

J-Energy Preserving Well-Posed Linear Systems

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Abstract

This is a short survey of the notion of a well-posed linear system. We start by describing the most basic concepts, proceed to discuss dissipative and conservative systems, and finally introduce *J*-energy preserving systems, i.e., systems that preserve energy with respect to some generalized inner products (possibly semi-definite or indefinite) in the input, state, and output spaces. The class of well-posed linear systems contains most linear time-independent distributed parameter systems: internal or boundary control of PDE:s, integral equations, delay equations, etc. These systems have existed in an implicit form in the mathematics literature for a long time, and they are closely connected to the scattering theory by Lax and Phillips and to the model theory by Sz.-Nagy and Foias. The theory has been developed independently by many different schools, and only recently have these different approaches begun to converge. One of the most interesting objects of present study is the Riccati equation theory for this class of infinite-dimensional systems (H^2 - and H^∞ -theory).

Keywords

Well-posed linear system, system node, transfer function, Lax–Phillips semi-group, dissipative system, conservative system, model theory, conservative realization, *J*-energy preserving system, Lyapunov equation, Riccati equation.

1 Well-Posed Linear Systems

Many infinite-dimensional linear time-independent continuous-time systems can be described by the equations

$$\begin{aligned}x'(t) &= Ax(t) + Bu(t), \\y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\x(0) &= x_0,\end{aligned}\tag{1}$$

on a triple of Hilbert spaces, namely, the input space U , the state space X , and the output space Y . We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operator

A is supposed to be the generator of a strongly continuous semigroup $t \mapsto \mathfrak{A}^t$. The generating operators A , B and C are usually unbounded, whereas D is always bounded.

By modifying this set of equations slightly we get the notion of a *well-posed linear system*. In the sequel, we think about the block matrix $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ as *one single (unbounded) operator* from $X \times U$ to $X \times Y$, and write (1) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0. \quad (2)$$

The operator S completely determines the system. Thus, we may identify the system with such an operator, which we call the *node* of the system.

There are some *necessary conditions* which a node S must satisfy in order to generate a well-posed linear system. First of all, S must be closed and densely defined as an operator from $X \times U$ into $X \times Y$. Let us denote the domain of S by $\mathcal{D}(S)$. Then S can be split into $S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$, where S_1 maps $\mathcal{D}(S)$ into X and S_2 maps $\mathcal{D}(S)$ into Y . By analogy to the finite-dimensional case, let us denote $S_1 = A\&B$ and $S_2 = C\&D$, so that $S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ (the reader who finds this notation confusing may throughout replace $A\&B$ by S_1 and $C\&D$ by S_2). It is not true, in general, that $A\&B$ and $C\&D$ (defined on $\mathcal{D}(S)$) can be decomposed into $A\&B = \begin{bmatrix} A & B \end{bmatrix}$ and $C\&D = \begin{bmatrix} C & D \end{bmatrix}$; this is possibly only in the case where $\mathcal{D}(S)$ can be written as the product of one subset of X times another subset of U . However, as we shall see below, certain extended versions of $A\&B$ and $C\&D$ can be decomposed as indicated above, so that $A\&B$ and $C\&D$ are the restrictions to $\mathcal{D}(S)$ of $\begin{bmatrix} A & B \end{bmatrix}$ respectively $\begin{bmatrix} C & D \end{bmatrix}$ for suitably defined operators A , B , C , and D . The first of these two decompositions is more fundamental than the second, so we have incorporated it in the following definition of a *system node* (the second decomposition is commented on after Definition 1.4).

Definition 1.1. We call S a *system node* on the three Hilbert spaces (U, X, Y) if it satisfies condition (S) below:

- (S) $S := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : X \times U \supset \mathcal{D}(S) \rightarrow X \times Y$ is a closed linear operator. Here $A\&B$ is the restriction to $\mathcal{D}(S)$ of $\begin{bmatrix} A & B \end{bmatrix}$, where A is the *generator of a C_0 semigroup*, which has been extended to an operator in $\mathcal{L}(X; X_{-1})$, where X_{-1} is the completion of X under the norm $\|x\|_{X_{-1}} := \|(\alpha I - A)^{-1}x\|_X$ (α is an arbitrary number in the resolvent set of A). The operator B is an arbitrary operator in $\mathcal{L}(U; X_{-1})$, and $C\&D$ is an arbitrary linear operator from $\mathcal{D}(S)$ to Y . In addition, we require that

$$\mathcal{D}(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times U \mid Ax + Bu \in X \right\}.$$

In the sequel we shall simply write $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and ignore the fact that $\begin{bmatrix} A & B \end{bmatrix}$ is defined on all of $X \times U$ (with values in X_{-1}) and not just on $\mathcal{D}(S)$ (with values in X). Often X_{-1} is defined in a different but equivalent way as the dual of $\mathcal{D}(A^*)$ (we identify the dual of X with X itself). Let us also remark that we can replace the assumption that S is closed by the equivalent assumption that $C\&D$ is *continuous from $\mathcal{D}(S)$ (with the graph norm) to Y* .

As the following lemma shows, every system node induces a “dynamical system” of a certain type:

Lemma 1.2. *Let S be a system node on (U, X, Y) . Then, for each $x_0 \in X$ and $u \in W^{2,1}(\mathbf{R}^+; U)$ with $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathcal{D}(S)$, the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0, \quad (3)$$

has a unique solution (x, y) satisfying $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S)$ for all $t \geq 0$, $x \in C^1(\mathbf{R}^+; X)$, and $y \in C(\mathbf{R}^+; Y)$.

Definition 1.3. By the *linear system* Σ generated by a system node S we understand the family Σ of maps defined by

$$\Sigma_0^t \begin{bmatrix} x_0 \\ \pi_{[0,t]}u \end{bmatrix} := \begin{bmatrix} x(t) \\ \pi_{[0,t]}y \end{bmatrix},$$

parametrized by $t \geq 0$, where x_0 , $x(t)$, u , and y are as in Lemma 1.2 and $\pi_{[0,t]}u$ and $\pi_{[0,t]}y$ are the restrictions of u and y to $[0, t]$. The *transfer function* of Σ is defined by

$$\widehat{\mathfrak{D}}(z) := C \& D \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}, \quad z \in \rho(A).$$

By taking *Laplace transforms* in (3) we find that if u is Laplace transformable with transform \hat{u} , then the output y is also Laplace transformable with transform

$$\hat{y}(z) = C(zI - A)^{-1}x_0 + \widehat{\mathfrak{D}}(z)\hat{u}(z)$$

for $\Re z$ large enough. Here

$$Cx := C \& D \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ for } x \in \mathcal{D}(A).$$

Thus, our definition of the transfer function is equivalent to the standard definition in the classical case.

So far we have defined Σ_0^t only for the class of smooth data given in Lemma 1.2. In order to extend Σ_0^t to a larger class of data we need an extra *well-posedness assumption*.

Definition 1.4. A system Σ generated by a system node S is *well-posed* if the following additional condition holds:

(WP) For some $t > 0$ there is a finite constant $K(t)$ such that the solution (x, y) in Lemma 1.2 satisfies

$$\|x(t)\|^2 + \|y\|_{L^2(0,t)}^2 \leq K(t)(\|x_0\|^2 + \|u\|_{L^2(0,t)}^2).$$

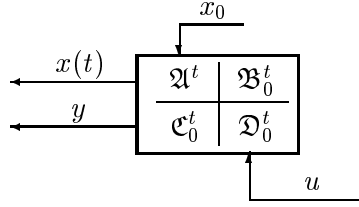


Figure 1: Input/state/output diagram of Σ

It is not difficult to show that if this condition holds for *one* $t > 0$, then it holds of *all* $t > 0$. If a system is well-posed, then Σ can be *extended by continuity* to a family of operators

$$\Sigma_0^t := \left[\begin{array}{c|c} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \hline \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{array} \right]$$

from $X \times L^2([0, t]; U)$ to $X \times L^2([0, t]; Y)$. (We still denote the extended family by Σ .)

As shown in [42], the system node S of a *well-posed system* can be always split into

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A: X \rightarrow X_{-1}$, $B: U \rightarrow X_{-1}$, $C: W \rightarrow Y$, and $D: U \rightarrow Y$. Here

$$\mathcal{D}(A) =: X_1 \subset W \subset X \subset X_{-1}$$

and $W = (\alpha I - A)^{-1}(X + BU)$ where $\alpha \in \rho(A)$. The operators A and B are unique, but C and D are *not always unique* (only the restriction of C to $X_1 = \mathcal{D}(A)$ is unique). The operators A , B , and C may be *unbounded*, but D is always bounded. If Σ is well-posed, then $\widehat{\mathfrak{D}}$ is bounded on some right half-plane, and $\widehat{\mathfrak{D}}$ can be written in the familiar form

$$\widehat{\mathfrak{D}}(z) = C(zI - A)^{-1}B + D, \quad z \in \rho(A).$$

For more details, explanations and examples we refer the reader to [1], [2, 3], [4], [10], [27, 28], [33, 34, 35, 36, 38, 40], [42, 43], [46, 47, 48, 49, 50, 51] and [55] (and the references therein). Note that different groups of authors use different notations. For example, in an alternative set of notations (introduced by George Weiss) the system itself and the transfer function are denoted by

$$\Sigma_t := \left[\begin{array}{c|c} \mathbb{T}_t & \mathbb{F}_t \\ \hline \mathbb{V}_t & \mathbb{F}_t \end{array} \right] := \left[\begin{array}{c|c} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \hline \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{array} \right], \quad \mathbf{G}(s) := \widehat{\mathfrak{D}}(s).$$

In the notation of Grabowski and Callier [18] the system node and the transfer function are denoted by

$$\begin{bmatrix} A & Ad \\ \hline c^t & 0 \end{bmatrix} := S, \quad \hat{g}(s) := \widehat{\mathfrak{D}}(s).$$

Translating the notation used by Arov and Nudelman [4] to our notation we obtain

$$\begin{aligned} \alpha &:= \Sigma, & \mathfrak{N}^- &:= U, & \mathfrak{N}^+ &:= Y, \\ \begin{bmatrix} B & L \\ M & K \end{bmatrix} &:= \begin{bmatrix} A & B \\ C & D \end{bmatrix}, & N &:= C \& D, & \theta_\alpha(s) &:= \widehat{\mathfrak{D}}(-is). \end{aligned}$$

In addition to the references mentioned above there is there is a large range of relevant literature that *we do not cite explicitly*. This applies, in particular, to most of the *Russian literature*, represented by Adamjan, Arov, Brodskiĭ, Kreĭn, Livšic, Nudelman, Potapov, and Šmulijan, among others. Another large block of relevant results is found in the western *operator theory* and *scattering theory* group, represented by Ball, de Branges, Douglas, Lax, Fuhrman, Helson, Helton, Phillips, Rosenblum, and Rovnyak. A third group of missing results are those that have been developed in *stochastic identification theory*, represented by Byrnes, Georgiou, Gilliam, Lindquist, and Picci. (The above lists of researchers are far from complete.)

2 Lax–Phillips Scattering

A generalized Lax–Phillips scattering model is a semigroup \mathfrak{T} defined on

$$\mathcal{Y} \times X \times \mathcal{U} = L^2(\mathbf{R}^-; Y) \times X \times L^2(\mathbf{R}^+; U)$$

with certain additional properties. We call \mathcal{U} the incoming subspace, X the central state space, and \mathcal{Y} the outgoing subspace. In the classical cases treated in [21, 22] \mathfrak{T} is required to be unitary (the conservative case) or a contraction semigroup (the dissipative case). Below we use the following notation:

$$\begin{aligned} (\pi_J u)(s) &:= \begin{cases} u(s), & s \in J, \\ 0, & \text{otherwise.} \end{cases} \\ (\pi_+ u)(s) &:= \begin{cases} u(s), & s \in \mathbf{R}^+, \\ 0, & s \in \mathbf{R}^-, \end{cases} \\ (\tau^t u)(s) &:= u(t+s), \quad s, t \in \mathbf{R}. \\ \tau_+^t &:= \pi_+ \tau^t, \quad t \geq 0, \end{aligned}$$

Theorem 2.1. *Let $\mathcal{Y} = L^2(\mathbf{R}^-; Y)$ and $\mathcal{U} = L^2(\mathbf{R}^+; U)$. For all $t \geq 0$ we define on $\mathcal{Y} \times X \times \mathcal{U}$ the operator \mathfrak{T}^t by*

$$\mathfrak{T}^t = \begin{bmatrix} \tau^t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tau_+^t \end{bmatrix} \begin{bmatrix} I & \mathfrak{C}_0^t & \mathfrak{D}_0^t \\ 0 & \mathfrak{A}^t & \mathfrak{B}_0^t \\ 0 & 0 & I \end{bmatrix}. \quad (4)$$

Then \mathfrak{T} is a strongly continuous semigroup. If x and y are the state trajectory and the output function of Σ with initial state $x_0 \in X$ and input function $u_0 \in \mathcal{U}$, and if we define $y(t) = y_0(t)$ for $t < 0$, then for all $t \geq 0$,

$$\begin{bmatrix} \tau^t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tau^t \end{bmatrix} \begin{bmatrix} \pi_{(-\infty, t]} y \\ x(t) \\ \pi_{[t, \infty)} u_0 \end{bmatrix} = \mathfrak{T}^t \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix}. \quad (5)$$

Formula (5) shows that at any time $t \geq 0$, the first component of $\mathfrak{T}_t \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix}$ represents the past output, the second component represents the present state, and the third component represents the future input.

The preceding theorem is taken from [42], and it is also found in [40]. Special cases of this result (where either the input or the output is missing) appear in [12] and [17]. The roots of Theorem 2.1 are very old, and the preceding references represent only *a fraction of all the relevant ones*. The setting which we describe above corresponds to the one found in *orthogonal scattering theory*. *Non-orthogonal scattering* has also been studied intensively. This case is important in, for example, the state space construction in stochastic identification theory.

3 Dissipative and Conservative Systems

Below we shall interpret the words “dissipative” and “conservative” in a rather restricted sense. Many authors use these words to represent (some subclass of) the more general classes of (R, P, J) -dissipative and (R, P, J) -conservative systems that we define in Section 5. In particular, the following definition is a special case of the definitions in the two classical papers [56, 57] by Willems.

Definition 3.1. A system Σ generated by a system node S is *dissipative* if the following *energy inequality* holds:

(D) For all $t > 0$, the solution (x, y) in Lemma 1.2 satisfies

$$\|x(t)\|^2 + \|y\|_{L^2(0,t)}^2 \leq \|x_0\|^2 + \|u\|_{L^2(0,t)}^2.$$

Thus, *every dissipative system is well-posed*: The dissipativity inequality (D) implies the well-posedness inequality (WP). Physically, dissipativity means that *there are no internal energy sources*.

Theorem 3.2. *The following conditions are equivalent:*

1. Σ is dissipative.
2. The corresponding Lax–Phillips model is a contraction semigroup.
3. For all $t > 0$, the operator $\Sigma_0^t = \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{bmatrix}$ is a contraction from $X \times L^2([0, t]; U)$ to $X \times L^2([0, t]; Y)$.

Dissipativity can also be characterized by some algebraic operator inequalities involving the system node $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ (which are **Linear Matrix Inequalities** in the finite-dimensional case). See [42] for the general case and [57] for the matrix case.

Definition 3.3. A system Σ generated by a system node S is *energy-preserving* if the following *energy balance equation* holds:

(E) For all $t > 0$, the solution (x, y) in Lemma 1.2 satisfies

$$\|x(t)\|^2 + \|y\|_{L^2(0,t)}^2 = \|x_0\|^2 + \|u\|_{L^2(0,t)}^2.$$

Note that every energy conserving system is dissipative, hence well-posed. Physically, a system is energy preserving if *there are no internal energy sources or sinks*.

Definition 3.4. A system Σ generated by a system node S is *conservative* if both the original system and the dual system are energy preserving.

The *dual system* is the one generated by S^* . (If S is a system node, then so is S^* .) A finite-dimensional system is *conservative if and only if it is energy preserving and $U = Y$* . Some related (but more complicated) results are true also in infinite-dimensions.

Theorem 3.5. *The following conditions are equivalent:*

1. Σ is conservative.
2. The corresponding Lax–Phillips model is a unitary semigroup.
3. For all $t > 0$, the operator $\Sigma_0^t = \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{bmatrix}$ is a unitary operator from $X \times L^2([0, t]; U)$ to $X \times L^2([0, t]; Y)$.

Preservation of energy can also be characterized by some algebraic operator identities involving the system node $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$: differentiating the energy balance equation with respect to t (by Lemma 1.2, this is possible) we find that, for all $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(S) = \{\begin{bmatrix} x \\ u \end{bmatrix} \in X \times U \mid Ax + Bu \in X\}$,

$$\langle Ax + Bu, x \rangle_X + \langle x, Ax + Bu \rangle_X + \langle Cx + Du, Cx + Du \rangle_Y = \langle u, u \rangle_U. \quad (6)$$

(Here we have used the fact that we can write $C\&D \begin{bmatrix} x \\ u \end{bmatrix} = Cx + Du$ for $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(S)$ since Σ is well-posed.) In the finite-dimensional case this set of equations decouples into the three independent equations

$$\begin{aligned} A + A^* + C^*C &= 0, \\ B + C^*D &= 0, \\ (B^* + D^*C &= 0,) \\ D^*D &= I. \end{aligned} \quad (7)$$

(The third equation above is the adjoint of the second.) In the infinite-dimensional case such a decoupling is much more difficult. This problem is discussed further in Section 6.

The results presented in this section are taken from [4], [24], [42], and [52], and they are also found in [40].

4 The Universal Model of a Contraction Semigroup

There is a classical problem in mathematics:

Let \mathfrak{A} be an arbitrary contraction semigroup on some Hilbert space X . Is it always possible to find a unitary dilation $\tilde{\mathfrak{A}}$ of \mathfrak{A} defined on some larger space \tilde{X} ?

By this we meant the following: X is a subset of \tilde{X} , $\tilde{\mathfrak{A}}$ is a unitary semigroup on \tilde{X} , and for all $t \geq 0$ and $x \in X$,

$$\mathfrak{A}^t x = \pi_X \tilde{\mathfrak{A}}^t x,$$

where π_X is the orthogonal projection of \tilde{X} onto X . We also say that \mathfrak{A} a *compression* of $\tilde{\mathfrak{A}}$. The answer to this question is:

Theorem 4.1. *Given an arbitrary contraction semigroup \mathfrak{A} on a Hilbert space X , it is always possible to find a conservative system Σ whose semigroup is the given semigroup \mathfrak{A} . The corresponding Lax–Phillips semigroup is a unitary dilation of \mathfrak{A} . The system Σ is unique (modulo unitary similarity transformations in the input and output spaces) if we require $\widehat{\mathfrak{D}}$ to be strictly contractive.*

Strict contractivity of $\widehat{\mathfrak{D}}$ means that there is no nontrivial subspace of U on which $\widehat{\mathfrak{D}}$ reduces to a isometric constant. This additional restriction on the transfer function is a *minimality requirement*: by factoring out $\mathcal{N}(B)$ from U and $\mathcal{R}(C)^\perp$ from Y , we can always reduce a conservative system to a system whose transfer function is strictly contractive. The corresponding Lax–Phillips semigroup is then a *minimal unitary dilation* of \mathfrak{A} .

The preceding theorem also has a “converse”:

Theorem 4.2. *Every contractive causal shift invariant operator \mathfrak{D} from $L^2(\mathbf{R}^+; U)$ to $L^2(\mathbf{R}^+; Y)$ has a conservative realization, i.e., there exists a conservative system Σ with this input/output map. The system Σ is unique (modulo unitary similarity transformations in its state space) if we require \mathfrak{A} to be completely non-unitary.*

Complete non-unitarity of the semigroup \mathfrak{A} means that there is no nontrivial reducing subspace on which \mathfrak{A} is unitary. Again the requirement that \mathfrak{A} is completely non-unitary is a *minimality requirement*: We can always reduce a conservative system to a system with a completely non-unitary semigroup by factoring out the intersection of the unreachable and unobservable subspaces.

By combining the preceding results with a further representation result for the Lax–Phillips semigroup corresponding to a conservative system we arrive at the following universal model of a completely non-unitary contraction semigroup:

Theorem 4.3. *Every completely non-unitary contraction semigroup \mathfrak{A} on some Hilbert space X is unitarily equivalent to a compression of the (bilateral) shift operator on some subspace \tilde{X} of $L^2(\mathbf{R}; Z)$ (and there are formulas for how to find the space Z and the subspace \tilde{X}).*

The results presented in this section are taken from the book [44] by Sz.-Nagy and Foias, and the same book also contains a wealth of additional material on conservative systems, including results on *strong stability*, *controllability*, *observability*, and *transfer functions being inner* (or co-inner or bi-inner). This book is written primarily in *discrete time*, but the results apply equally well in continuous time. Recall that well-posedness does not cause any problems when the systems are *dissipative*. The *Cayley transform* maps a well-defined subclass of all discrete time dissipative [or conservative] systems *one-to-one onto* the class of continuous time dissipative [or conservative] linear systems, and it preserves controllability, observability, strong stability, inner transfer functions, etc. In other words, it preserves almost all the important properties of a system. The only exceptions are exponential stability and all properties related to the behavior of the system over a finite time interval (such as exact controllability/observability in finite time).

Of course, there is also a fair amount of additional literature on this subject, in particular, a huge *Russian literature*. See [53] for a detailed description of a

particular case of the general construction presented in Theorem 4.1. Theorem 4.2 is also found in [4] (in continuous time). Theorem 4.3 is found in [44] (both in discrete and continuous time). A modern presentation of the results presented up to now (in continuous time) is found in the book manuscript [40].

5 (R, P, J) -Energy Preserving Systems

The standard energy inequality of a dissipative system can also be written with the help of inner products:

$$\langle x(t), x(t) \rangle_X + \int_0^t \langle y(s), y(s) \rangle_Y ds \leq \langle x_0, x_0 \rangle_X + \int_0^t \langle u(s), u(s) \rangle_U ds.$$

We can make this equation more general by introducing self-adjoint cost operators: the input cost operator $R: U \rightarrow U$, the state cost operator $P: X \rightarrow X$, and the output cost operator $J: Y \rightarrow Y$.

Definition 5.1. A system Σ generated by a system node S is (R, P, J) -dissipative if the following (R, P, J) -energy inequality holds:

(JD) For all $t > 0$, the solution (x, y) in Lemma 1.2 satisfies

$$\langle x(t), Px(t) \rangle_X + \int_0^t \langle y(s), Jy(s) \rangle_Y ds \leq \langle x_0, Px_0 \rangle_X + \int_0^t \langle u(s), Ru(s) \rangle_U ds.$$

The system Σ is (R, P, J) -energy preserving if the above inequality holds as an equality:

(JE) For all $t > 0$, the solution (x, y) in Lemma 1.2 satisfies

$$\langle x(t), Px(t) \rangle_X + \int_0^t \langle y(s), Jy(s) \rangle_Y ds = \langle x_0, Px_0 \rangle_X + \int_0^t \langle u(s), Ru(s) \rangle_U ds.$$

Finally, Σ is (R, P, J) -conservative if both Σ and the dual system Σ^d are (R, P, J) -energy preserving.

It is important to observe that *in the above cases well-posedness is not guaranteed, nor is it necessarily relevant*. This possible lack of well-posedness causes some additional difficulties, but it is still possible to extend some of the results mentioned in Section 3 to the class of systems described in Definition 5.1. For simplicity, let us restrict the discussion to the case where the system is (R, P, J) -energy preserving. Differentiating the (R, P, J) -energy balance equation with respect to t (and using Lemma 1.2) we obtain the following *Lyapunov equation*,

$$\begin{aligned} \langle Ax + Bu, Px \rangle_X + \langle x, P(Ax + Bu) \rangle_X \\ + \langle C\&D \begin{bmatrix} x \\ u \end{bmatrix}, J[C\&D] \begin{bmatrix} x \\ u \end{bmatrix} \rangle_Y = \langle u, Ru \rangle_U, \end{aligned} \tag{8}$$

valid for all $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(S)$. In the finite-dimensional case this set of equations decouples into the three independent equations

$$\begin{aligned} A^*P + PA + C^*JC &= 0, \\ PB + C^*JD &= 0, \\ (B^*P + D^*JC &= 0,) \\ D^*JD &= R. \end{aligned} \tag{9}$$

Again, in the infinite-dimensional case such a decoupling is much more difficult (see the discussion in Section 6).

(R, P, J) -energy preserving systems appear in optimal control, for example in minimization problems (H^2 -optimal control) and in minimax problems (H^∞ -optimal control). The operator J is known, but the original system is not energy preserving in any sense. Instead we want to find a feedback operator K , an input cost operator R , and a state cost operator (Riccati operator) P such that the corresponding *closed loop system is (R, P, J) -energy preserving*. By this we mean the following. We add another combined observation/feedthrough operator $K\&L$ to the bottom of the system node S (similar to the original $C\&D$), and we then feed this output back into the system, i.e., we take u to be of the form

$$u = v + K\&L \begin{bmatrix} x \\ u \end{bmatrix},$$

where v is the closed loop input. To simplify the discussion, let us assume that we can split $K\&L$ into $K\&L = \begin{bmatrix} K & L \end{bmatrix}$, and that we can take $L = 0$ (and let us also ignore the fact that the splitting of $K\&L$ into $\begin{bmatrix} K & L \end{bmatrix}$ need not be unique). Then $K\&L \begin{bmatrix} x \\ u \end{bmatrix} = Kx$, so that

$$u = v + Kx.$$

As we mentioned above, the operator K should be chosen in such a way that closed loop system is (R, P, J) -energy preserving for some (unknown) cost operators R and P . By replacing u in the Lyapunov equation (8) by $v = u - Kx$ we get the *Riccati equation*

$$\begin{aligned} \langle Ax + Bu, Px \rangle_X + \langle x, P(Ax + Bu) \rangle_X \\ + \langle C\&D \begin{bmatrix} x \\ u \end{bmatrix}, J[C\&D] \begin{bmatrix} x \\ u \end{bmatrix} \rangle_Y &= \langle u - Kx, R(u - Kx) \rangle_U, \end{aligned} \tag{10}$$

which is valid for all $\begin{bmatrix} x \\ u - Kx \end{bmatrix} \in \mathcal{D}(S)$. In the finite-dimensional case this set of equations decouples into the three equations

$$\begin{aligned} A^*P + PA + C^*JC &= K^*RK, \\ PB + C^*JD &= -K^*R, \\ (B^*P + D^*JC &= -RK,) \\ D^*JD &= R. \end{aligned} \tag{11}$$

Especially in the case of the positive real lemma (which will be discussed below) the system (11) is often referred to as the *Popov-Kalman-Szegö-Yakubovic* system. In

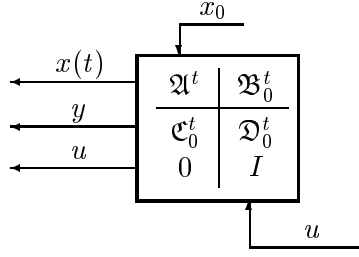


Figure 2: Input added to output of Σ

the infinite-dimensional case the interpretation of these equations raises some serious questions.

By appropriate choice of R , P , and J we get many of the standard one-block and two-block “optimal control” results. This time we work with the extended system of Figure 2, where we have added a copy of the input to the output (in addition the systems used in the full information H^∞ problem has two inputs instead of one):

1. $y := \begin{bmatrix} y \\ u \end{bmatrix}$, $C := \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D := \begin{bmatrix} 0 \\ I \end{bmatrix}$, $J := \begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}$. This is the *standard LQR Riccati equation*. Here $R \geq 0$ is the same operator which appears in the definition of J , and both $R \geq 0$ and $J \geq 0$. We further require $P \geq 0$, and (11) becomes

$$\begin{aligned} A^*P + PA + C^*C &= K^*RK, \\ RK &= -B^*P. \end{aligned}$$

Usually R is assumed to be invertible, in which case we can eliminate K to get the standard regulator Riccati equation

$$A^*P + PA + C^*C = PBR^{-1}B^*P.$$

2. $y := \begin{bmatrix} y \\ u \end{bmatrix}$, $C := \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D := \begin{bmatrix} D \\ I \end{bmatrix}$, $J := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$. This is the *normalized coprime factorization problem*. We still have $J \geq 0$, $P \geq 0$, $R \geq 0$, and (11) becomes

$$\begin{aligned} A^*P + PA + C^*C &= K^*RK, \\ RK &= -(B^*P + D^*C), \\ R &= I + D^*D. \end{aligned}$$

Clearly R is invertible, and we may eliminate R and K to get the normalized coprime factorization Riccati equation

$$A^*P + PA + C^*C = (PB + C^*D)(I + D^*D)^{-1}(B^*P + D^*C).$$

3. $y := \begin{bmatrix} y \\ u \end{bmatrix}$, $C := \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D := \begin{bmatrix} D \\ I \end{bmatrix}$, $J := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$. This is the *bounded real lemma*. Here J is indefinite and (11) becomes

$$\begin{aligned} A^*P + PA + C^*C &= K^*RK, \\ RK &= -(B^*P + D^*C), \\ R &= D^*D - \gamma^2 I. \end{aligned}$$

Usually γ is chosen so that $R \leq 0$ is invertible and $P \geq 0$. Eliminating R and K from the above system we get the bounded real lemma Riccati equation

$$A^*P + PA + C^*C = -(PB + C^*D)(\gamma^2 I - D^*D)^{-1}(B^*P + D^*C).$$

4. $y := \begin{bmatrix} y \\ u \end{bmatrix}$, $C := \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D := \begin{bmatrix} D \\ I \end{bmatrix}$, $J := -\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$: This is the *positive real lemma*. The operator J is again indefinite, and (11) becomes

$$\begin{aligned} A^*P + PA &= K^*RK, \\ RK &= -B^*P + C, \\ R &= -(D + D^*). \end{aligned}$$

Clearly $R \leq 0$, and we require that $P \geq 0$. If $D + D^*$ is invertible, then we can eliminate R and K to get the positive real lemma Riccati equation

$$A^*P + PA + C^*C = -(PB - C)(D + D^*)^{-1}(B^*P - C).$$

5. $u := \begin{bmatrix} w \\ u \end{bmatrix}$, $y := \begin{bmatrix} y \\ u \end{bmatrix}$, $B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, $C := \begin{bmatrix} C \\ 0 \end{bmatrix}$, $D := \begin{bmatrix} D_{11} & D_{12} \\ 0 & I \end{bmatrix}$, $J := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$. This is the *H^∞ full information problem*. In this problem $P \geq 0$, but J and R are indefinite. In the special case where $D_{12} = 0$ the system (11) becomes (the feedback operator K has two components $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$ since we now have two inputs)

$$\begin{aligned} A^*P + PA + C^*C &= \begin{bmatrix} K_1^* & K_2^* \end{bmatrix} R \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \\ R \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} &= -\left(\begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} P + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C \right), \\ R &= \begin{bmatrix} D_{11}^* D_{11} & 0 \\ 0 & -\gamma^2 I \end{bmatrix}. \end{aligned}$$

(If $D_{12} \neq 0$ then the solution contains an additional feed-forward term from the disturbance to the control input.) For further comments on this example we refer the reader to standard text books on H^∞ control.

One important problem which is missing from the above list is the *Nehari problem*. It is related to the full information H^∞ -problem discussed above, but it does not quite fit into the general framework of this section (a minimax approach to this problem is presented in [41]). Another related problem is the computation of the *n :th singular value* of the Hankel operator. In the last problem even the appropriate Riccati operator P is indefinite.

6 Open Questions and Some Solutions

Presently research is going on to extend the finite-dimensional Riccati equation theory to the setting of an infinite-dimensional well-posed linear system. Much has been done, but even more remains to be done. There are several problems

which makes the the infinite-dimensional theory significantly more difficult than the finite-dimensional one. The first question to ask is: *what do we really want out of the Riccati equation?* Depending on the answer (optimal control, spectral factorization, computation of invariant subspaces, optimal identification, etc.) the “correct formulation” of the problem might be different. Once we have decided the purpose of the Riccati equation, we are faced with the problem that *the closed loop system may not be a system*, or it may *not be well-posed*. If it is a system, then *what is the “correct form” of the Riccati equation?* For example, to compute the feedback operator K from the equation $K = -R^{-1}(B^*P + D^*JC)$ we need to know for which $x \in \mathcal{D}(C)$ it is true that $Px \in \mathcal{D}(B^*)$. In addition the *non-uniqueness of the splitting of the observation/feedthrough operators $C&D$* into $[C \ D]$ and $K&L$ into $[K \ L]$ comes into play. All the operators A , B , C , D , and K appear in the Riccati equation (L is supposed to be zero), but which are the “correct” versions of these operators? Can we always take $L = 0$? Is there always a “correct splitting” of $C&D$ and $K&L$ which makes the Riccati equation valid? Such a splitting does not depend only on the system itself, but also on the given cost operator J .

The difficulties described above have been approached in different ways. Some results on the “*correct splitting*” of $C&D$ into $[C \ D]$ (i.e., a splitting which makes the Riccati equation valid with $L = 0$) are found in [13] (this reference does not require the optimal closed loop system to be well-posed). If $\widehat{\mathcal{D}}(\infty)$ exists, then the system is called *regular*, and it is possible to take $D = \widehat{\mathcal{D}}(\infty)$. In general this splitting is not compatible with the Riccati equation, and we have to add some correction terms to the Riccati equation, similar to those seen in the discrete time theory. In particular, the formula for R becomes more complicated. See [25], [31, 32, 33, 35, 36, 37, 39], and [55]. These results are largely based on *spectral factorization*, and so are [5, 6, 7] and [14, 15, 16]. When the closed loop system is not well-posed we can use the (non-well-posed) *compensators with internal loop* introduced in [11] (see [25]). Another approach is to use the *Cayley transform* to map the continuous time system into a discrete time system and use the discrete time Riccati equation theory (joint work with Malinen is in progress). In discrete time all the operators become bounded, but some extra correction terms enter the Riccati equations. In many cases *additional smoothing properties* have been used (Pritchard–Salamon and parabolic cases); see, e.g., [8], [9], [19], [45], [26], [30, 29], and [54].

Acknowledgement. While working on this manuscript I had several intensive discussions with Jarmo Malinen, who spotted many misleading or clumsy formulations and suggested alternative approaches. Some further comments by one of the editors also helped me to improve the presentation.

References

- [1] A. Adamajan and D. Z. Arov. On unitary couplings of semiunitary operators. In *Eleven Papers in Analysis*, volume 95 of *American Mathematical Society Translations*, pages 75–129, Providence, R.I., 1970. American Mathematical Society.

- [2] D. Z. Arov. Passive linear stationary dynamic systems. *Siberian Math. J.*, 20:149–162, 1979.
- [3] D. Z. Arov. Passive linear systems and scattering theory. In *Dynamical Systems, Control Coding, Computer Vision*, volume 25 of *Progress in Systems and Control Theory*, pages 27–44, Basel Boston Berlin, 1999. Birkhäuser Verlag.
- [4] D. Z. Arov and M. A. Nudelman. Passive linear stationary dynamical scattering systems with continuous time. *Integral Equations Operator Theory*, 24:1–45, 1996.
- [5] F. M. Callier and J. Winkin. Spectral factorization and LQ-optimal regulation for multivariable distributed systems. *Internat. J. Control*, 52:55–75, 1990.
- [6] F. M. Callier and J. Winkin. LQ-optimal control of infinite-dimensional systems by spectral factorization. *Automatica*, 28:757–770, 1992.
- [7] F. M. Callier and J. Winkin. The spectral factorization problem for multivariable distributed parameter systems. *Integral Equations Operator Theory*, 34:270–292, 1999.
- [8] R. F. Curtain and A. Ichikawa. The Nehari problem for infinite-dimensional linear systems of parabolic type. *Integral Equations Operator Theory*, 26:29–45, 1996.
- [9] R. F. Curtain and J. C. Oostveen. The Nehari problem for nonexponentially stable systems. *Integral Equations Operator Theory*, 31:307–320, 1998.
- [10] R. F. Curtain and G. Weiss. Well posedness of triples of operators (in the sense of linear systems theory). In *Control and Optimization of Distributed Parameter Systems*, volume 91 of *International Series of Numerical Mathematics*, pages 41–59, Basel, 1989. Birkhäuser-Verlag.
- [11] R. F. Curtain, G. Weiss, and M. Weiss. Stabilization of irrational transfer functions by controllers with internal loop. Preprint, 1997.
- [12] K.-J. Engel. On the characterization of admissible control- and observation operators. *Systems Control Lett.*, 34:225–227, 1998.
- [13] F. Flandoli, I. Lasiecka, and R. Triggiani. Algebraic Riccati equations with non-smoothing observation arising in hyperbolic and Euler–Bernoulli boundary control problems. *Ann. Mat. Pura Appl. (4)*, 153:307–382, 1988.
- [14] P. Grabowski. On the spectral-Lyapunov approach to parametric optimization of distributed-parameter systems. *IMA J. Math. Control Inform.*, 7:317–338, 1991.
- [15] P. Grabowski. The LQ controller synthesis problem. *IMA J. Math. Control Inform.*, 10:131–148, 1993.

- [16] P. Grabowski. The LQ controller problem: an example. *IMA J. Math. Control Inform.*, 11:355–368, 1994.
- [17] P. Grabowski and F. M. Callier. Admissible observation operators. Semigroup criteria of admissibility. *Integral Equations Operator Theory*, 25:182–198, 1996.
- [18] P. Grabowski and F. M. Callier. Boundary control systems in factor form: transfer functions and input-output maps. *Integral Equations Operator Theory*, 2001.
- [19] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theorems. I Abstract Parabolic Systems*, volume 74 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge and New York, 2000.
- [20] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theorems. II Abstract Hyperbolic-Like Systems over a Finite Horizon*, volume 75 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge and New York, 2000.
- [21] P. D. Lax and R. S. Phillips. *Scattering Theory*. Academic Press, New York, 1967.
- [22] P. D. Lax and R. S. Phillips. Scattering theory for dissipative hyperbolic systems. *J. Funct. Anal.*, 14:172–235, 1973.
- [23] J. Malinen. Discrete time H^∞ algebraic Riccati equations. Doctoral dissertation, Helsinki University of Technology, 2000.
- [24] J. Malinen, O. J. Staffans, and G. Weiss. When is a linear system conservative? In preparation, 2001.
- [25] K. Mikkola. Infinite-dimensional H^∞ and H^2 regulator problems and their algebraic Riccati equations with applications to the Wiener class. Doctoral dissertation, Helsinki University of Technology, 2001.
- [26] J. Oostveen. *Strongly stabilizable distributed parameter systems*, volume 20 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000.
- [27] D. Salamon. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Trans. Amer. Math. Soc.*, 300:383–431, 1987.
- [28] D. Salamon. Realization theory in Hilbert space. *Math. Systems Theory*, 21:147–164, 1989.
- [29] A. J. Sasane and R. F. Curtain. Inertia theorems for operator Lyapunov inequalities. *Systems Control Lett.*, pages 127–132, 2001.

- [30] A. J. Sasane and R. F. Curtain. Optimal Hankel norm approximation for the Pritchard-Salamon class of infinite-dimensional systems. *Integral Equations Operator Theory*, 2001.
- [31] O. J. Staffans. Quadratic optimal control of stable systems through spectral factorization. *Math. Control Signals Systems*, 8:167–197, 1995.
- [32] O. J. Staffans. On the discrete and continuous time infinite-dimensional algebraic Riccati equations. *Systems Control Lett.*, 29:131–138, 1996.
- [33] O. J. Staffans. Quadratic optimal control of stable well-posed linear systems. *Trans. Amer. Math. Soc.*, 349:3679–3715, 1997.
- [34] O. J. Staffans. Coprime factorizations and well-posed linear systems. *SIAM J. Control Optim.*, 36:1268–1292, 1998.
- [35] O. J. Staffans. Quadratic optimal control of well-posed linear systems. *SIAM J. Control Optim.*, 37:131–164, 1998.
- [36] O. J. Staffans. Feedback representations of critical controls for well-posed linear systems. *Internat. J. Robust Nonlinear Control*, 8:1189–1217, 1998.
- [37] O. J. Staffans. On the distributed stable full information H^∞ minimax problem. *Internat. J. Robust Nonlinear Control*, 8:1255–1305, 1998.
- [38] O. J. Staffans. Admissible factorizations of Hankel operators induce well-posed linear systems. *Systems Control Lett.*, 37, 1999.
- [39] O. J. Staffans. Quadratic optimal control through coprime and spectral factorization. *European J. Control*, 5:167–179, 1999.
- [40] O. J. Staffans. *Well-Posed Linear Systems: Part I*. Book manuscript, available at <http://www.abo.fi/~staffans/>, 2001.
- [41] O. J. Staffans and K. M. Mikkola. A minimax formulation of the infinite-dimensional Nehari problem. In *Mathematical Theory of Networks and Systems, Proceedings of the MTNS98 symposium in Padova, Italy*, pages 539–542, 1998.
- [42] O. J. Staffans and G. Weiss. Transfer functions of regular linear systems. Part II: the system operator and the Lax-Phillips semigroup. Manuscript, 2001.
- [43] O. J. Staffans and G. Weiss. Transfer functions of regular linear systems. Part III: inversions and duality. Manuscript, 2001.
- [44] B. Sz.-Nagy and C. Foiaş. *Harmonic Analysis of Operators on Hilbert Space*. North-Holland, Amsterdam London, 1970.
- [45] B. van Keulen. *H_∞ -Control for Distributed Parameter Systems: A State Space Approach*. Birkhäuser Verlag, Basel Boston Berlin, 1993.
- [46] G. Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27:527–545, 1989.

- [47] G. Weiss. Admissible observation operators for linear semigroups. *Israel J. Math.*, 65:17–43, 1989.
- [48] G. Weiss. The representation of regular linear systems on Hilbert spaces. In *Control and Optimization of Distributed Parameter Systems*, volume 91 of *International Series of Numerical Mathematics*, pages 401–416, Basel, 1989. Birkhäuser-Verlag.
- [49] G. Weiss. Representations of shift-invariant operators on L^2 by H^∞ transfer functions: an elementary proof, a generalization to L^p , and a counterexample for L^∞ . *Math. Control Signals Systems*, 4:193–203, 1991.
- [50] G. Weiss. Transfer functions of regular linear systems. Part I: characterizations of regularity. *Trans. Amer. Math. Soc.*, 342:827–854, 1994.
- [51] G. Weiss. Regular linear systems with feedback. *Math. Control Signals Systems*, 7:23–57, 1994.
- [52] G. Weiss, O. J. Staffans, and M. Tucnak. Well-posed linear systems – a survey with emphasis on conservative systems. *Appl. Math. Comp. Sci.*, 11:7–34, 2001.
- [53] G. Weiss and M. Tucnak. How to get a conservative well-posed linear system out of thin air. Part I: well-posedness and energy balance. Submitted, 2001.
- [54] M. Weiss. Riccati equation theory for Pritchard–Salamon systems: a Popov function approach. *IMA J. Math. Control Inform.*, 14:1–37, 1997.
- [55] M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems. *Math. Control Signals Systems*, 10:287–330, 1997.
- [56] J. C. Willems. Dissipative dynamical systems Part I: General theory. *Arch. Rational Mech. Anal.*, 45:321–351, 1972.
- [57] J. C. Willems. Dissipative dynamical systems Part II: Linear systems with quadratic supply rates. *Arch. Rational Mech. Anal.*, 45:352–393, 1972.