# The State/Signal Resolvent Functions

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(joint work with Damir Z. Arov)

This is a slightly abbreviated version of my talk at the Oberwolfach meeting "Spectral Theory and Weyl Functions" at January 9, 2015. In the talk itself I presented an additional example which illustrates the difference between the resolvent set of a state/signal system and the resolvent sets of its input/state/output representations.

## 1. Input/state/output systems in time and frequancy domain

One way to model the dynamics of an i/s/o (input/state/output) system is to use an equation of the following form, where $S\colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ is a closed linear operator:

$$(1) \qquad \Sigma_{\mathrm{iso}}\colon \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S)\,, \\[2mm] \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+, \quad x(0) = x^0.$$

Here $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces, $x(t) \in \mathcal{X}$ is the state, $u(t) \in \mathcal{U}$ is the input, and $y(t) \in \mathcal{Y}$ is the output. By a classical future trajectory of $\Sigma_{\mathrm{iso}}$ we mean a triple of functions $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ which satisfies (1) for all $t \in \mathbb{R}^+$, with $x$ continuously differentiable with values in $\mathcal{X}$ and $\begin{bmatrix} u \\ y \end{bmatrix}$ continuous with values in $\begin{bmatrix} \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$. Different classes of i/s/o systems of this type are described in [Sta05].

A general i/s/o system can be seen as an extension of a standard finite-dimensional i/s/o system. If $S$ is bounded, the $S$ can be written in block matrix form $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, and (1) becomes

$$(2) \qquad \Sigma_{\mathrm{iso}}\colon \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad t \in \mathbb{R}^+, \quad x(0) = x^0.$$

In this case we say that $A$ is the main operator, $B$ is the control operator, $C$ is the observation operator, $D$ is the feedthrough operator. The case where $A$ generates a $C_0$ semigroup and $B$, $C$, and $D$ are bounded is described in the book [CZ95].

Let $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ be a classical future trajectory which is, for example, bounded. Multiplying the equation (1) by $\mathrm{e}^{-\lambda t}$ and integrating over $\mathbb{R}^+$ we find that the Laplace transforms $\begin{bmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{bmatrix}$ of $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ satisfy

$$(3) \qquad \widehat{\Sigma}_{\mathrm{iso}}\colon \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \qquad \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \mathrm{dom}\,(S), \quad \lambda \in \Omega,$$

where $\Omega$ is the open right half-plane.

In the sequel we concentrate our attention on the equation (3), where we allow $\Omega$ to be and arbitrary open subset of $\mathbb{C}$. In the setting described above one natural

choice is to take $\Omega$ to be the open right half-plane. In the study of discrete time i/s/o system one natural choice is to take $\Omega$ to be the open unit disk.

We arrived at the frequency domain equation (3) by taking Laplace transforms in the time domain equation (1). In the original time domain setting $x^0$ was the initial state, $u$ was the input, $x$ the "final" state, and $y$ the output. The analogous interpretation in the frequency domain would be to interpret $x^0$ and $\hat{u}$ as "given data" and $\hat{x}$ and $\hat{y}$ as "dependent data". In other words,

(i) $x^0$ and $\hat{u}(\lambda)$ should be "free" in the sense that $x^0$ can be an arbitrary vector in $\mathcal{X}$ and $\hat{u}$ can be an arbitrary analytic function in $\Omega$ with values in $\mathcal{U}$;

(ii) $\hat{x}(\lambda)$ and $\hat{y}(\lambda)$ should be determined uniquely by $x^0$ and $\hat{u}(\lambda)$.

**Definition 1.** (i) A point $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(\Sigma)$ of $\Sigma$, or equivalently, to the i/s/o resolvent set $\rho_{i/s/o}(S)$ of $S$, if for every $x^0 \in \mathcal{X}$ and for every $\hat{u}(\lambda) \in \mathcal{U}$ there is a unique pair of vectors $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix}$ satisfying (3), and $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix}$ depends continuously on $\begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}$.

(ii) The $\mathcal{L}\left(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}\right)$-valued matrix function $\widehat{\mathfrak{S}}\colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ with domain $\rho(\Sigma)$ which at the point $\lambda$ maps $\begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}$ into $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix}$ is called the i/s/o resolvent matrix of $\Sigma$ (or of $S$).[1]

Since $\widehat{\mathfrak{S}}(\lambda) \in \mathcal{L}\left(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}\right)$ for every $\lambda \in \rho(\Sigma)$ this operator has a block matrix representation

$$(4) \qquad \widehat{\mathfrak{S}}(\lambda) := \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}, \qquad \lambda \in \rho(\Sigma).$$

The components of this operator are called as follows:

- $\widehat{\mathfrak{A}}$ is the s/s (state/state) resolvent function,
- $\widehat{\mathfrak{B}}$ is the i/s (input/state) resolvent function,
- $\widehat{\mathfrak{C}}$ is the s/o (state/output) resolvent function,
- $\widehat{\mathfrak{D}}$ is the i/o (input/output) resolvent function.

The different components of $\widehat{\mathfrak{S}}$ are known under different names in the literature. The operator $\widehat{\mathfrak{A}}$ is the standard resolvent of the main operator $A$ of the system, i.e., $\widehat{\mathfrak{A}}(\lambda) = (\lambda - A)^{-1}$, where $Ax = P_{\mathcal{X}} S \begin{bmatrix} x \\ 0 \end{bmatrix}$ with $\operatorname{dom}(A) = \left\{ x \in \mathcal{X} \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in \operatorname{dom}(S) \right\}$ ("top left corner" of $S$). If $\Sigma_{\mathrm{iso}}$ has been constructed from a conservative boundary triplet as described in [AKS12a, AKS12b], then $\widehat{\mathfrak{B}}$ is the so called "Gamma field" and $\widehat{\mathfrak{D}}$ is the "Weyl function". Two other names for $\widehat{\mathfrak{D}}$ are "the transfer function" (used in control theory) and the "characteristic function of the main operator" (used in operator theory).

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[1]It is, of course, possible to define $\widehat{\mathfrak{S}}(\lambda)$ also for $\lambda \notin \rho(\Sigma_{\mathrm{iso}})$ by means of its graph determined by (3). For such $\lambda$ the operator $\widehat{\mathfrak{S}}(\lambda)$ will still be closed, but it will be unbounded or multi-valued or not defined on all of $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$.

**Theorem 2.** *Let* $S \colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ *be an operator with dense domain. Then* $\rho_{i/s/o}(S) \neq \emptyset$ *if and only if $S$ is an "operator node" in the sense of* [Sta05, Definition 4.7.2].

**Lemma 3.**　　(i) *The i/s/o resolvent matrix* $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ *is analytic in* $\rho(\Sigma)$.

　　(ii) $\widehat{\mathfrak{S}}$ *satisfies the i/s/o resolvent identity*

$$
(5) \qquad \widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = (\mu - \lambda) \begin{bmatrix} \widehat{\mathfrak{A}}(\mu) \\ \widehat{\mathfrak{C}}(\mu) \end{bmatrix} \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \end{bmatrix}
$$

　　　*for all $\mu,\ \lambda \in \rho(\Sigma)$.*

　(iii) *If $\rho_{i/s/o}(S) \neq \emptyset$, then $\rho_{i/s/o}(S) = \rho(A)$, where $A$ is the main operator of $S$.*

Thus in particular, if $\rho_{i/s/o}(S) \neq \emptyset$, then $\rho(A) \neq \emptyset$. However, the condition $\rho(A) \neq \emptyset$ does not imply that $\rho_{i/s/o}(S) \neq \emptyset$.

## 2. State/signal systems in the time domain

One option to model the dynamics of an electrical circuit with lumped elements is to use a finite-dimensional i/s/o system of the following type. The state $x(t)$ is an $N$-vector whose components are the currents in the coils and the voltages over the capacitors. If the circuit has $M$ terminals, then we can, e.g., use the currents entering these terminals as inputs, and the voltages over the terminals as the outputs. The equation (1) describing the dynamics of the system can be deribed from the Kirchoff's and Ohm's laws. However, we could just as well have picked the voltages to be the inputs and the currents to be the outputs. This would give a different i/s/o system, but *the underlying physical system remains the same!* Thus, every electrical circuit can be used to construct *an infinite family of i/s/o systems* (by choosing different combinations of voltages and currents as inputs and outputs).

The following question arises: *Is there a simple equation which describes the curcuit itself* (instead of an infinite family of i/s/o systems)?

Another special case of an infinite-dimensional i/s/o system is the following *boundary control system*:

$$
(6) \qquad \Sigma_{\text{iso}} \colon \begin{cases} \dot{x}(t) = Lx(t), \\ \Gamma_0 x(t) = u(t), \qquad t \in \mathbb{R}^+, \quad x(0) = x^0. \\ \Gamma_1 x(t) = y(t), \end{cases}
$$

Here $L$ is, e.g., a partial differential operator in some Lipschitz domain in $\mathbb{R}^n$, and $\Gamma_0$ and $\Gamma_1$ are two boundary mappings, e.g., $\Gamma_0 =$ Neumann trace and $\Gamma_1 =$ Dirichlet trace. See [Sta05] for details. Above we may interpret $u$ as the input and $y$ the output, or the other way around. Or we could replace $\Gamma_0$ and $\Gamma_1$ by some other boundary mappings. *Different choices of inputs and outputs lead completely different i/s/o system of the type* (1). Thus, to every boundary control system of the type (6) there corresponds an *infinite family of i/s/o systems*.

The same question as above rises in this case, too: Is there a simple way to describe the boundary control system itself (instead of using an infinite family of i/s/o systems)?

In the case of the boundary control system (6) the solution is obvious: We simply combine the two variables $u$ and $y$ into a common "interaction" signal $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ which contains both the input and the output, define $\Gamma = \left[\begin{smallmatrix} \Gamma_0 \\ \Gamma_1 \end{smallmatrix}\right]$, and write (6) in the form

$$(7) \qquad \Sigma_{\text{iso}} : \begin{cases} \dot{x}(t) = Lx(t), \\ \Gamma x(t) = w(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x^0.$$

In the case of the more general i/s/o system (1) the solution is less obvious, but such a solution still exists. One way to proceed is the following. We again take the signal space $\mathcal{W}$ to be the product space $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, move the output equation in (1) into the domain of a new generator $F : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \to \mathcal{X}$ (whose domain no longer is dense in $\mathcal{W}$), and rewrite (1) in the form

$$(8) \qquad \Sigma : \begin{cases} \left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right] \in \text{dom}\,(F), \\[2mm] \dot{x}(t) = F\left(\left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right]\right), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x^0,$$

where the *state/signal generator* $F$ is given by

$$(9) \qquad \begin{aligned} \text{dom}\,(F) &= \left\{ \left[\begin{smallmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \text{dom}\,(S), \; y_0 = \begin{bmatrix} 0 & 1_{\mathcal{Y}} \end{bmatrix} S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \right\}, \\ F\left[\begin{smallmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{smallmatrix}\right] &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]. \end{aligned}$$

The above representation can be further "simplified" by using the *graph representation of* (1). We still take $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, let $\mathfrak{K}$ be the product space $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$, and rewrite (8) in the form

$$(10) \qquad \Sigma : \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t \in \mathbb{R}^+, \quad x(0) = x^0,$$

where the *generating subspace* $V$ is the (reordered) graph of $S$ (or of $F$):

$$(11) \qquad \begin{aligned} V &= \left\{ \left[\begin{smallmatrix} z_0 \\ x_0 \\ u_0 \\ y_0 \end{smallmatrix}\right] \in \mathfrak{K} \,\middle|\, \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \text{dom}\,(S), \; \left[\begin{smallmatrix} z_0 \\ y_0 \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \right\} \\ &= \left\{ \left[\begin{smallmatrix} z_0 \\ x_0 \\ u_0 \\ y_0 \end{smallmatrix}\right] \in \mathfrak{K} \,\middle|\, \left[\begin{smallmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{smallmatrix}\right] \in \text{dom}\,(F), \; z_0 = F\left[\begin{smallmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{smallmatrix}\right] \right\}. \end{aligned}$$

A *classical future trajectory* of (8) or (10) is a pair of continuous functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$, with $x$ continuously differentiable, which satisfies (8) or (10). It follows from the above construction that $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a classical future trajectory of the i/s/o system $\Sigma_{\text{iso}}$ if and only if $\left[\begin{smallmatrix} x, \\ \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \end{smallmatrix}\right]$ is a classical future trajectory of the corresponding s/s system $\Sigma$.

## 3. The s/s resolvent set and the characteristic node bundle

If $x$, $\dot{x}$, and $w$ in (10) are Laplace transformable, then it follows from (10) (since we assume $V$ to be closed) that the Laplace transforms $\hat{x}$ and $\hat{w}$ of $x$ and $w$ satisfy

$$(12) \qquad \widehat{\Sigma}: \begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V, \qquad \lambda \in \Omega,$$

where $\Omega$ is some open right half-plane. (To prove this it suffices to multiply by $e^{-\lambda t}$ and integrate by parts in the $\dot{x}$-component.) This formula can be rewritten in the form

$$(13) \qquad \begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) := \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V.$$

**Definition 4.** The above family of subspaces $\widehat{\mathfrak{E}} : \{\widehat{\mathfrak{E}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathfrak{K} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ is called the *characteristic node bundle of* $\Sigma$. We refer to each of the subspaces $\widehat{\mathfrak{E}}(\lambda)$ as the *fiber of* $\widehat{\mathfrak{E}}$ *at the point* $\lambda \in \mathbb{C}$.

Thus, $\widehat{\mathfrak{E}}$ is an *"analytic subspace-valued function"* defined on $\mathbb{C}$.

It follows from (8) and (10) that (3) holds for some $\hat{x}(\lambda)$, $\hat{u}(\lambda)$, and $\hat{y}(\lambda)$ if and only if (12) holds with $\hat{w}(\lambda) = \begin{bmatrix} \hat{u}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix}$. If $\lambda \in \rho_{\mathrm{iso}}(S)$, then

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \widehat{\mathfrak{S}}(\lambda) \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix},$$

which can be rewritten in the form

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \begin{bmatrix} \hat{u}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \begin{bmatrix} 0 \\ \widehat{\mathfrak{C}}(\lambda) \end{bmatrix} & \begin{bmatrix} 1_{\mathcal{U}} \\ \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix}.$$

Here $\begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ can be arbitrary, and we get the following result:

**Lemma 5.** *Let $\Sigma_{\mathrm{iso}}$ be an i/s/o representation of the s/s system $\Sigma$, and suppose that $\lambda \in \rho(\Sigma_{\mathrm{iso}})$. Then the fiber $\widehat{\mathfrak{E}}(\lambda)$ of the characteristic node bundle $\widehat{\mathfrak{E}}$ at $\lambda$ has the representation*

$$(14) \qquad \widehat{\mathfrak{E}}(\lambda) = \mathrm{im}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \begin{bmatrix} 0 \\ \widehat{\mathfrak{C}}(\lambda) \end{bmatrix} & \begin{bmatrix} 1_{\mathcal{U}} \\ \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \end{bmatrix}\right)$$

Note that this can be interpreted as a *graph representation of* $\widehat{\mathfrak{E}}(\lambda)$ *over the first copy of* $\mathcal{X}$ *and the input space* $\mathcal{U}$.

Suppose that $\lambda \in \rho(\Sigma_{\mathrm{iso}})$ for some i/s/o representation $\Sigma_{\mathrm{iso}}$ of $\Sigma$. Then it is easy to see that $\widehat{\mathfrak{E}}(\lambda)$ has the following properties:

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(i) $\begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \Rightarrow x = 0$;

(ii) For every $z \in \mathcal{X}$ there exists some $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ such that $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda)$.

(iii) The projection of $\widehat{\mathfrak{E}}(\lambda)$ onto its first and third components is closed in $\begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$.

**Definition 6.** Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node with node bundle $\widehat{\mathfrak{E}}$. Then the *resolvent set* $\rho(\Sigma)$ of $\Sigma$ consists of all those points $\lambda \in \mathbb{C}$ for which conditions (i)–(iii) above hold.

**Theorem 7.** *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. Then $\rho(\Sigma)$ is the union of the resolvent sets of all i/s/o representations of $\Sigma$.*

## 4. The characteristic signal bundle

In i/s/o systems theory one is often intersted in the "pure i/o behavior", which one gets by "ignoring the state". More precisely, one takes the inital state $x_0 = 0$, and only looks at the relationship between the input $u$ and the output $y$, ignoring the state $x$. If we take $x_0 = 0$ in (3) and ignore $\hat{x}$, then the full frequency domain relation $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix}$ is replaced by the i/o relation $\hat{y}(\lambda) = \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda)$, where $\widehat{\mathfrak{D}}(\lambda)$ is the i/o resolvent function of $\Sigma_{\mathrm{iso}}$.

The same procedure can be carried out in the case of a s/s system: Taking $x_0 = 0$ and ignoring the value of $\hat{x}(\lambda)$ in (13) we see that $\hat{w}(\lambda) \in \widehat{\mathfrak{F}}(\lambda)$, where

$$(15) \qquad \widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \;\middle|\; \begin{bmatrix} 0 \\ z \\ w \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \text{ for some } z \in \mathcal{X} \right\}.$$

**Definition 8.** The family of subspaces $\widehat{\mathfrak{F}} : \{\widehat{\mathfrak{F}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathcal{W}$ is called the *characteristic signal bundle*. We refer to each of the subspaces $\widehat{\mathfrak{F}}(\lambda)$ as the *fiber of* $\widehat{\mathfrak{F}}$ *at the point* $\lambda \in \mathbb{C}$.

Whereas the characteristic node bundle $\widehat{\mathfrak{E}}$ is analytic everywhere in $\mathbb{C}$ (i.e., the fibers depend on $\lambda$ in an analytic way), the same is not true for the singal bundle $\widehat{\mathfrak{F}}$. Even the dimension of the fibers $\widehat{\mathfrak{F}}(\lambda)$ may change from one point to another. However, the following result is true:

**Lemma 9.** *The characteristic signal bundle $\widehat{\mathfrak{F}}$ is analytic in $\rho(\Sigma)$.*

## 5. Details and Proofs

An introduction to what I have been explaining above is written down in [Sta14]. Proofs are given in [AS15]. The connection to boundary triplets and generalized boundary triplets is explained in [AKS12a, AKS12b].

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