator A surrounded by another operator and its adjoint). The transfer function of this node is easily computed, and it turns out to be $\widehat{\mathfrak{D}}(s) = -sA(s - A)^{-1}$, $s \in \mathbf{C}^+$. This example is impedance energy preserving if and only if A generates an isometric semigroup (i.e, A is skew-symmetric), and it is impedance conservative if and only if A generates a unitary group (i.e, A is skew-adjoint).

5 Well-Posed Impedance Passive Systems

Many impedance passive systems are well-posed. There is a simple way of characterizing such systems:

Theorem 5.1. *An impedance passive system node is well-posed if and only if its transfer function* $\widehat{\mathfrak{D}}$ *is bounded on some (or equivalently, on every) vertical line in* $\mathbf{C}^+$*. When this is the case, the growth bound of the system is zero, and, in particular,* $\widehat{\mathfrak{D}}$ *is bounded on every right half-plane* $\mathbf{C}^+_\epsilon = \{s \in \mathbf{C} \mid \Re s > \epsilon\}$ *with* $\epsilon > 0$.

Proof. Suppose that the system node S is both well-posed and impedance passive. The growth bound of this system is then zero, and this implies that $\widehat{\mathfrak{D}}$ is bounded in every half-plane $\mathbf{C}^+_\epsilon$ with $\epsilon > 0$; see, e.g., [20, Section 4.6].

Conversely, suppose that $\|\widehat{\mathfrak{D}}(s)\| \le M$ for some $M < \infty$ and all s with $\Re s = \alpha > 0$. Let us transform the vertical line $\Re s = \alpha$ to the imaginary axis by using an exponential shift of the type described in Lemma 2.5: we replace S by $S_\alpha = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] - \left[\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right]$. This has the effect of replacing the original transfer function $\widehat{\mathfrak{D}}$ by $\widehat{\mathfrak{D}}_\alpha$ given by $\widehat{\mathfrak{D}}_\alpha(s) = \widehat{\mathfrak{D}}(s+\alpha)$. In particular, $\|\widehat{\mathfrak{D}}_\alpha(s)\| \le M$ for all s with $\Re s = 0$. Moreover, it follows immediately from Lemma 2.5 and Definition 4.1 that S_α is impedance passive. We claim that S_α is well-posed. Take a C^∞ input function u supported on $[0, t]$, and let the initial state (of the shifted system) be zero. The Fourier transforms of u and the corresponding output function y are the restriction of the Laplace transforms of these functions to the imaginary axis. Moreover, $\hat{y}(s) = \widehat{\mathfrak{D}}_\alpha(s)\hat{u}(s)$, so $|\hat{y}(s)| \le M|\hat{u}(s)|$ for all s with $\Re s = 0$. This implies that $y \in L^2(\mathbf{R}^+; V)$ and that

$$\|y\|_{L^2(0,t)} \le \|y\|_{L^2(\mathbf{R}^+)} \le M\|u\|_{L^2(0,t)}$$

(where M does not depend on u or t). By the Cauchy–Schwarz inequality,

$$2\int_0^t \Re\langle y(t), u(t)\rangle dt \le 2\|y\|_{L^2(0,t)}\|u\|_{L^2(0,t)} \le 2M\|u\|^2_{L^2(0,t)}.$$

By **(PI)**,

$$|x(t)|^2 \le 2M\|u\|^2_{L^2(0,t)}.$$

Thus, for all C^∞ input functions functions u supported on $[0, t]$, the state trajectory and the output function with initial state zero satisfy

$$\|y\|^2_{L^2(0,t)} + |x(t)|^2 \le (M^2 + 2M)\|u\|^2_{L^2(0,t)}.$$

Since C^∞ is dense in L^2, this implies that both the input map and the input-output maps are well-posed. The same argument applied to the dual system shows that the output map is well-posed, as well. The semi-group is always well-posed. Thus, the whole shifted system is well-posed, hence so is the original one. □

Theorem 5.2. *Let Σ be a well-posed system with system node S and transfer function $\widehat{\mathfrak{D}}$. In addition, suppose that $1+\widehat{\mathfrak{D}}$ is invertible on some right half-plane and that $(1+\widehat{\mathfrak{D}})^{-1}$ is bounded on this half-plane. (The last condition is, in particular true, if Σ is a well-posed impedance passive system.) Then the following claims are true:*

(i) *There is a unique well-posed system $\Sigma^\times$ with the following property: if x is the state trajectory and $y \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$ is the output function of Σ with initial state x_0 and input function $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$, and if we use the same initial state x_0 and the input function $u^\times = \frac{1}{\sqrt{2}}(u+y)$ for the system $\Sigma^\times$, then the state trajectory $x^\times$ of $\Sigma^\times$ coincides with the state trajectory x of Σ, and the output function of $\Sigma^\times$ is given by $y^\times = \frac{1}{\sqrt{2}}(u-y)$.*

(ii) *The system $\Sigma^\times$ is scattering passive (or energy preserving or conservative) if and only if Σ is impedance passive (or energy preserving or conservative).*

(iii) *The system node $S^\times$ can be determined from its main operator $A^\times$, control operator $B^\times$, observation operator $C^\times$, and transfer function $\widehat{\mathfrak{D}}^\times$, which can be computed from the following formulas, valid for all $\alpha \in \rho(A)\cap\rho(A^\times)$,*[9]

$$
\begin{aligned}
&\begin{bmatrix} (\alpha - A^\times)^{-1} & \frac{1}{\sqrt{2}}(\alpha - A^\times_{|X})^{-1}B^\times \\ \frac{1}{\sqrt{2}}C^\times(\alpha - A^\times)^{-1} & \frac{1}{2}(1+\widehat{\mathfrak{D}}^\times(\alpha)) \end{bmatrix} \\
&\quad = \left(\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \right)^{-1} \\
&\quad = \begin{bmatrix} (\alpha - A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\
&\qquad + \begin{bmatrix} (\alpha - A_{|X})^{-1}B \\ 1 \end{bmatrix} (1+\widehat{\mathfrak{D}}(\alpha))^{-1} \begin{bmatrix} -C(\alpha - A)^{-1} & 1 \end{bmatrix}
\end{aligned}
\tag{38}
$$

[9] $A^\times_{|X}$ is the extension of $A^\times$ to an operator in $\mathcal{L}(X;X^\times_{-1})$, where $X^\times_{-1}$ is the analogue of X_{-1} with A replaced by $A^\times$.

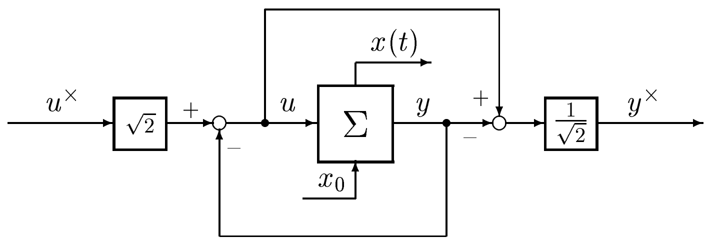

Figure 1: The diagonal transform

In particular, $1 + \widehat{\mathfrak{D}}(\alpha)$ *is invertible and*

$$\widehat{\mathfrak{D}}^{\times}(\alpha) = (1 - \widehat{\mathfrak{D}}(\alpha))(1 + \widehat{\mathfrak{D}}(\alpha))^{-1}$$

for all $\alpha \in \rho(A) \cap \rho(A^{\times})$.

(iv) *The transfer function* $\widehat{\mathfrak{D}}^{\times}$ *of* $S^{\times}$ *has the property that* $1+\widehat{\mathfrak{D}}^{\times}$ *is always invertible on some right half-plane and that* $(1+\widehat{\mathfrak{D}}^{\times})^{-1}$ *is bounded on this half-plane. If we repeat the same transform with* S *replaced by* $S^{\times}$*, then we recover the original system. Thus, in particular, the system* $\Sigma^{\times}$ *is impedance passive (or energy preserving or conservative) if and only if* Σ *is scattering passive (or energy preserving or conservative). Furthermore,* (38) *also holds if we interchange* S *and* $S^{\times}$.

Figure 1 contains a diagram of the transform described in this theorem. Following [8], we shall refer to the above transform as the *diagonal transform*. There is also a non-well-posed version of this theorem to which we shall return in [19].

Proof. (i) Let us begin with the uniqueness. A system is uniquely defined if we know its state trajectory and its output for all initial states $x_0 \in X$ and all input functions $u^{\times} \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$, so uniqueness follows as soon as we have shown that, given any $x_0 \in X$, we can produce every possible function $u^{\times} \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$ by choosing $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+;U)$ appropriately and defining $u^{\times} = \frac{1}{\sqrt{2}}(u+y)$. This will become evident from the proof below.

Clearly, by slightly modifying the system Σ we can replace its original output function y by $u^{\times} = \frac{1}{\sqrt{2}}(u+y)$: to do this it suffices to keep $A\&B$ unchanged but replace the original control/feedthrough operator $C\&D$ by $\widetilde{C\&D} = \frac{1}{\sqrt{2}}\big(C\&D + \begin{bmatrix}0 & 1\end{bmatrix}\big)$. The transfer function of the resulting system $\widetilde{\Sigma}$ is $\frac{1}{\sqrt{2}}(1+\widehat{\mathfrak{D}})$. The extra assumption that we imposed on Σ implies that this transfer function has a bounded inverse on some right half-plane, which means that $\widetilde{\Sigma}$ is flow-invertible (see [22]). In particular, this means that given any $x_0 \in X$ and $u^\times \in L^2_{\rm loc}(\mathbf{R}^+;U)$, we can find an input function $u \in L^2_{\rm loc}(\mathbf{R}^+;U)$ so that $u^\times = \frac{1}{\sqrt{2}}(u+y)$ (as needed in the uniqueness proof). Let us denote the flow-inverted system by $\widetilde{\Sigma}^\times$ and its system node by $\widetilde{S}^\times = \begin{bmatrix} \widetilde{A\&B}^\times \\ \widetilde{C\&D}^\times \end{bmatrix}$. The only difference between $\widetilde{\Sigma}$ and $\widetilde{\Sigma}^\times$ is that we have interchanged the meaning of the input and the output: the relationships between all the different signals are the same, but whereas the input of Σ is u and the output is $u^\times$, the input of $\widetilde{\Sigma}$ is $u^\times$ and the output is u. From $\widetilde{\Sigma}^\times$ we easily get the final system $\Sigma^\times$: we keep the top row $\widetilde{A\&B}^\times$ of the system node unchanged but replace $\widetilde{C\&D}^\times$ by (notice that $y^\times = \sqrt{2}\,u - u^\times$)

$$C\&D^\times = \sqrt{2}\,\widetilde{C\&D}^\times - \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The transfer function of $\Sigma^\times$ then becomes

$$\widehat{\mathfrak{D}}^\times = \sqrt{2}\left(\frac{1}{\sqrt{2}}\big(1+\widehat{\mathfrak{D}}\big)\right)^{-1} - 1 = \big(1-\widehat{\mathfrak{D}}\big)\big(1+\widehat{\mathfrak{D}}\big)^{-1}.$$

(ii) With the same notations as above, a short mechanical computation shows that

$$|u^\times|^2 - |y^\times|^2 = 2\Re\langle y, u\rangle.$$

Hence, it follows from Definitions 3.2 and 4.1 that Σ is impedance passive if and only if $\Sigma^\times$ is scattering passive. For the same reason, Σ is impedance energy preserving if and only $\Sigma^\times$ is scattering energy preserving. Finally, by applying this result to the dual system (the construction of the system $\Sigma^\times$ described above commutes with the duality transform) we find that Σ is impedance conservative if and only if $\Sigma^\times$ is scattering conservative.

(iii) This follows from the above construction, Lemma 4.3 and the formulas in [22].

(iv) The invertibility of $1+\widehat{\mathfrak{D}}^\times$ follows from the fact that $1+\widehat{\mathfrak{D}}^\times = 2(1+\widehat{\mathfrak{D}})^{-1}$, and the remaining claims from the fact that $u^\times = \frac{1}{\sqrt{2}}(u+y)$ and $y^\times = \frac{1}{\sqrt{2}}(u-y)$ if and only if $u = \frac{1}{\sqrt{2}}(u^\times + y^\times)$ and $u = \frac{1}{\sqrt{2}}(u^\times - y^\times)$. □

In our following theorem we need some additional notions that we have not used so far, namely the reachable and unobservable subspaces of a system node S. By the *reachable* subspace of S we mean the closure in X of the set of all possible values of $x(t)$ in Lemma 2.3 if we take $x_0 = 0$ (and let u and t vary). Its orthogonal complement is the *unreachable subspace*. By the *unobservable* subspace of S we mean the closure of the set of all $x_0 \in X_1$ for which the output y in Lemma 2.3 with initial state x_0 and zero input function u is identically zero. Its orthogonal complement is the *observable subspace.* It is well-known that the orthogonal complement of the reachable subspace of S is the unobservable subspace of the dual system node S^* (and the same statement is true if we interchange S and S^*). A system is *simple* if the intersection of the unreachable and unobservable subspaces is $\{0\}$.

Theorem 5.3. *Every positive analytic function on $\mathbf{C}^+$ which is proper (i.e., it is bounded on some right half-plane) has a simple well-posed impedance conservative realization, which is unique modulo a unitary similarity transform in the state space.*

Proof. Since $\widehat{\mathfrak{D}}$ is a positive, $(1+\widehat{\mathfrak{D}}(\alpha))^{-1}$ exists and is bounded on $\mathbf{C}^+$. Define $\widehat{\mathfrak{D}}^\times(\alpha) = (1-\widehat{\mathfrak{D}}(\alpha))(1+\widehat{\mathfrak{D}}(\alpha))^{-1}$. This is a contractive analytic function on $\mathbf{C}^+$, so by it has a simple scattering conservative realization $\Sigma^\times$, which is unique modulo a unitary similarity transform in its state space (see, e.g., [2, Theorem 6.4] or [20, Chapter 11]). From here we get a simple impedance conservative realization of $\widehat{\mathfrak{D}}$ by applying Theorem 5.2. The uniqueness claim remains true (the diagonal transform does not influence the unreachable and unobservable subspaces). □

We call a system node S (and the corresponding system Σ) on (U,X,Y) *(approximately) controllable* if the reachable subspace is all of X and *(approximately) observable* if the observable subspace is all of X. A system which is both controllable and observable is *minimal.* The realization described in Theorem 5.3 will not be minimal in general. However, from this realization we can derive a minimal realization, e.g., as follows (see [2, Section 7] or [20, Section 9.1] for details). We proceed in two steps. Let $\mathcal{R}$ be the reachable subspace of Σ. By 'restricting Σ to $\mathcal{R}$' we get a controllable system Σ_1 on $(U,\mathcal{R},Y)$ whose main operator is $A_1 = A_{|\mathcal{R}}$, control operator is $B_1 = B$, observation operator is $C_1 = C_{|\mathcal{R}}$, and transfer function $\widehat{\mathfrak{D}}$ is the same as the original transfer function. It is not difficult to show that if the original system Σ is conservative (scattering or impedance), then the new system is a energy preserving (scattering or impedance), and that it is unique among all controllable energy preserving (scattering or impedance) realizations of $\widehat{\mathfrak{D}}$ modulo a unitary similarity transform in the state space. If Σ_1 is observable, then we have obtained a minimal passive (and even energy preserving) realization. If not, then we let $\mathcal{O}_1$ be the observable subspace of Σ_1, denote the orthogonal projection of $\mathcal{R}$ onto $\mathcal{O}_1$ by π, and 'project Σ_1 onto $\mathcal{O}_1$' to get the minimal system Σ_2 whose main operator is

$A_2 = \pi A_1 = \pi A_{|\mathcal{R}}$, control operator is $B_1 = \pi B_1 = \pi B$, observation operator is $C_2 = C_{1|\mathcal{R}} = C_{|\mathcal{R}}$, and transfer function $\widehat{\mathfrak{D}}$ is still the same as the original transfer function. This system is passive (scattering or impedance) whenever Σ_1 is passive. Thus, we arrive at the following result:

Corollary 5.4. *Every proper positive analytic function on* $\mathbf{C}^+$ *has a minimal well-posed impedance passive realization.*

The above realization is not unique (for example, we could, instead first have projected the system onto the observable subspace to get a system whose adjoint is energy preserving, and then restricted the new system to the reachable subspace), but it is possible to make it unique by requiring it to be 'optimal' in a certain sense.[10] See [2, Section 7] and [32, Section 4] for details.

6 A Feedback Interpretation

The diagonal transform in Theorem 5.2 has a natural output feedback interpretation. In that transform we introduce a new input signal $u^\times$, choose the input of the original system Σ to be $u = \sqrt{2}\,u^\times - y$, and regard the new output signal to be $y^\times = \frac{1}{\sqrt{2}}(u - y)$. If we ignore the trivial scaling factors $\sqrt{2}$ and $1/\sqrt{2}$, then the replacement of u by the new input $u^\times$ is a typical negative identity state feedback, whereas the replacement of y by $y^\times$ just amounts to the addition of an extra feedthrough term to the resulting closed loop system. Recall that if Σ is a well-posed linear system on (U, X, Y), then $K \in \mathcal{L}(Y; U)$ is called an admissible feedback operator if the replacement of the input signal u by $u = u^K + Ky$ leads to a new well-posed linear system with input signal u^K. In the special case where $U = Y$ considered above we may use *negative identity output feedback*, i.e., we let $K = -1$. Thus, Theorem 5.2 implies the following result:

Corollary 6.1. *Let* Σ *be a well-posed impedance passive system induced by a system node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *on* (U, X, U). *Then* -1 *is an admissible feedback operator for* Σ, *and the closed loop system corresponding to this feedback operator is energy stable (in the sense of Definition 2.6).*

[10] The above construction produces Arov's optimal realization. This is the minimal realization which uses the norm in the state space induced by Willems' *available storage*. If we instead first project onto the observable subspace and then restrict to the reachable subspace, then we get Arov's $*$-optimal realization. This is the realization which uses the norm induced by Willems' *required supply* function.

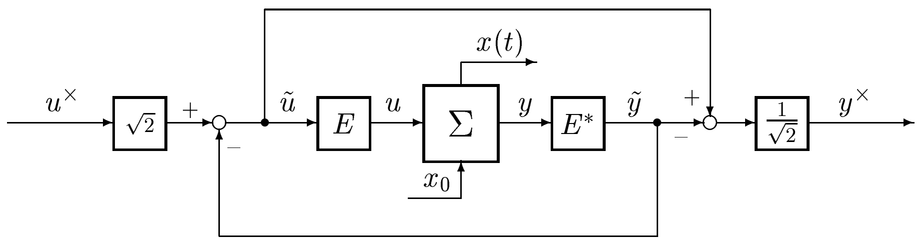

Figure 2: Modified diagonal transform

Instead of using negative identity feedback we may use *any scalar negative feedback*. As a matter of fact, even every strictly positive operator-valued feedback will make the closed loop system energy stable. To see this we may argue as follows. If $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an impedance passive system node, then so is

$$S_E = \begin{bmatrix} I & 0 \\ 0 & E^* \end{bmatrix} \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & E \end{bmatrix},$$

where E is an arbitrary bounded linear operator on U; this follows from Theorem 4.2. Moreover, S_E is impedance energy preserving or conservative whenever S has this property. If E is invertible, then the converse is also true: passivity, or energy preservation, or conservativity of S_E implies that S has the same property. We can apply Theorem 5.2 to the system node S_E instead of the system node S to get the scattering passive system drawn in Figure 2 (this is true independently of whether E is invertible or not). If E is invertible, then this effectively amounts to the feedback connection with feedback operator $EE^* \gg 0$ drawn in Figure 3. In this figure we have ignored the feedforward connection in Figure 2, and we have used the invertibility of E to replace the output $\tilde{y}$ by the output $y = (E^*)^{-1}\tilde{y}$ (the mapping from $u^\times$ to y in Figure 2 need not be well-posed in general, but $\tilde{y}$ is always a well-posed output). The invertibility of E is also needed if we want to replace the input $u^\times$ in Figure 3 by the new independent input $v = [\sqrt{2}E]^{-1}u^\times$.

Acknowledgments. This paper would not have been written, had it not been for Ruth Curtain. She kept asking me all the right questions about positive real functions, first at IWOTA2000 and then at the PDPS workshop in the summer of 2001, and she made me realize that there were many aspects of these functions that I did not understand properly. An additional significant source of inspiration was the paper [29] by George

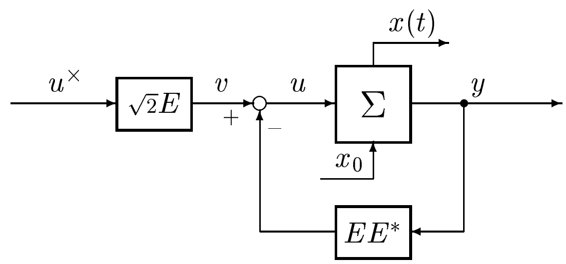

Figure 3: Equivalent feedback connection

Weiss and Marius Tucsnak discussed above. Much of my present knowledge of scattering conservative system comes out of numerous discussions with Jarmo Malinen and George Weiss.

References

[1] D. Z. Arov. Passive linear systems and scattering theory. In *Dynamical Systems, Control Coding, Computer Vision*, volume 25 of *Progress in Systems and Control Theory*, pages 27–44, Basel Boston Berlin, 1999. Birkhäuser Verlag.

[2] D. Z. Arov and M. A. Nudelman. Passive linear stationary dynamical scattering systems with continuous time. *Integral Equations Operator Theory*, 24:1–45, 1996.

[3] J. A. Ball and N. Cohen. De Branges-Rovnyak operator models and systems theory: a survey. In *Topics in Matrix and Operator Theory*, volume 50 of *Operator Theory: Advances and Applications*, pages 93–136, Basel Boston Berlin, 1991. Birkhäuser Verlag.

[4] R. F. Curtain and G. Weiss. Well posedness of triples of operators (in the sense of linear systems theory). In *Control and Optimization of Distributed Parameter Systems*, volume 91 of *International Series of Numerical Mathematics*, pages 41–59, Basel Boston Berlin, 1989. Birkhäuser-Verlag.

[5] P. Grabowski and F. M. Callier. Boundary control systems in factor form: transfer functions and input-output maps. *Integral Equations Operator Theory*, 41:1–37, 2001.

[6] P. Grabowski and F. M. Callier. Circle criterion and boundary control systems in factor form: input-output approach. *Int. J. Appl. Math. Comput. Sci.*, 11:1387–1403, 2001.

[7] J. W. Helton. Systems with infinite-dimensional state space: the Hilbert space approach. *Proceedings of the IEEE*, 64:145–160, 1976.

[8] M. S. Livšic. *Operators, Oscillations, Waves (Open Systems)*, volume 34 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1973.

[9] J. Malinen, O. J. Staffans, and G. Weiss. When is a linear system conservative? In preparation, 2002.

[10] R. Ober and S. Montgomery-Smith. Bilinear transformation of infinite-dimensional state-space systems and balanced realizations of nonrational transfer functions. *SIAM J. Control Optim.*, 28:438–465, 1990.

[11] D. Salamon. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.*, 300:383–431, 1987.

[12] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, 21:147–164, 1989.

[13] Y. L. Smuljan. Invariant subspaces of semigroups and the Lax-Phillips scheme. Dep. in VINITI, N 8009-1386, Odessa, 49p., 1986.

[14] O. J. Staffans. Quadratic optimal control of stable well-posed linear systems. *Trans. Amer. Math. Soc.*, 349:3679–3715, 1997.

[15] O. J. Staffans. Coprime factorizations and well-posed linear systems. *SIAM J. Control Optim.*, 36:1268–1292, 1998a.

[16] O. J. Staffans. Admissible factorizations of Hankel operators induce well-posed linear systems. *Systems Control Lett.*, 37:301–307, 1999.

[17] O. J. Staffans. J-energy preserving well-posed linear systems. *Int. J. Appl. Math. Comput. Sci.*, 11:1361–1378, 2001.

[18] O. J. Staffans. Passive and conservative infinite-dimensional impedance and scattering systems (from a personal point of view). To appear in the Proceedings of MTNS02, 2002.

[19] O. J. Staffans. Stabilization by collocated feedback. Submitted, 2002.

[20] O. J. Staffans. *Well-Posed Linear Systems: Part I.* Book manuscript, available at http://www.abo.fi/˜staffans/, 2002.

[21] O. J. Staffans and G. Weiss. Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup. *Trans. Amer. Math. Soc.*, 2002. To appear.

[22] O. J. Staffans and G. Weiss. Transfer functions of regular linear systems. Part III: inversions and duality. Submitted, 2002.

[23] G. Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27:527–545, 1989a.

[24] G. Weiss. Admissible observation operators for linear semigroups. *Israel J. Math.*, 65:17–43, 1989b.

[25] G. Weiss. Transfer functions of regular linear systems. Part I: characterizations of regularity. *Trans. Amer. Math. Soc.*, 342:827–854, 1994a.

[26] G. Weiss. Regular linear systems with feedback. *Math. Control Signals Systems*, 7:23–57, 1994b.

[27] G. Weiss. Optimal control of systems with a unitary semigroup and with colocated control and observation. *Systems Control Lett.*, 2002.

[28] G. Weiss, O. J. Staffans, and M. Tucsnak. Well-posed linear systems – a survey with emphasis on conservative systems. *Int. J. Appl. Math. Comput. Sci.*, 11:7–34, 2001.

[29] G. Weiss and M. Tucsnak. How to get a conservative well-posed linear system out of thin air. Part I: well-posedness and energy balance. Submitted, 2001.

[30] M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems. *Math. Control Signals Systems*, 10:287–330, 1997.

[31] J. C. Willems. Dissipative dynamical systems Part I: General theory. *Arch. Rational Mech. Anal.*, 45:321–351, 1972.

[32] J. C. Willems. Dissipative dynamical systems Part II: Linear systems with quadratic supply rates. *Arch. Rational Mech. Anal.*, 45:352–393, 1972.

[33] A. H. Zemanian. *Realizability theory for continuous linear systems*, volume 97 of *Mathematics in Science and Engineering*. Academic Press, New York, London, 1972.

[34] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice Hall, Englewood Cliffs, New Jersey, 1996.