Quadratic Optimal Control of a Parabolic Equation

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Abstract

We apply a spectral factorization approach to the infinite horizon quadratic cost minimization problem for a stable parabolic equation, and show that this approach leads to the same conclusions as the classical approach based on the algebraic Riccati equation. In particular, we find that the so called regular spectral factorization assumption is valid in this case.

1 Introduction

We study the infinite horizon quadratic cost minimization problem for a parabolic equation, and look for a feedback representation of the function u which minimizes the cost function

$$Q(x_0, u) = \int_0^\infty \left(\|y(t)\|^2 + \|u(t)\|^2 \right) \, dt, \tag{1}$$

where y is the output of the system

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \quad && t \geq 0, \\ y(t) &= Cx(t), && t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{2}$$

Here $u(t) \in U$, $x(t) \in Z$, and $y(t) \in Y$, where the input space U, the state space Z, and the output space U are separable Hilbert spaces. The operator A generates an exponentially stable analytic semigroup $\mathcal{A}$ on Z, and the observation operator C is not too unbounded compared to the control operator B. More precisely, there exists some $\gamma < 1$ such that $C(-A)^{-\gamma}B$ is a bounded linear operator from U to Y, where $(-A)^{-\gamma}$ represents the usual fractional power of A [Pazy, 1983, Section 2.6].

There is a well-developed state space solution of this problem, based on the differential and algebraic Riccati equations. The major part of this theory up to the early 90's is summarized in Bensoussan et al. [1992], Lasiecka and Triggiani [1991], and some more recent developments are found in, e.g., Lasiecka et al. [1995, 1997], Pandolfi [1997], Triggiani [1994].

Recently another similar but seemingly different theory has been developed for the solution of the same problem (without the parabolicity assumption) within the class of regular well-posed linear systems in the sense of Salamon and Weiss. See Staffans [1997, 1998b] and Weiss and Weiss [1997]. The latter theory is based on spectral factorization, and it suffers from the fact that some of the more specific conclusions (including all those that refer to the algebraic Riccati equation) utilize a "regular spectral factorization hypothesis" which can be difficult to verify. We show that the regular spectral factorization hypothesis is satisfied in the parabolic case (1)–(2), which means that the spectral factorization approach of Staffans [1997, 1998b] and Weiss and Weiss [1997] applies to this problem in its full strength. The key ingredient in the regularity proof is the "boot-strap argument" introduced in [Lasiecka and Triggiani, 1983, pp. 52–53]). The conclusions that we get are essentially the same as the conclusions obtained with the more classical approach, restricted to stable systems.

2 The Parabolic Equation

We let A generate an exponentially stable analytic semigroup $\mathcal{A}$ in Z. For each $\alpha \in \mathbf{R}$ we let $Z_\alpha = (-A)^{-\alpha} Z$ be the domain of $(-A)^\alpha$, with norm $\|x\|_{Z_\alpha} = \|(-A)^\alpha x\|_Z$ and inner product $\langle x_1, x_1 \rangle_{Z_\alpha} = \langle x_1, (-A^*)^\alpha (-A)^\alpha x_2 \rangle_Z$. Then the restrictions of A to Z_α for $\alpha > 0$ and the extensions of A to Z_α for $\alpha < 0$ (which we still denote by A) generate analytic semigroups in Z_α, for all $\alpha \in \mathbf{R}$. These semigroups are all similar to each other, and they commute with A^β for all $\beta \in \mathbf{R}$. We therefore denote all of them by the same letter $\mathcal{A}$. The generator of the semigroup $\mathcal{A}$ on Z_α is then $A \in \mathcal{L}(Z_{\alpha+1}; Z_\alpha)$. Moreover, for each $t > 0$ and $\alpha \in \mathbf{R}$, $\mathcal{A}^t$ maps Z^α into $\cap_{\beta \in \mathbf{R}} Z_\beta$, and for each $\alpha \geq 0$, there exist constants $K > 0$ and $\epsilon > 0$

such that

$$\|A^{\alpha}\mathcal{A}^{t}\| \le Kt^{-\alpha}\mathrm{e}^{-\epsilon t}, \quad t>0, \tag{3}$$

where the norm represents the operator norm in any one of the spaces Z_{α} [Pazy, 1983, Theorem 6.13]. The same construction can be repeated with A replaced by A^* to give another scale of Hilbert spaces $Z^*_{\alpha} = (-A^*)^{-\alpha}Z$ with similar properties. We identify Z^*_{α} with the dual of $Z_{-\alpha}$ by using Z as the pivot space. Note that $Z_0 = Z^*_0 = Z$.

The assumptions on the operators B and C in (2) are $B \in \mathcal{L}(U; Z_{\alpha_B})$ and $C \in \mathcal{L}(Z_{\alpha_C}; Y)$. Here α_B and α_C are two fixed numbers satisfying $\alpha_B \le \alpha_C < \alpha_B + 1$. For each $\alpha \in \mathbf{R}$, $x_0 \in Z_{\alpha}$, and $u \in L^1(\mathbf{R}; U)$ we define

$$\begin{aligned} (\mathcal{B}u)(t) &= \int_{-\infty}^{t} \mathcal{A}^{t-s}Bu(s)\,ds, && t \in \mathbf{R}, \\ (\mathcal{C}x_0)(t) &= C\mathcal{A}^{t}x_0, && t \in \mathbf{R}^+, \\ (\mathcal{D}u)(t) &= C(\mathcal{B}u)(t), && t \in \mathbf{R}. \end{aligned} \tag{4}$$

This results in a well-posed linear system:

Proposition 1 *Let A generate an exponentially stable analytic semigroup $\mathcal{A}$ in Z, and let $B \in \mathcal{L}(U; Z_{\alpha_B})$ and $C \in \mathcal{L}(Z_{\alpha_C}; Y)$, where $\alpha_B \le \alpha_C < \alpha_B + 1$. Define $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ as in (4), and fix any β satisfying $\alpha_C - 1/2 < \beta < \alpha_B + 1/2$. Then $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is a stable strictly causal regular well-posed linear system on (U, X, Y), where $X = Z_{\beta}$. The generating operators of this system are A, B, and C.*

Here "well-posed" means that the initial value $x_0 \in X$ (where X = the state space) and the input $u \in L^2([0,\infty; U))$ is mapped continuously into the final state $x(t) \in X$ and output $y \in L^2([0,\infty; Y))$. Regularity means that this system has a well-defined feedthrough operator, and the strict causality means that this feedthrough operator is zero. See, e.g., Curtain and Weiss [1989], Staffans [1997, 1998a], and Weiss [1994a,b] for details. The easy proof of this proposition is given in Staffans [1998e] (and it is also found implicitly in Lasiecka and Triggiani [1991]). It is based on (3) and Young's inequality [Stein and Weiss, 1971, p. 178] (the convolution of an L^p-function with an L^q-function belongs to L^r with $1/r = 1/p + 1/q - 1$, and it is continuous if $1/p + 1/q = 1$). (The exponential stability is not yet important here, but we will use it later.)

3 LQ Optimal Control

Since Ψ is an exponentially stable regular well-posed linear system, we can apply both the PDE approach of Bensoussan et al. [1992] and Lasiecka and Triggiani [1991] and the spectral factorization approach of Staffans [1997, 1998b] and Weiss and Weiss [1997] to solve the quadratic cost minimization problem where (1) is minimized over all $u \in L^2(\mathbf{R}^+; U)$ subject to (2). The starting point is the following simple result:

Proposition 2 *Make the same assumptions and introduce the same notations as in Proposition 1. Then, for each $x_0 \in X = Z_{\beta}$, there is a unique control $u^{\mathrm{opt}} \in L^2(\mathbf{R}^+; U)$ which minimizes the cost (1) subject to (2). Denote the corresponding minimal cost by $Q^{\mathrm{opt}}(x_0)$. Then Q^{opt} is a positive (possibly unbounded) quadratic functional on Z which can be written in the form*

$$Q^{\mathrm{opt}}(x_0) = \min_{u \in L^2(\mathbf{R}^+; U)} Q(x_0, u) = \langle x_0, \Pi x_0 \rangle_Z,$$

where the inner product is computed in Z, and Π (the Riccati operator) is a positive (possibly unbounded) operator in Z.

A more interesting task is to find a feedback representation of $u^{\mathrm{opt}}(x_0)$, and this requires a much deeper analysis.

Theorem 1 *Make the same assumptions and introduce the same notations as in Proposition 1, and fix any ϵ satisfying $0 < \epsilon \le 1 + \alpha_B - \alpha_C$. Then B, C, and Π have the following "smoothness" properties:*

$$\begin{aligned} &B \in \mathcal{L}(U; Z_{\alpha_B}) \subset \mathcal{L}(U; Z_{\alpha_C - 1 + \epsilon}), \\ &C \in \mathcal{L}(Z_{\alpha_C}; Y), \\ &B^* \in \mathcal{L}(Z^*_{-\alpha_B}; U) \subset \mathcal{L}(Z^*_{-\alpha_C + 1 - \epsilon}; U), \\ &C^* \in \mathcal{L}(Y; Z^*_{-\alpha_C}), \\ &\Pi \in \mathcal{L}(Z_{\alpha_C - 1/2 + \epsilon}; Z^*_{-\alpha_C + 1/2 - \epsilon}), \\ &\Pi \in \mathcal{L}(Z_{\alpha_C}; Z^*_{-\alpha_C + 1 - \epsilon}), \\ &\Pi \in \mathcal{L}(Z_{\alpha_C - 1 + \epsilon}; Z^*_{-\alpha_C}). \end{aligned} \tag{5}$$

The Riccati operator Π is the unique stabilizing solution (in the sense of Mikkola [1997]) of the algebraic Riccati equation

$$A^*\Pi + \Pi A + C^*C = \Pi BB^*\Pi, \tag{6}$$

valid in $\mathcal{L}(Z_{\alpha_C + \epsilon}; Z^*_{-\alpha_C - \epsilon})$. The optimal control $u^{\mathrm{opt}}(x_0)$ can be expressed in feedback form with feedback operator

$$K = -B^*\Pi \in \mathcal{L}(Z_{\alpha_C}; U). \tag{7}$$

More precisely, the operator

$$A_{\circlearrowleft} = A + BK \tag{8}$$

generates an exponentially stable analytic semigroup $\mathcal{A}_{\circlearrowleft}$ on Z_{β} for all β with $\alpha_C - 1 \le \beta \le \alpha_B + 1$, and $u^{\mathrm{opt}}(x_0) = K\mathcal{A}_{\circlearrowleft}x_0$ for all $x_0 \in Z_{\beta}$ with $\alpha_C - 1/2 < \beta < \alpha_B + 1/2$.

This is a slightly enhanced version of [Lasiecka and Triggiani, 1991, Theorem 2.1] (which is based on Da Prato and Ichikawa [1985], Flandoli [1987], and Lasiecka and Triggiani [1987]). In the proof of this theorem we may, without loss of generality, take $\alpha_C = 0$ (as is done throughout in Lasiecka and Triggiani [1991]), or $-1/2 < \alpha_B \le 0 \le \alpha_C < 1/2$ (as is done in Staffans [1998e]). This is achieved with a simple change of pivot space from Z to Z_β for some $\beta \neq 0$; see [Bensoussan et al., 1992, Vol. I, Section 2.5] or [Staffans, 1998e, Proposition 1]. Today two different proofs of this theorem are available: the original proof given in Lasiecka and Triggiani [1991] (combined with some additional straightforward estimates), and the proof given in Staffans [1998e] which is based on spectral factorization combined with Lasiecka's and Triggiani's boot-strap argument.

4 Spectral Factorization

In the proof of Theorem 1 given in Staffans [1998e] the following result plays a key role:

Theorem 2 *Make the same assumptions and introduce the same notations as in Proposition 1. Define K by (7), and, for all $\alpha \in \mathbf{R}$, $x_0 \in Z_\alpha$, and $u \in L^1(\mathbf{R}; U)$, define*

$$\begin{aligned}(\mathcal{K}x_0)(t) &= K\mathcal{A}^t x_0, \qquad t \in \mathbf{R}^+,\\ (\mathcal{F}u)(t) &= K(\mathcal{B}u)(t), \quad t \in \mathbf{R}.\end{aligned} \tag{9}$$

Then $\left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \left[\begin{smallmatrix}\mathcal{C}\\ \mathcal{K}\end{smallmatrix}\right] & \left[\begin{smallmatrix}\mathcal{D}\\ \mathcal{F}\end{smallmatrix}\right]\end{smallmatrix}\right]$ is a strictly causal regular well-posed linear system on (U, X, Y), and also the adjoint of this system is regular. Moreover, $I - \mathcal{F}$ is the (unique) spectral factor of $I + \mathcal{D}^*\mathcal{D}$ with identity feedthrough operator. The inverse of this spectral factor is $I + \mathcal{F}_\circlearrowleft$, where

$$(\mathcal{F}_\circlearrowleft u)(t) = K \int_{-\infty}^{t} \mathcal{A}_\circlearrowleft^{t-s} Bu(s)\, ds, \quad t \in \mathbf{R}. \tag{10}$$

The statement that $\mathcal{X} = I - \mathcal{F}$ is a spectral factor of $I - \mathcal{D}^*\mathcal{D}$ means that $\mathcal{X}$ is a bounded causal shift-invariant operator on $L^2(\mathbf{R}; U)$ with a bounded inverse, and that $\mathcal{X}^*\mathcal{X} = I + \mathcal{D}^*\mathcal{D}$; see Staffans [1997, 1998b], Weiss and Weiss [1997] for details. This theorem can be deduced from Theorem 1, but the approach used in Staffans [1998e] is the opposite one: first Theorem 2 is proved, and then Theorem 1 is derived from Theorem 2. The existence of a spectral factor $\mathcal{X}$ follows from a very general result: every bounded strictly positive time invariant operator on $L^2(\mathbf{R}; U)$ has a spectral factor [Staffans, 1997, Lemma 18]. Moreover, if we define $\mathcal{F} = I - \mathcal{X}$, then it is possible to show that there is a output map $\mathcal{K}$ such that $\left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \left[\begin{smallmatrix}\mathcal{C}\\ \mathcal{K}\end{smallmatrix}\right] & \left[\begin{smallmatrix}\mathcal{D}\\ \mathcal{F}\end{smallmatrix}\right]\end{smallmatrix}\right]$ is a stable well-posed linear system on (U, X, Y), that the closed loop system that we get by feeding the second output back into the input is also well-posed and stable, and that u^{opt} is equal to the second output of this closed loop system in the absence of an external input [Staffans, 1997, Theorem 27]. This also gives us the first estimate $\Pi \in \mathcal{L}(Z_{\alpha_C - 1/2+\epsilon}; Z^*_{-\alpha_C+1/2-\epsilon})$ on Π in (5). For this part of the theory the parabolic nature of (2) is irrelevant.

The parabolicity comes into play when we want to derive the algebraic Riccati equation (6), because for the derivation of this equation we need, for example, both $\mathcal{X}$ and $\mathcal{X}^*$ to be regular in the sense of Weiss [1994a]. If U is finite-dimensional, then it is easy to prove this regularity by appealing to a classical spectral factorization result in the Wiener algebra [Clancey and Gohberg, 1981, Theorem 6.3, p. 63 and Corollary 1.1, p. 75]. To carry out this step in the case where U is infinite-dimensional we use the boot-strap argument introduced by Lasiecka and Triggiani in [Lasiecka and Triggiani, 1983, pp. 52–53]. The same argument gives us the second estimate $\Pi \in \mathcal{L}(Z_{\alpha_C}; Z^*_{-\alpha_C+1-\epsilon})$ on Π in (5), and, via duality, the third estimate $\Pi \in \mathcal{L}(Z_{\alpha_C-1+\epsilon}; Z^*_{-\alpha_C})$. The claims about the analyticity of the closed loop semigroup $\mathcal{A}_\circlearrowleft$ follows from standard perturbation results for analytic generators [Lunardi, 1995, Propositions 2.2.15 and 2.4.1].

It is a interesting open problem to find additional examples on the existence of a regular spectral factor. See Curtain and Staffans [1998].

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