Well-Posed State/Signal Systems in Continuous Time

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***Abstract*— We introduce the notion of a well-posed linear state/signal system in continuous time, thereby complementing the corresponding discrete time theory developed by Damir Arov and the second author. A linear state/signal system has a state space $\mathcal{X}$ and a signal space $\mathcal{W}$, where the state space acts like an internal memory, and the signal space allows interactions with the surrounding world. A state/signal system resembles an input/state/output system apart from the fact that inputs and outputs are not separated from each other. This system is well-posed if it is possible to decompose the signal space $\mathcal{W}$ into a direct sum of an input space $\mathcal{U}$ and an output space $\mathcal{Y}$ so that this decomposition results in a well-posed input/state/output system. Such a decomposition of $\mathcal{W}$ is called an *admissible input/output pair*. We give different characterizations of well-posedness and input/output admissibility, in terms of either classic or generalized trajectories of the system. Finally we mention some work in progress.**

A *classical trajectory* on a time interval $[0, T]$ of a *linear continuous time-invariant s/s (= state/signal) node* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with a Banach *state space* $\mathcal{X}$ and a Banach *signal space* $\mathcal{W}$ consists of a continuously differentiable state component $x(t) \in \mathcal{X}$ and a continuous signal component $w(t) \in \mathcal{W}$. The evolution of these trajectories is determined by a condition of the type

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in [0, T], \qquad x(0) = x_0, \tag{1}$$

where x_0 is a given initial state at time zero and V is the so called *generating subspace* of Σ. We throughout require V to satisfy (at least) the following three conditions:

(i) V is a closed subspace of $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$;
(ii) If $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$ then $z = 0$;
(iii) There is a $T > 0$ such that for each $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$ there exists at least one classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on $[0, T]$ with $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$.

Classical trajectories on the time interval $[0, \infty)$ are defined in the same way.

For the rest of this note we fix some $p \in [1, \infty)$. By a *generalized trajectory* of Σ on the time interval $[0, T]$ we mean a pair of functions $x \in C([0, T]; \mathcal{X})$ and $w \in L^p([0, T]; \mathcal{W})$ which can be approximated by a sequence of classical trajectories $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ in the sense that $x_n \to x$ in $C([0, T]; \mathcal{X})$ and $w_n \to w \in L^p([0, T]; \mathcal{W})$ as $n \to \infty$. By a generalized trajectory of Σ on the time interval $[0, \infty)$ we mean a pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ whose restriction to any finite interval $[0, T]$ is a generalized trajectory on $[0, T]$. We denote the set of classical trajectories of Σ on $[0, T]$ and $[0, \infty)$ by $\mathfrak{V}[0, T]$ and $\mathfrak{V}$, respectively, and the set of generalized trajectories of Σ on $[0, T]$ and $[0, \infty)$ by $\mathfrak{W}[0, T]$ and $\mathfrak{W}$, respectively. Finally, $\mathfrak{V}_0[0, T] = \left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0, T] \;\middle|\; \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} = 0 \right\}$ and $\mathfrak{W}_0[0, T] = \left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[0, T] \;\middle|\; x(0) = 0 \right\}$.

In Definition 1 below we introduce the notion of a *well-posed* s/s node. We begin by decomposing the signal space $\mathcal{W}$ of Σ into a direct sum $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$. Let $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}$ be the projection in $\mathcal{W}$ onto $\mathcal{U}$ along $\mathcal{Y}$. For each $w \in L^p([0, T]; \mathcal{W})$ we define $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ point-wise, i.e., $(\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w)(t) = \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(t)$ almost everywhere.

Definition 1: The s/s node $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is *well-posed* if there exists a $T > 0$ and a direct sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of $\mathcal{W}$ such that the following three conditions hold:

(iv) The set $\{x(0) \mid \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0, T]\}$ is dense in $\mathcal{X}$;
(v) The set $\{\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \mid \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}_0[0, T]\}$ is dense in $L^p([0, T]; \mathcal{U})$;
(vi) there exists a finite constant K such that all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}([0, T])$ satisfy

$$\begin{aligned} &\|x(t)\|_{\mathcal{X}} + \|w\|_{L^p([0,t];\mathcal{W})} \\ &\qquad \le K\big(\|x(0)\|_{\mathcal{X}} + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|_{L^p([0,t];\mathcal{U})}\big) \end{aligned} \tag{2}$$

for all $t \in [0, T]$.

A decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of $\mathcal{W}$ satisfying conditions (v) and (vi) above for some $T > 0$ is called an *admissible* i/o (input/output) pair for Σ.

In general a well-posed s/s node has more than one admissible i/o pair. The following result can be used to test when a given decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ is admissible for Σ.

Theorem 2: Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a well-posed s/s node, and let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$. Then the following conditions are equivalent:

(i) $(\mathcal{U}, \mathcal{Y})$ is an admissible i/o pair for Σ, i.e., conditions (v) and (vi) in Definition 1 hold for some $T > 0$;

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(ii) Conditions (v) and (vi) in Definition 1 hold for all $T > 0$;

(iii) The map $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \to \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ is a bijection from $\mathfrak{W}_0[0,T]$ to $L^p([0,T];\mathcal{U})$ for some $T > 0$;

(iv) The map $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \to \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ is a bijection from $\mathfrak{W}_0[0,T]$ to $L^p([0,T];\mathcal{U})$ for all $T > 0$.

It can be shown that if $\Sigma = (V;\mathcal{X},\mathcal{W})$ is well-posed, then $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0[0,T]$ for all $T > 0$. A slightly weaker version of this condition is used in our following result, which characterizes well-posedness and admissibility in terms of generalized trajectories (as opposed to the set $\mathfrak{V}[0,T]$ of classical trajectories used in Definition 1).

Theorem 3: Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a s/s node, i.e., suppose that V satisfies conditions (i)–(iii) listed below (1). In addition suppose that $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0[0,T]$ for some $T > 0$. Let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$. Then the following conditions are equivalent:

(i) Σ is well-posed and $(\mathcal{U},\mathcal{Y})$ is an admissible i/o pair for Σ;

(ii) there exists some $T > 0$ such that the map $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \to (x(0), \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w)$ is a bijection from $\mathfrak{W}[0,T]$ to $\mathcal{X} \times L^p([0,T];\mathcal{U})$.

(iii) there exists some $T > 0$ such that

 (a) for each $x_0 \in \mathcal{X}$ there exists at least one $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[0,T]$ such that $x(0) = x_0$;

 (b) the map $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \to \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ is a bijection from $\mathfrak{W}_0[0,T]$ to $L^p([0,T];\mathcal{U})$.

Clearly the generating subspace V determines both the set of classical trajectories $\mathfrak{V}[0,T]$ and the set of generalized trajectories $\mathfrak{W}[0,T]$ of Σ on $[0,T]$ uniquely. A partial converse is true: $\mathfrak{V}[0,T]$ determines V uniquely since it can be shown that $V = \left\{ \left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T] \right\}$.

Apparently it need not be true that V is uniquely determined by the set of generalized trajectories $\mathfrak{W}[0,T]$ or $\mathfrak{W}$ on $[0,T]$ or $[0,\infty)$. However, in many cases these families of generalized trajectories are more important than the corresponding families of classical trajectories. We therefore introduce the notion of a well-posed state signal *system* $\Sigma = (\mathfrak{W};\mathcal{X},\mathcal{W})$, by which we mean the *family of generalized trajectories* $\mathfrak{W}$ *on* $[0,\infty)$ *of some well-posed s/s node* with state space $\mathcal{X}$ and signal space $\mathcal{W}$. Thus, a well-posed linear system $\Sigma = (\mathfrak{W};\mathcal{X},\mathcal{W})$ may be generated by more than one well-posed s/s node $(V;\mathcal{X},\mathcal{W})$. We prove that among these nodes there always exists a unique *maximal* one $(V';\mathcal{X},\mathcal{W})$. Here maximality means that if both $(V;\mathcal{X},\mathcal{W})$ and $(V';\mathcal{X},\mathcal{W})$ generate the same system $(\mathfrak{W};\mathcal{X},\mathcal{W})$, then necessarily $V \subset V'$. We show that $(V;\mathcal{X},\mathcal{W})$ is maximal if and only if it is true that every generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ which has the smoothness of a classical trajectory is actually classical.

We next describe the relationship between well-posed s/s systems and the class of well-posed i/s/o (input/state/output) systems found in, e.g., [Sta05]. Let $\Sigma = (V;\mathcal{X},\mathcal{W})$ be a well-posed s/s node, and let $(\mathcal{U},\mathcal{Y})$ be an admissible i/o pair for Σ. Then the admissibility of the decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ implies that for each $x_0 \in \mathcal{X}$ and each $u \in L^p_{\mathrm{loc}}([0,\infty);\mathcal{U})$ there is a unique generalized trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ of Σ on $[0,\infty)$ such that $x(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = u$. Moreover, the trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ depends continuously on x_0 and u, and the map $(x_0, u) \to (x, \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w)$ defines a well-posed i/s/o system $\Sigma_{i/s/o}$ in the sense of [Sta05], with $\mathcal{U}$ as input space and $\mathcal{Y}$ as output space. We call this $\Sigma_{i/s/o}$ the *i/s/o representation of* Σ corresponding to the i/o pair $(\mathcal{U},\mathcal{Y})$. The converse is also true: To each well-posed i/s/o system $\Sigma_{i/s/o}$ with a Banach input space $\mathcal{U}$ and a Banach output space $\mathcal{Y}$ there corresponds a unique well-posed s/s node $\Sigma = (V;\mathcal{X},\mathcal{U} \times \mathcal{Y})$ such that V is maximal and such that $\Sigma_{i/s/o}$ is the i/s/o representation of Σ corresponding to the i/o pair $(\mathcal{U},\mathcal{Y})$. The maximal generating subspace V can be interpreted as the graph of the system node which generates $\Sigma_{i/s/o}$. (We refer the reader to [Sta05] for the definition of a system node.)

Above we have introduced the notion of an i/s/o representation of a well-posed s/s system. Two additional types of useful representations exist, namely *driving-variable representations* and *output-nulling representations*. A driving-variable representation is a well-posed i/s/o system in which an additional driving input variable is used to generate all the generalized trajectories of Σ (i.e., we get a trajectory of Σ by simply dropping the driving variable). An output-nulling representation is another well-posed i/s/o system which produces generalized trajectories of Σ whenever an additional output error variable vanishes. These representations can be used in the same way as the corresponding discrete time driving-variable and output-nulling representations were used in [AS05]–[AS07c].

Work in progress includes the study of interconnections of well-posed s/s systems and the theory of continuous time passive well-posed s/s systems in the spirit of [AS07a]–[AS07c].

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