On the Distributed Stable Full Information H^∞ Minimax Problem

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Abstract. We study the distributed parameter suboptimal full information H^∞ problem for a stable well-posed linear system with control u, disturbance w, state x, and output y. Here u, w, and y are L^2-signals on $(0, \infty)$ with values in the Hilbert spaces U, W, and Y, and the state x is a continuous function of time with values in the Hilbert space H. The problem is to determine if there exists a (dynamic) γ-suboptimal feedforward compensator, i.e., a compensator $\mathcal{U}$ such that the choice $u = \mathcal{U}w$ makes the norm of the input/output map from w to y less than a given constant γ. A sufficient condition for the existence of a γ-suboptimal compensator is that an appropriately extended input/output map of the system has a (J, S)-inner-outer factorization of a special type, and if the control and disturbance spaces are finite-dimensional and the system has an L^1 impulse response, then this condition is also necessary. Moreover, in this case there exists a central state feedback/feedforward controller, which can be used to give a simple parameterization of the set of all γ-suboptimal compensators. Our proofs use a game theory approach.

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game, (J, S)-spectral factorization, (J, S)-inner-outer factorization, (J, S)-lossless factorization.

Contents

1 Introduction

This is the second out of three papers that take the first steps in the development a quite general state space theory for the full information H^∞ problem, and on a longer perspective, for the general suboptimal H^∞ problem. In our setting the transfer functions need not be rational or meromorphic; they are just plain H^∞ without any extra rationality or smoothness assumptions. We are interested in state space results as opposed to pure frequency domain or input/output results. The outlines of our proofs follow the standard frequency domain route (based on spectral factorization) that has also been used for the finite dimensional (rational) H^∞ problem, but we have transformed the frequency domain arguments to the time domain and added some state space ingredients. The key addition is the factorization of the Hankel operator induced by the input/output map as the product of the controllability and observability maps, and this makes it possible to connect the state space and the frequency domain theories to each other.

Because of the quite general class of systems that we allow, we need to extend a large number of more or less well known finite dimensional results. Some of the extensions are known, others are straightforward, and some are neither known nor straightforward. One particular feature is that we bypass all those finite-dimensional results that lean on the fact that it is possible to normalize certain feedforward terms to be either zero or the identity operator. The primary motivation for this is that these feedforward terms need not be well-defined in general. However, at the same time it leads to a simplification in the sense that there is no need to perform a number of preliminary normalizations before applying the final result. We believe that the results presented here are interesting even in the finite dimensional setting due to our somewhat different point of view.

The general problem that we study here is of the following type. Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be a stable well-posed linear system with control input space U, disturbance input space W, state space H, and output space Y. To this system we adjoin the indefinite cost function

$$Q(x_0, u, w) = \int_{\mathbf{R}^+} \langle y(s), Jy(s) \rangle_Y \, ds, \tag{1.1}$$

where $J = J^*$ is an indefinite operator on Y, and

$$y = \mathcal{C}x_0 + \mathcal{D}_1\pi_+u + \mathcal{D}_2\pi_+w \tag{1.2}$$

is the output of Ψ with initial value $x_0 \in H$, control $u \in L^2(\mathbf{R}^+; U)$, and disturbance $w \in L^2(\mathbf{R}^+; W)$. The goal is to find out if there exists a (causal dynamic) feedforward compensator $\mathcal{U}$ which makes Q a uniformly concave function of $w \in L^2(\mathbf{R}^+; W)$ if we take $x_0 = 0$ and $u = \mathcal{U}\pi_+w$ (see Figure 1 with $x_0 = 0$ and $\tilde{u} = 0$), and if this is the case, then we want to find a simple parameterization of all such compensators. In other words, we want to give a simple description of the set of all possible causal time-invariant linear mapping $\mathcal{U}\colon L^2(\mathbf{R}^+; W) \to L^2(\mathbf{R}^+; U)$ such that

$$Q(0, \mathcal{U}\pi_+w, w) \le -\epsilon \|w\|^2_{L^2(\mathbf{R}^+;W)} \tag{1.3}$$

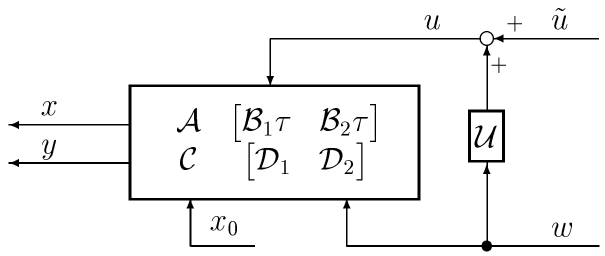

Figure 1: Dynamic feedforward compensator

for some $\epsilon > 0$ and all $w \in L^2(\mathbf{R}^+; W)$. These compensators will be called *uniformly suboptimal.*

The preceding formulation is a simplification and, at the same time, a slight extension of the stable suboptimal full information H^∞ minimization problem: when is it possible to find a (dynamic causal) feedforward compensator $\mathcal{U}$ which makes the norm of the input/output operator mapping $w \in L^2(\mathbf{R}^+; W)$ into the output $y \in L^2(\mathbf{R}^+; Y)$ strictly less than than a prescribed constant γ if we take $x_0 = 0$ and $u = \mathcal{U}\pi_+ w$ (see Figure 1)? This input/output map is equal to $(\mathcal{D}_1\mathcal{U} + \mathcal{D}_2)\pi_+$, so the problem is to find out if it is possible to choose $\mathcal{U}$ in such a way that the norm of the operator $(\mathcal{D}_1\mathcal{U} + \mathcal{D}_2)\pi_+ \colon L^2(\mathbf{R}^+; W) \to L^2(\mathbf{R}^+; Y)$ is strictly less than γ.

To connect this problem to the first one we reformulate it as follows: we adjoin a copy of the disturbance w to the output y, creating a larger system

$$\Psi^{\mathrm{FI}} = \begin{bmatrix} \mathcal{A} & \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \end{bmatrix} \\ \begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix} & \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \\ 0 & I \end{bmatrix} \end{bmatrix} \tag{1.4}$$

with output $y = \left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right]$. This extended system has the same input space $U \times W$ and the same state space H as the original system, but the output space is now $Y \times W$. For this extended system we define the cost function Q as above, with $J = \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$. In terms of the original system this cost can be written in the form

$$Q(x_0, u, w) = \|y\|^2_{L^2(\mathbf{R}^+;Y)} - \gamma^2 \|w\|^2_{L^2(\mathbf{R}^+;W)}, \tag{1.5}$$

hence

$$Q(0, \mathcal{U}\pi_+ u, w) = \|(\mathcal{D}_1\mathcal{U} + \mathcal{D}_2)\pi_+ w\|^2_{L^2(\mathbf{R}^+;Y)} - \gamma^2 \|w\|^2_{L^2(\mathbf{R}^+;W)}.$$

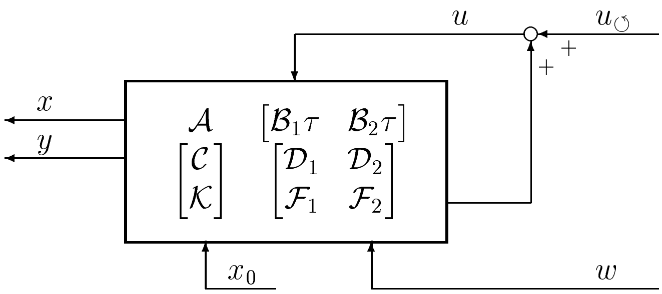

Figure 2: State feedback/feedforward controller

We find that $Q(0, \mathcal{U}\pi_+ w, w)$ is uniformly concave in w if and only if $\|\mathcal{D}_1\mathcal{U} + \mathcal{D}_2\| < \gamma$, so the problem (1.1)–(1.2) contains the stable suboptimal full information H^∞ minimization problem as a special case.

In order to avoid some degenerate situations we shall most of the time make the following *nondegeneracy assumption.*

Hypothesis 1.1 *The function $Q(0, u, 0)$ is uniformly convex in $u \in L^2(\mathbf{R}^+; U)$, i.e., $Q(0, u, 0) \geq \epsilon \|u\|^2_{L^2(\mathbf{R}^+;W)}$, for some $\epsilon > 0$ and all $u \in L^2(\mathbf{R}^+; U)$.*

It is well known that, under Hypothesis 1.1, in the full information case with finite-dimensional U, W, H, and Y, there are several other conditions which are equivalent to the existence of a uniformly suboptimal compensator. More precisely, the following conditions (I)–(VI) are then equivalent (we have formulated conditions (I)–(V) in such a way that they apply to the more general problem (1.1)–(1.2) as well; the precise definitions of the notions used here will be given later).

Conditions 1.2

(I) There exists a (dynamic) uniformly suboptimal feedforward compensator $\mathcal{U}$ (see Figure 1);

(II) For each $x_0 \in H$ and $w \in L^2(\mathbf{R}^+; W)$, the function $u \mapsto Q(x_0, u, w)$ is uniformly convex on $L^2(\mathbf{R}^+; U)$, and, for each $x_0 \in H$, the function $w \mapsto Q^{\min}(x_0, w) = \min_{u \in L^2(\mathbf{R}^+;U)} Q(x_0, u, w)$ is uniformly concave on $L^2(\mathbf{R}^+; U)$;

(III) *The input/output map $\mathcal{D}$ has a (J,T)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where both $\mathcal{X}$ and $\mathcal{X}_{11}$ have a bounded causal inverse. Here $T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ and $\mathcal{X} = \left[\begin{smallmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{smallmatrix}\right]$, where the block form is the one induced by the natural splitting of $U \times W$ into its components U and W.*

(IV) *There exists a so called central stabilizing state feedback/feedforward controller (see Figure 2) such that the corresponding closed loop cost function is uniformly concave with respect to $w \in L^2(\mathbf{R}^+; W)$ for all $x_0 \in H$ and all $u_\circlearrowleft \in L^2(\mathbf{R}^+; U)$.*

(V) *There exists a stabilizing state feedback/feedforward controller (see Figure 2) such that the corresponding closed loop cost function is uniformly concave with respect to $w \in L^2(\mathbf{R}^+; W)$ for all $x_0 \in H$ and all $u_\circlearrowleft \in L^2(\mathbf{R}^+; U)$.*

(VI) *The full information H^∞ Riccati equation has a stabilizing solution.*

Our first main result is the following infinite-dimensional analogue of this equivalence:

Theorem 1.3 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be a stable well-posed linear system on $(U \times W, H, Y)$, let $J = J^* \in \mathcal{L}(Y)$, and define Q by (1.1)–(1.2). Then (III) $\Rightarrow$ (IV) $\Rightarrow$ (V) $\Rightarrow$ (I). If Hypothesis 1.1 holds, then (I) $\Rightarrow$ (II), and (II) together with Hypothesis 4.6 implies (III).*

This theorem is a summary of Lemmas 2.3 and 2.7, Theorem 6.4, Definition 7.1, and Corollary 7.3.

It is an interesting fact that this theorem is valid both for the full information problem and for the more general problem (1.1)–(1.2). In this theorem the implications (IV) $\Rightarrow$ (V) $\Rightarrow$ (I) $\Rightarrow$ (II) are trivial. Hypothesis 4.6 which is used in the proof of the implication (II) $\Rightarrow$ (III) can be regarded an extra regularity assumption on the input/output map. For example, it it satisfied in the case where U and W are finite-dimensional and the system has an L^1 impulse response (see Lemma 4.4).

The two implications missing in Theorem 1.3, namely (III) $\Rightarrow$ (VI) $\Rightarrow$ (III), are true only under some extra "technical" assumption on the input/output map. The implication (III) $\Rightarrow$ (VI) was established in Staffans [1998c] (a summary is given at the end of Section 10) and the implication (VI) $\Rightarrow$ (III) in Mikkola [1997].

The major part of this work is devoted to the proof of Theorem 1.3. Our proofs of the two nontrivial steps (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) use a game theory approach. This differs from the approach taken in most text books, such as Green and Limebeer [1995] and Zhou et al. [1996], which invoke the Riccati equation for these steps, but without explaining the true physical meaning of the Riccati operator. There the Riccati operator is typically simply seen as one out of several auxiliary operators that happens to be the solution to a certain Riccati equation, and its role as the minimax value of a two player dynamical zero sum game with quadratic cost function is all but ignored. We feel that, in order to gain some insight in the physical meaning of the full information Riccati equation, it is necessary to have a good understanding of the underlying game.

This particular game is a two player game with decision variables $u \in L^2(\mathbf{R}^+;U)$ (controlled by the minimizing player; the control engineer) and $w \in L^2(\mathbf{R}^+;W)$ (controlled by the maximizing player; the nature), and cost function $Q(x_0,u,w)$. Here x_0 plays a role of a parameter which influences the value of the game. As usual (see, for example Başar and Bernard [1991] or Başar and Olsder [1995]), the *open loop lower value* $\underline{Q}(x_0)$ and *upper value* $\overline{Q}(x_0)$ of this game are defined by

$$\underline{Q}(x_0) = \sup_{w\in L^2(\mathbf{R}^+;W)} \inf_{u\in L^2(\mathbf{R}^+;U)} Q(x_0,u,w), \tag{1.6}$$

$$\overline{Q}(x_0) = \inf_{u\in L^2(\mathbf{R}^+;U)} \sup_{w\in L^2(\mathbf{R}^+;W)} Q(x_0,u,w). \tag{1.7}$$

Trivially, $\underline{Q}(x_0) \le \overline{Q}(x_0)$.

If $\underline{Q}(x_0) = \overline{Q}(x_0)$, then $\underline{Q}(x_0) = \overline{Q}(x_0)$ is called the *open loop value* of the game at the point $x_0 \in H$. Since the cost function is quadratic and there are no hard constraints, a necessary condition for the game to have a finite open loop value is that $Q(x_0,u,w)$ is convex in u and concave in w, and a sufficient condition for this to happen is that $Q(x_0,u,w)$ is uniformly convex in u and uniformly concave in w. In the latter case the infima and suprema in (1.6) are achieved for some $u^{\mathrm{crit}} \in L^2(\mathbf{R}^+;U)$ and $w^{\mathrm{crit}} \in L^2(\mathbf{R}^+;W)$ (depending on x_0), and this pair $(u^{\mathrm{crit}},w^{\mathrm{crit}})$ is an *open loop saddle point* (or Nash equilibrium) for the game in the sense that

$$Q(x_0,u^{\mathrm{crit}},w) \le Q(x_0,u^{\mathrm{crit}},w^{\mathrm{crit}}) \le Q(x_0,u,w^{\mathrm{crit}}), \tag{1.8}$$

for all $u \in L^2(\mathbf{R}^+;U)$ and $w \in L^2(\mathbf{R}^+,w)$.

In general we cannot expect the system under study to have an open loop saddle point. For example, the cost function (1.5) for the full information H^∞ minimization problem is concave in w iff $\|\mathcal{D}_2\| \leq \gamma$, and it is uniformly concave in w iff $\|\mathcal{D}_2\| < \gamma$. Thus, if the system has an open loop saddle point then the control objective is almost satisfied already by the open loop system (where $u \equiv 0$). Moreover, if the system has a uniform open loop saddle point then the full control objective is satisfied by the open loop system, and the solution to the H^∞ minimization problem is trivial. We therefore need to introduce a more sophisticated type of saddle point for this game.

If condition (II) in Theorem 1.3 holds, then obviously the lower value $\underline{Q}$ is finite, whereas the upper value can be $+\infty$. In the sequel we simply ignore the upper value, which is of no importance. In the computation of the lower value we fix $x_0 \in H$ and $w \in L^2(\mathbf{R}^+; W)$, then compute the control $u^{\min}(x_0, w)$ that minimizes $Q(x_0, u, w)$, and finally maximize $Q^{\min}(x_0, w) = Q(x_0, u^{\min}(x_0, w), w)$ over $w \in L^2(\mathbf{R}^+; W)$. This leads to the standard solution of the (open loop) Stackelberg game with w as the leader (with no information available about the state x and the control u, only information about the initial state x_0), and u as the follower (with complete information available).

The Stackelberg solution is straightforward, but it does not yet lead to the final solution due to the fact that it has the wrong information structure. In the full information H^∞ problem with cost function (1.5) we are really asking whether it is possible to find a *causal* compensator for which the closed loop system has a saddle point. In the Stackelberg solution the control u has too much information available: it is not required to be causal, but can depend on both "past" and "future" values of w. The correct information structure is to allow u to depend only on the initial value x_0 and on "past" and "present" values of the disturbance w. Thus, we are really looking for a "feedback saddle point" (or feedback Nash equilibrium), where the disturbance loop is open but the control loop is closed, and there is a possible feedforward term from the disturbance to the control.

It is possible to get a certain type of feedback/feedforward representation of the Stackelberg equilibrium by applying [Staffans 1998c, Theorem 5.1], since this equilibrium is J-critical in the sense of [Staffans 1998c, Definition 3.1]. (This amounts to finding a (J, S)-inner-outer factorization $\mathcal{N}\mathcal{X}$ of the input/output map $\mathcal{D}$ with invertible outer factor $\mathcal{X}$; cf. (III).) However, that theorem does not gives us the correct information structure either in the sense that it gives solutions which employ feedback through both the control input and the disturbance input; i.e., in addition to the desired feedback/feedforward term $u = \mathcal{K}_1 x_0 + \mathcal{F}_{11}\pi_+ u + \mathcal{F}_{12}\pi_+ w$ there is another feedback/feedforward term $w = \mathcal{K}_2 x_0 + \mathcal{F}_{21}\pi_+ u + \mathcal{F}_{22}\pi_+ w$ entering through the disturbance input. The requirement is that even without this additional term the system should be well-behaved.

Thus, the next problem that we have to solve is whether it is possible to disconnect the feedback entering through the disturbance input, without loss of well-posedness and stability, to turn the double feedback solution given above into a feedback saddle point, where the only information available to w is the initial state x_0 (i.e., w is an open loop input), whereas u is allowed to use information about x_0 and causal information about the disturbance w. (This is where the invertibility condition on $\mathcal{X}_{11}$ in (III) and the normalization of S to $S = T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ comes into play.) When this is possible we get a uniformly suboptimal compensator of the type mentioned in (IV), i.e., a central state feedback/feedforward controller.

We prove Theorem 1.3 in Sections 2–7, breaking it up into smaller pieces. More precisely, we show that under the regularity assumptions listed in Theorem 1.3, (V) $\Rightarrow$ (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) $\Rightarrow$ (V). Parameterizations of the sets of all suboptimal and all uniformly suboptimal compensators is developed in Sections 8 and 9. In Section 10 we introduce an extra regularity condition which makes it possible to separate the feedback part of a state feedback/feedforward controller from its feedforward part, and derive the algebraic Riccati equation satisfied by the Riccati operator. Finally, in Section 11 we discuss how the factorizations that we use in this work are related to the more common (J, S)-lossless factorizations.

From time to time we make quite heavy use of Staffans [1998c], and we expect the reader to have access to this paper. We also refer the reader to the same paper for a short review of the theory of well-posed linear systems, and recommend Staffans [1997 1998ab] for additional reading.

We use the following set of notations.

$\mathcal{L}(U;Y)$, $\mathcal{L}(U)$: The set of bounded linear operators from U into Y or from U into itself, respectively.

I: The identity operator.

A^*: The (Hilbert space) adjoint of the operator A.

dom(A): The domain of the (unbounded) operator A.

range(A): The range of the operator A.

$\mathbf{R},\ \mathbf{R}^+,\ \mathbf{R}^-$: $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^+ = [0, \infty)$, and $\mathbf{R}^- = (-\infty, 0]$.

$L^2(J; U)$: The set of U-valued L^2-functions on the interval J.

$TI(U; Y),\ TI(U)$: The set of bounded linear time-invariant operators from $L^2(\mathbf{R}; U)$ into $L^2(\mathbf{R}; Y)$, or from $L^2(\mathbf{R}; U)$ into itself.

$TIC(U; Y),\ TIC(U)$: The set of causal operators in $TI(U; Y)$ or $TI(U)$.

$\langle \cdot, \cdot \rangle_H$: The inner product in the Hilbert space H.

$\tau(t)$: The time shift operator $\tau(t)u(s) = u(t+s)$ (this is a left-shift when $t > 0$ and a right-shift when $t < 0$).

π_J: $(\pi_J u)(s) = \begin{cases} u(s) & \text{if } s \in J, \\ 0 & \text{if } s \notin J \end{cases}$, here $J \subset \mathbf{R}$ is an interval .

$\pi_+,\ \pi_-$: $\pi_+ = \pi_{\mathbf{R}^+}$ and $\pi_- = \pi_{\mathbf{R}^-}$.

$A \geq B,\ A >> B$: See Definition 2.5.

Throughout this paper U, W, H, and Y are separable Hilbert spaces, although many of the results that we prove are valid in nonseparable Hilbert spaces as well. (We make explicit use of the separability assumption only in the proof of Lemma 5.4, and we conjecture this lemma is true even in the nonseparable case.) The operators J and S satisfy $J = J^* \in \mathcal{L}(Y)$ and $S = S^* \in \mathcal{L}(U \times W)$. The operator T is given by $T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{L}(U \times W)$, where the block form is the one induced by the natural splitting of $U \times W$ into its components U and W. We frequently write V for $U \times W$.

We extend an L^2-function u defined on a subinterval J of $\mathbf{R}$ to the whole real line by requiring u to be zero outside of J, and we denote the extended function by $\pi_J u$. Thus, we use the same symbol π_J both for the embedding operator $L^2(J) \to L^2(\mathbf{R})$ and for the corresponding orthogonal projection operator $L^2(\mathbf{R}) \to \text{range}(\pi_J)$. With this interpretation, $\pi_+ L^2(\mathbf{R}) = L^2(\mathbf{R}^+) \subset L^2(\mathbf{R})$ and $\pi_- L^2(\mathbf{R}) = L^2(\mathbf{R}^-) \subset L^2(\mathbf{R})$.

Square brackets [ ] are used to denote optional parts of a statement. Such a statement remains valid if all the text within square brackets is omitted, and also if the appropriate parts of the statement are replaced by the text in the brackets.

2 The Implications (V) $\Rightarrow$ (I) $\Rightarrow$ (II)

In this section we present some basic definitions and preliminary results, and establish the (easy) implications (V) $\Rightarrow$ (I) $\Rightarrow$ (II) in Theorem 1.3.

Definition 2.1 *The operator $\mathcal{U} \in TIC(W;U)$ is a suboptimal (dynamic feedforward) compensator for $\Psi = \begin{bmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{bmatrix}$ if the (open loop) cost function $Q(x_0, u, w)$ defined in* (1.1) *satisfies*

$$Q(0, \mathcal{U}\pi_+ w, w) \leq 0$$

for all $w \in L^2(\mathbf{R}^+; W)$ (cf. Figure 1). It is a uniformly suboptimal compensator for Ψ if

$$Q(0, \mathcal{U}\pi_+ w, w) \leq -\epsilon \|w\|^2_{L^2(\mathbf{R}^+;W)}$$

for some $\epsilon > 0$ and all $w \in L^2(\mathbf{R}^+; W)$.

Notice our use of the word "compensator" to represent this class of feedforward controllers. We use this word in order to distinguish this class of controllers from the following class of controllers of state feedback/feedforward type:

Definition 2.2

(i) The triple $(\mathcal{K}, \mathcal{F}_1, \mathcal{F}_2)$ is a stabilizing state feedback/feedforward controller for $\Psi = \begin{bmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{bmatrix}$ if $\begin{bmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \\ \mathcal{F}_1 & \mathcal{F}_2 \end{bmatrix} \end{bmatrix}$ is a stable well-posed linear system, and $(I - \mathcal{F}_1)$ has an inverse in $TIC(U)$. (This means that the feedback connection drawn in Figure 2 is also a stable well-posed linear system (cf. [Staffans 1997, Proposition 20].)

(ii) The corresponding closed loop cost function $Q^\frown$ is given by

$$Q^\frown(x_0, u_\circlearrowleft, w) = \int_{\mathbf{R}^+} \langle y(s), Jy(s) \rangle_Y \, ds,$$

where $y = \mathcal{C}^\frown x_0 + \mathcal{D}_1^\frown \pi_+ u_\circlearrowleft + \mathcal{D}_2^\frown \pi_+ w$ is the output in Figure 2 with initial state $x_0 \in H$, control $u_\circlearrowleft \in L^2(\mathbf{R}^+; U)$, and disturbance $w \in L^2(\mathbf{R}^+; W)$. Here (cf. [Staffans 1997, Proposition 20])

$$\begin{aligned} \mathcal{C}^\frown &= \mathcal{C} + \mathcal{D}_1 (I - \mathcal{F}_1)^{-1} \mathcal{K}_1, \\ \mathcal{D}_1^\frown &= \mathcal{D}_1 (I - \mathcal{F}_1)^{-1}, \\ \mathcal{D}_2^\frown &= \mathcal{D}_2 + \mathcal{D}_1 (I - \mathcal{F}_1)^{-1} \mathcal{F}_2. \end{aligned}$$

(iii) *A stabilizing state feedback/feedforward controller* $(\mathcal{K}, \mathcal{F}_1, \mathcal{F}_2)$ *is suboptimal if the closed loop cost function* $Q^{\frown}(x_0, u, w)$ *satisfies*

$$Q^{\frown}(0, 0, w) \leq 0$$

for all $w \in L^2(\mathbf{R}^+; W)$.

(iv) *A stabilizing state feedback/feedforward controller* $(\mathcal{K}, \mathcal{F}_1, \mathcal{F}_2)$ *is uniformly suboptimal if the closed loop cost function* $Q^{\frown}(x_0, u, w)$ *satisfies*

$$Q^{\frown}(0, 0, w) \leq -\epsilon \|w\|^2_{L^2(\mathbf{R}^+;W)}$$

for some $\epsilon > 0$ *and all* $w \in L^2(\mathbf{R}^+; W)$.

Lemma 2.3 *If* $(\mathcal{K}, \mathcal{F}_1, \mathcal{F}_2)$ *is a [uniformly] suboptimal state feedback/feedforward controller for* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$, *then* $\mathcal{U} = (I - \mathcal{F}_1)^{-1}\mathcal{F}_2$ *is a [uniformly] suboptimal compensator for this system.*

Proof. This follows from Definitions 2.1 and 2.2 and the fact that if we take $\mathcal{U} = (I - \mathcal{F}_1)^{-1}\mathcal{F}_2$, then $Q(0, \mathcal{U}\pi_+ w, w) = Q^{\frown}(0, 0, w)$. $\square$

Definition 2.4 *In the sequel we refer to the compensator* $\mathcal{U} = (I - \mathcal{F}_1)^{-1}\mathcal{F}_2$ *in Lemma 2.3 as the compensator induced by the static state feedback/feedforward controller* $(\mathcal{K}, \mathcal{F}_1, \mathcal{F}_2)$.

Thus, the implication (V) $\Rightarrow$ (I) is valid. We proceed to prove the implication (I) $\Rightarrow$ (II).

Definition 2.5 *The operator* $A = A^* \in \mathcal{L}(H)$ *is positive [uniformly positive] if* $\langle x, Ax\rangle \geq 0$ $[\langle x, Ax\rangle \geq \epsilon\|x\|^2$ *for some* $\epsilon > 0]$ *for all* $x \in H$. *It is [uniformly] negative if* $-A$ *is [uniformly] positive. The notations* $A \geq B$ *and* $B \leq A$ $[A >> B$ *and* $B << A]$ *mean that* $A - B$ *is [uniformly] positive.*

Our proof of the implication (I) $\Rightarrow$ (II) uses the most elementary part of the following lemma, namely the existence of the minimizing function $u^{\min}(x_0, w)$ (the main part of this lemma will be needed later).

Lemma 2.6 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be a stable well-posed linear system. Then Hypothesis 1.1 holds iff $\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+ >> 0$ on $L^2(\mathbf{R}^+;U)$. In this case, for each fixed $x_0 \in H$ and $w \in L^2(\mathbf{R}^+;W)$, the function $u \mapsto Q(x_0,u,w)$ is uniformly convex on $L^2(\mathbf{R}^+;U)$, and there is a unique function $u^{\min}(x_0,w)$ that minimizes $Q(x_0,u,w)$ with respect to u. This function $u^{\min}$ and the corresponding output $y^{\min}$ and state $x^{\min}$ are given by*

$$
\begin{aligned}
u^{\min}(x_0,w) &= -\pi_+(\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^* J\left(\mathcal{C}x_0 + \mathcal{D}_2\pi_+ w\right), && (2.1)\\
y^{\min}(x_0,w) &= \mathcal{C}x_0 + \mathcal{D}_1\pi_+ u^{\min}(x_0,w) + \mathcal{D}_2\pi_+ w \\
&= \left(I - \mathcal{D}_1\pi_+(\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^* J\right)\left(\mathcal{C}x_0 + \mathcal{D}_2\pi_+ w\right) \\
&= (I - P_1)\left(\mathcal{C}x_0 + \mathcal{D}_2\pi_+ w\right), && (2.2)\\
x^{\min}(t,x_0,w) &= \mathcal{A}(t)x_0 + \mathcal{B}_1\tau(t)\pi_+ u^{\min}(x_0,w) + \mathcal{B}_2\tau(t)\pi_+ w \\
&= \mathcal{A}(t)x_0 + \mathcal{B}_2\tau(t)\pi_+ w \\
&\quad - \mathcal{B}_1\tau(t)\pi_+(\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^* J\left(\mathcal{C}x_0 + \mathcal{D}_2\pi_+ w\right), && (2.3)
\end{aligned}
$$

where $P_1 = \mathcal{D}_1\pi_+(\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^* J$ is the projection onto the range of $\mathcal{D}_1\pi_+$ along the null space of $\pi_+\mathcal{D}_1^* J$. The minimal cost $Q^{\min}(x_0,w) = Q(x_0,x^{\min}(x_0,w),w) = \min_{u\in L^2(\mathbf{R}^+;U)} Q(x_0,u,w)$ is given by

$$
\begin{aligned}
Q^{\min}(x_0,w) &= \langle y^{\min}(x_0,w), Jy^{\min}(x_0,w)\rangle_{L^2(\mathbf{R}^+;Y)} \\
&= \langle \left(\mathcal{C}x_0 + \mathcal{D}_2\pi_+ w\right), J(I-P_1)\left(\mathcal{C}x_0 + \mathcal{D}_2\pi_+ w\right)\rangle_{L^2(\mathbf{R}^+;Y)}. && (2.4)
\end{aligned}
$$

In particular, there is a constant $K < \infty$ such that

$$
Q^{\min}(x_0,w) \geq -K\left\|\mathcal{C}x_0 + \mathcal{D}_2\pi_+ w\right\|^2_{L^2(\mathbf{R}^+;Y)}. \tag{2.5}
$$

Moreover, the minimal output $y^{\min}$ satisfies

$$
\pi_+\mathcal{D}_1^* Jy^{\min}(x_0,w) = \pi_+\mathcal{D}_1^* J\left(\mathcal{C}x_0 + \mathcal{D}_1\pi_+ u^{\min}(x_0,w) + \mathcal{D}_2\pi_+ w\right) = 0. \tag{2.6}
$$

Proof. We begin by observing that $Q(0,u,0) = \langle \mathcal{D}_1\pi_+ u, J\mathcal{D}_1\pi_+ u\rangle_{L^2(\mathbf{R}^+;Y)}$, and this function is uniformly convex in $u \in L^2(\mathbf{R}^+;U)$ iff $\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+ >> 0$ on $L^2(\mathbf{R}^+;U)$. Fix some arbitrary $x_0 \in H$ and $w \in L^2(\mathbf{R}^+;W)$. Then the quadratic term of $Q(x_0,u,w)$ with respect to u is still equal to $\langle \mathcal{D}_1\pi_+ u, J\mathcal{D}_1\pi_+ u\rangle_{L^2(\mathbf{R}^+;Y)}$, so even for nonzero x_0 and w, it is true that $Q(x_0,u,w)$ is uniformly convex with respect to u iff $\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+ >> 0$. In this case there is a unique minimizing control $u^{\min}(x_0,w) \in L^2(\mathbf{R}^+;U)$. To show that the corresponding output $y^{\min}$ satisfies (2.6) we argue as follows. Without loss of generality, let us suppose that U is a real Hilbert space (if not, then we replace the inner product in U by the real inner product $\Re\langle\cdot,\cdot\rangle$), and let us compute the Fréchet derivative of the cost function $Q(x_0,u,w)$ with respect to u at the optimal $u^{\min}$. For each variation $\eta \in L^2(\mathbf{R}^+;U)$, we have

$$\begin{aligned} dQ(x_0,u^{\min})\eta &= 2\left\langle \mathcal{C}x_0 + \mathcal{D}_1\pi_+u^{\min} + \mathcal{D}_2\pi_+w, J\mathcal{D}_1\pi_+\eta\right\rangle_{L^2(\mathbf{R}^+;U)} \\ &= 2\left\langle y^{\min}, J\mathcal{D}_1\pi_+\eta\right\rangle_{L^2(\mathbf{R}^+;U)} \\ &= 2\left\langle \mathcal{D}_1^*Jy^{\min}, \eta\right\rangle_{L^2(\mathbf{R}^+;U)}. \end{aligned}$$

This is zero for all $\eta \in L^2(\mathbf{R}^+;U)$ iff (2.6) holds. Clearly, (2.1) follows from (2.6). By substituting this value for $u^{\min}$ into $x^{\min} = \mathcal{A}x_0 + \mathcal{B}_1\tau\pi_+u^{\min} + \mathcal{B}_2\tau\pi_+w$, $y^{\min} = \mathcal{C}x_0 + \mathcal{D}_1\pi_+u^{\min} + \mathcal{D}_2\pi_+w$, and $Q(x_0,u^{\min},w)$ (and making a straightforward computation) we get the remaining formulas. $\square$

Lemma 2.7 *Suppose that the stable well-posed linear system* $\Psi = \begin{bmatrix} \mathcal{A} & [\mathcal{B}_1\ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1\ \mathcal{D}_2] \end{bmatrix}$ *has a [uniformly] suboptimal compensator* $\mathcal{U}$*, and that Hypothesis 1.1 holds. Then, for each* $x_0 \in H$ *and* $w \in L^2(\mathbf{R}^+;W)$*, the function* $u \mapsto Q(x_0,u,w)$ *is uniformly convex on* $L^2(\mathbf{R}^+;U)$ *and, for each* $x_0 \in H$*, the function* $w \mapsto Q^{\min}(x_0,w) = \min_{u\in L^2(\mathbf{R}^+;U)} Q(x_0,u,w)$ *is [uniformly] concave on* $L^2(\mathbf{R}^+;W)$*.*

In other words, the implication (I) $\Rightarrow$ (II) in Theorem 1.3 is true.

Proof. The uniform convexity of $Q(x_0,u,w)$ with respect to $u \in L^2(\mathbf{R}^+;U)$ follows from Hypothesis 1.1 and Lemma 2.6. Let $\mathcal{U}$ be a [uniformly] suboptimal compensator. Then there is some $\epsilon \ge 0$ [or $\epsilon > 0$ in the uniform case] such that

$$Q(0,\mathcal{U}\pi_+w,w) \le -\epsilon\|w\|^2_{L^2(\mathbf{R}^+;W)}, \qquad w \in L^2(\mathbf{R}^+;W).$$

Clearly this implies that

$$Q^{\min}(0,w) \le -\epsilon\|w\|^2_{L^2(\mathbf{R}^+;W)}, \qquad w \in L^2(\mathbf{R}^+;W).$$

hence $Q^{\min}(0,w)$ is [uniformly] concave in $w \in L^2(\mathbf{R}^+;W)$. As in the proof of Lemma 2.6, this implies that, for all $x_0 \in H$, $Q^{\min}(x_0,w)$ is [uniformly] concave in $w \in L^2(\mathbf{R}^+;W)$. $\square$

3 Minimax J-Coercivity

The convexity-concavity property that we have encountered in (II) of Theorem 1.3 and also in Lemma 2.7 is important enough to get a name of its own.

Definition 3.1 *A stable system* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ *on* $(U \times W, H, Y)$ *is*

(i) J*-coercive if the Toeplitz operator* $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ *is invertible in* $\mathcal{L}(L^2(\mathbf{R}^+; U \times W))$*,*

(ii) minimax J*-coercive if, for each* $x_0 \in H$ *and* $w \in L^2(\mathbf{R}^+; W)$*, the function* $u \mapsto Q(x_0, u, w)$ *is uniformly convex on* $L^2(\mathbf{R}^+; U)$ *and, for each* $x_0 \in H$*, the function* $w \mapsto Q^{\min}(x_0, w) = \min_{u \in L^2(\mathbf{R}^+;U)} Q(x_0, u, w)$ *is [uniformly] concave on* $L^2(\mathbf{R}^+; W)$*.*

In the context of Weiss [1997], our J-minimax coercivity notion is closely related to Weiss' "analytic signature condition", which is a combination of Hypothesis 1.1 and condition (I); cf. Lemma 2.7.

We have the following more technical alternative characterization of minimax J-coercivity.

Lemma 3.2 *The system* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ *is minimax* J*-coercive iff*

$$\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ >> 0$$

on $L^2(\mathbf{R}^+; U)$ *and*

$$\pi_+\mathcal{D}_2^*\left(J - J\mathcal{D}_1\pi_+(\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^*J\right)\mathcal{D}_2\pi_+ << 0 \tag{3.1}$$

on $L^2(\mathbf{R}^+; W)$*. Here* $(\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}$ *stands for the inverse of* $\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+$ *in* $\mathcal{L}(L^2(\mathbf{R}^+; U))$ *(which exists since* $\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ >> 0$*).*

Proof. By Lemma 2.6, the function $u \mapsto Q(x_0, u, w)$ is uniformly convex on $L^2(\mathbf{R}^+; U)$ for each fixed $x_0 \in H$ and $w \in L^2(\mathbf{R}^+; W)$ iff $\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ >> 0$. Denote the operator on the left hand side of (3.1) by $\mathcal{E}$. Then, by (2.4), the quadratic term in the functional $w \mapsto Q^{\min}(x_0, w)$ is $\langle w, \mathcal{E}w\rangle_{L^2(\mathbf{R}^+;W)}$. Thus, this functional is uniformly concave iff $\mathcal{E} << 0$. $\square$

Minimax J-coercive systems have the following properties.

Lemma 3.3 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be minimax J-coercive. Then the following claims are true.*

(i) Ψ is J-coercive (i.e., $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ has a bounded inverse). In particular, the time-invariant operator $\mathcal{D}^*J\mathcal{D}$ is invertible in $TI(U \times W)$.

(ii) The inverse $\mathcal{E} = \left[\begin{smallmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{smallmatrix}\right]$ of $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ is given by

$$\begin{aligned}
\mathcal{E}_{22} &= (\pi_+\mathcal{D}_2^*\left(J - J\mathcal{D}_1\pi_+(\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^*J\right)\mathcal{D}_2\pi_+)^{-1}, \\
\mathcal{E}_{21} &= -\mathcal{E}_{22}\pi_+\mathcal{D}_2^*J\mathcal{D}_1\pi_+(\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}, \\
\mathcal{E}_{12} &= -(\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^*J\mathcal{D}_2\pi_+\mathcal{E}_{22}, \\
\mathcal{E}_{11} &= (\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1} + \mathcal{E}_{12}\mathcal{E}_{22}^{-1}\mathcal{E}_{12}.
\end{aligned}$$

(iii) The time-invariant operator $\mathcal{D}_1^*J\mathcal{D}_1$ is uniformly positive on $L^2(\mathbf{R}; U)$, and the time-invariant operator $\mathcal{D}_2^*\left(J - J\mathcal{D}_1(\mathcal{D}_1^*J\mathcal{D}_1)^{-1}\mathcal{D}_1^*J\right)\mathcal{D}_2$ is uniformly negative on $L^2(\mathbf{R}; W)$.

Proof. (i) Since $\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ >> 0$, hence invertible, we conclude from the Schur decomposition

$$\begin{aligned}
&\pi_+ \begin{bmatrix} \mathcal{D}_1^* \\ \mathcal{D}_2^* \end{bmatrix} J \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \end{bmatrix} \pi_+ \\
&\quad = \begin{bmatrix} I & 0 \\ \pi_+\mathcal{D}_2^*J\mathcal{D}_1\pi_+(\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}\pi_+ & I \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ & 0 \\ 0 & \pi_+\mathcal{D}_2^*\left(J - J\mathcal{D}_1\pi_+(\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^*J\right)\mathcal{D}_2\pi_+ \end{bmatrix} \\
&\qquad \times \begin{bmatrix} I & (\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^*J\mathcal{D}_2\pi_+ \\ 0 & I \end{bmatrix}
\end{aligned}$$

that this Toeplitz operator in invertible. Thus $\mathcal{D}$ is J-coercive. The invertibility of $\mathcal{D}^*J\mathcal{D}$ follows from [Staffans 1998c, Lemma 4.4(iii)].

(ii) We get (ii) by inverting each of the operators in the Schur decomposition given above, and multiplying the results.

(iii) The uniform positivity of $\mathcal{D}_1^*J\mathcal{D}_1$ follows from [Staffans 1998c, Lemma 4.4(ii)]. To get the second claim we use the same lemma to get for some $\epsilon > 0$,

$$\begin{aligned}
&-\epsilon\tau(-t)\pi_+\tau(t) \\
&\quad \geq \tau(-t)\pi_+\mathcal{D}_2^*\left(J - J\mathcal{D}_1\pi_+(\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+)^{-1}\pi_+\mathcal{D}_1^*J\right)\mathcal{D}_2\pi_+\tau(t) \\
&\quad \geq \tau(-t)\pi_+\mathcal{D}_2^*\left(J - J\mathcal{D}_1(\mathcal{D}_1^*J\mathcal{D}_1)^{-1}\mathcal{D}_1^*J\right)\mathcal{D}_2\pi_+\tau(t).
\end{aligned}$$

Let $t \to -\infty$, and use [Staffans 1998c, Lemma 4.4(i)] to conclude that

$$\mathcal{D}_2^* \left(J - J\mathcal{D}_1(\mathcal{D}_1^* J\mathcal{D}_1)^{-1}\mathcal{D}_1^* J \right) \mathcal{D}_2 << 0. \qquad \square$$

By combining Lemmas 3.2 and 3.3 we get still another characterization of minimax J-coercivity.

Lemma 3.4 *The system* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ *is minimax J-coercive iff the following three conditions hold:*

(i) Ψ *is J-coercive, i.e.,* $\pi_+\mathcal{D}^* J\mathcal{D}\pi_+$ *is invertible,*

(ii) $\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+ >> 0$ *on* $L^2(\mathbf{R}^+; U)$,

(iii) $\mathcal{E}_{22} << 0$ *on* $L^2(\mathbf{R}^+; W)$, *where* $\mathcal{E} = \left[\begin{smallmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{smallmatrix}\right]$ *is the inverse of* $\pi_+\mathcal{D}^* J\mathcal{D}\pi_+$.

As we mentioned in Section 1, to investigate the validity of the implication (II) $\Rightarrow$ (III) we employ a minimax argument, where we maximize the function $Q^{\min}(x_0, w)$ with respect to w to find the "worst possible disturbance". As the following lemma shows, this maximization is straightforward.

Lemma 3.5 *Let* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ *be minimax J-coercive. Define* $u^{\min}$, $x^{\min}$, $y^{\min}$, *and* $Q^{\min}$ *as in Lemma 2.6. Then, for each fixed* $x_0 \in H$, *there is a unique function* $w^{\mathrm{crit}}(x_0)$ *that maximizes* $Q^{\min}(x_0, w)$ *with respect to* $w \in L^2(\mathbf{R}^+; W)$. *Define* $x^{\mathrm{crit}}(x_0) = x^{\min}(x_0, w^{\mathrm{crit}}(x_0))$, $u^{\mathrm{crit}}(x_0) = u^{\min}(x_0, w^{\mathrm{crit}}(x_0))$, $y^{\mathrm{crit}}(x_0) = y^{\min}(x_0, w^{\mathrm{crit}}(x_0))$, *and* $Q^{\mathrm{crit}} = Q^{\min}(x_0, w^{\mathrm{crit}}(x_0))$. *Then* $x^{\mathrm{crit}}(x_0)$, $y^{\mathrm{crit}}(x_0)$, $u^{\mathrm{crit}}(x_0)$, *and* $w^{\mathrm{crit}}(x_0)$ *are given by*

$$\begin{bmatrix} u^{\mathrm{crit}}(x_0) \\ w^{\mathrm{crit}}(x_0) \end{bmatrix} = -\pi_+(\pi_+\mathcal{D}^* J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^* J\mathcal{C}x_0, \tag{3.2}$$

$$\begin{aligned} y^{\mathrm{crit}}(x_0) &= \mathcal{C}x_0 + \mathcal{D}\pi_+ \begin{bmatrix} u^{\mathrm{crit}}(x_0) \\ w^{\mathrm{crit}}(x_0) \end{bmatrix} \\ &= \left(I - \mathcal{D}\pi_+(\pi_+\mathcal{D}^* J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^* J \right) \mathcal{C}x_0 \\ &= (I - P)\mathcal{C}x_0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} x^{\mathrm{crit}}(t, x_0) &= \mathcal{A}(t)x_0 + \mathcal{B}\tau(t)\pi_+ \begin{bmatrix} u^{\mathrm{crit}}(x_0) \\ w^{\mathrm{crit}}(x_0) \end{bmatrix} \\ &= \mathcal{A}(t)x_0 - \mathcal{B}\tau(t)\pi_+(\pi_+\mathcal{D}^* J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^* J\mathcal{C}x_0, \end{aligned} \tag{3.4}$$

where $P = \mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J$ *is the projection onto the range of* $\mathcal{D}\pi_+$ *along the null space of* $\pi_+\mathcal{D}^*J$*. The minimax cost is given by*

$$\begin{aligned} Q^{\mathrm{crit}}(x_0) &= \max_{w\in L^2(\mathbf{R}^+;W)} Q^{\min}(x_0,w). \\ &= \langle y^{\mathrm{crit}}(x_0), Jy^{\mathrm{crit}}(x_0)\rangle_{L^2(\mathbf{R}^+;Y)} \\ &= \langle x_0, \mathcal{C}^*J(I-P)\mathcal{C}x_0\rangle_H. \end{aligned} \tag{3.5}$$

Moreover, the minimax output y^{crit} *satisfies*

$$\pi_+\mathcal{D}^*Jy^{\mathrm{crit}}(x_0) = \pi_+\mathcal{D}^*J\left(\mathcal{C}x_0 + \mathcal{D}\pi_+\begin{bmatrix} u^{\min}(x_0) \\ w^{\mathrm{crit}}(x_0)\end{bmatrix}\right) = 0. \tag{3.6}$$

Proof. Arguing in the same way as in the proof of Lemma 2.6 we find that (since $J_1 = J(I-P_1)$ is self-adjoint)

$$\begin{aligned} 0 &= \mathcal{D}_2^*J(I-P_1)(\mathcal{C}x_0 + \mathcal{D}_2\pi_+w^{\mathrm{crit}}) \\ &= \mathcal{D}_2^*Jy^{\min}(x_0,w^{\mathrm{crit}}) \\ &= \mathcal{D}_2^*Jy^{\mathrm{crit}}(x_0). \end{aligned}$$

By (2.6), also $\mathcal{D}_1^*Jy^{\mathrm{crit}}(x_0) = 0$, hence (3.6) holds. The rest of the proof follows the same lines as the proof of Lemma 2.6. $\square$

It follows from (3.6) and [Staffans 1998c, Lemma 3.2] that the pair $\left[\begin{smallmatrix} u^{\mathrm{crit}}(x_0) \\ w^{\mathrm{crit}}(x_0)\end{smallmatrix}\right]$ is a *J-critical control* pair in the sense of [Staffans 1998c, Definition 3.1]. Thus, all those results of Staffans [1998c] that deal with stable systems can be applied to stable minimax J-coercive systems. In particular, we recall the following definition:

Definition 3.6 ([Staffans 1998c, Definition 3.5]) *Let* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \;\; \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \;\; \mathcal{D}_2]\end{smallmatrix}\right]$ *be a J-coercive stable well-posed linear system on* $(U\times W, H, Y)$*. Then we define*

$$\begin{aligned} \mathcal{A}_\circlearrowleft(t) &= \mathcal{A}(t) - \mathcal{B}\tau(t)\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\mathcal{C}, \\ \mathcal{C}_\circlearrowleft &= \left(I - \mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\right)\mathcal{C}, \\ \mathcal{K}_\circlearrowleft &= \begin{bmatrix} \mathcal{K}_{\circlearrowleft 1} \\ \mathcal{K}_{\circlearrowleft 2}\end{bmatrix} = -(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\mathcal{C}, \\ \Pi &= \mathcal{C}^*\left(J - J\mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\right)\mathcal{C}. \end{aligned}$$

The operator Π *is called the Riccati operator of* Ψ *(with respect to the operator* J*).*

Note that in order to define these operators, it suffices if Ψ is J-coercive; it need not be minimax J-coercive. According to Lemma 3.5, if Ψ is minimax J-coercive, then $x^{\mathrm{crit}} = \mathcal{A}_{\circlearrowleft}(t)x_0$, $y^{\mathrm{crit}} = \mathcal{C}_{\circlearrowleft}x_0$, $u^{\mathrm{crit}} = \mathcal{K}_{\circlearrowleft 1}x_0$, $w^{\mathrm{crit}} = \mathcal{K}_{\circlearrowleft 2}x_0$, and $Q^{\mathrm{crit}}(x_0) = \langle x_0, \Pi x_0\rangle_H$. Moreover, by (3.6)

$$\pi_+\mathcal{D}^*J\mathcal{C}_{\circlearrowleft} = 0. \tag{3.7}$$

4 (J,S)-Inner-Outer Factorizations and Feedback Representations

If Ψ is minimax J-coercive, then the construction in the preceding section gave us a unique minimax control/disturbance pair. As shown in Staffans [1998c], it is possible to get a feedback/feedforward representation for this pair if we can find a (J,S)-inner-outer factorization of the input/output map $\mathcal{D}$. This notion and some related notions are defined as follows (with $V = U \times W$).

Definition 4.1 ([Staffans 1998c, Definition 4.5]) *Let $J = J^* \in \mathcal{L}(Y)$, and let $S = S^* \in \mathcal{L}(V)$.*

(i) The operator $\mathcal{N} \in TIC(V;Y)$ is (J,S)-inner if $\mathcal{N}^*J\mathcal{N} = S$.

(ii) The operator $\mathcal{X} \in TIC(V)$ is outer if the image of $L^2(\mathbf{R}^+;V)$ under $\mathcal{X}\pi_+$ is dense in $L^2(\mathbf{R}^+;V)$.

(iii) The operator $\mathcal{X} \in TIC(V)$ is an (invertible) S-spectral factor of $\mathcal{D}^*J\mathcal{D} \in TI(V)$ if $\mathcal{X}$ is invertible in $TIC(V)$ and $\mathcal{D}^*J\mathcal{D} = \mathcal{X}^*S\mathcal{X}$.

(iv) The factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a (J,S)-inner-outer factorization of $\mathcal{D} \in TIC(V;Y)$ if $\mathcal{N} \in TIC(V;Y)$ is (J,S)-inner and $\mathcal{X} \in TIC(V)$ is outer.

(v) In each case we call S the sensitivity operator of $\mathcal{N}$ or of the factorization.

There is a simple connection between inner-outer factorizations and spectral factorizations.

Lemma 4.2 ([Staffans 1998c, Lemma 4.6(i)]) *If $\mathcal{X}$ is a S-spectral factor of $\mathcal{D}^*J\mathcal{D}$, then $\mathcal{N}\mathcal{X} = (\mathcal{D}\mathcal{X}^{-1})\,\mathcal{X}$ is a (J,S)-inner-outer factorization of $\mathcal{D}$. Conversely, if $\mathcal{N}\mathcal{X}$ is a (J,S)-inner-outer factorization of $\mathcal{D}$ and $\mathcal{X}$ is invertible in $TIC(V)$, then $\mathcal{X}$ is a S-spectral factor of $\mathcal{D}^*J\mathcal{D}$.*

In the classical case the existence of a S-spectral factor of $\mathcal{D}^*J\mathcal{D}$ is guaranteed whenever Ψ is J-coercive. In particular, in this case it follows from Lemmas 2.7 and 4.2 that $\mathcal{D}$ has a (J,S)-inner-outer factorization whenever Ψ is minimax J-coercive. This is no longer true in the infinite-dimensional case (see [Staffans 1998c, Remark 4.8]), except in special cases, such as the following.

Definition 4.3 *A system $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ on (V,H,Y) has an L^1 impulse response if $\mathcal{D}$ is a convolution operator of the form (for all $v \in L^2(\mathbf{R};V)$ and almost all $t \in \mathbf{R}$)*

$$(\mathcal{D}v)(t) = Dv(t) + \int_{-\infty}^{t} E(t-s)v(s)\,ds,$$

where $D \in \mathcal{L}(U;Y)$ and $E \in L^1(\mathbf{R}^+;\mathcal{L}(V;Y))$.

Lemma 4.4 ([Staffans 1998c, Corollary 4.10]) *If both U and W are finite-dimensional, and if $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ is J-coercive and has an L^1 impulse response, then $\mathcal{D}$ has a (J,S)-inner-outer factorization.*

As our next theorem shows, the existence of a well-posed state feedback/feedforward representation of the critical control/disturbance pair is equivalent to the existence of a (J,S)-inner-outer factorization of $\mathcal{D}$.

Theorem 4.5 ([Staffans 1998c, Theorem 5.1]) *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be a stable J-coercive well-posed linear system on $(U\times W,H,Y)$. Define u^{crit}, x^{crit}, y^{crit}, and Q^{crit} as in* (3.2)–(3.5)*, and let Π be the Riccati operator defined in Definition 3.6.*

(i) Suppose that $\mathcal{D}$ has a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. Then S is invertible in $\mathcal{L}(U\times W)$, $\mathcal{X}$ is invertible in $TIC(U\times W)$, and $\mathcal{X}$ is a S-spectral factor of $\mathcal{D}^*J\mathcal{D}$. Define $\mathcal{M} = \mathcal{X}^{-1}$ and $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix} =$

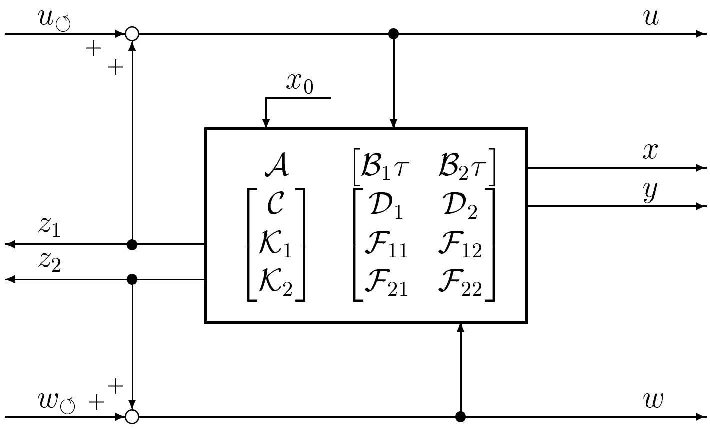

Figure 3: Closed loop feedback connection

$\begin{bmatrix}-S^{-1}\pi_+\mathcal{N}^*J\mathcal{C} & (I-\mathcal{X})\end{bmatrix}$. *Then* $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix}$ *is a stabilizing state feedback/feedforward controller for* Ψ*, i.e., the feedback connection drawn in Figure 3 defines a well-posed linear system* $\Psi_{\circlearrowleft}$*, given by*

$$\Psi_{\circlearrowleft} = \begin{bmatrix} \mathcal{A}_{\circlearrowleft} & \mathcal{B}_{\circlearrowleft} \\ \begin{bmatrix}\mathcal{C}_{\circlearrowleft}\\ \mathcal{K}_{\circlearrowleft}\end{bmatrix} & \begin{bmatrix}\mathcal{D}_{\circlearrowleft}\\ \mathcal{F}_{\circlearrowleft}\end{bmatrix}\end{bmatrix} = \begin{bmatrix} \mathcal{A}+\mathcal{B}\mathcal{M}\tau\mathcal{K} & \mathcal{B}\mathcal{M} \\ \begin{bmatrix}\mathcal{C}+\mathcal{N}\mathcal{K}\\ \mathcal{M}\mathcal{K}\end{bmatrix} & \begin{bmatrix}\mathcal{N}\\ \mathcal{M}-I\end{bmatrix}\end{bmatrix}.$$

Moreover, the state and outputs of this closed loop system are equal to $x^{\mathrm{crit}}(t,x_0)$*,* $y^{\mathrm{crit}}(x_0)$*, and* $\left[\begin{smallmatrix}u^{\mathrm{crit}}(x_0)\\ w^{\mathrm{crit}}(x_0)\end{smallmatrix}\right]$*, respectively, if we take the two closed loop inputs* $u_{\circlearrowleft}$ *and* $w_{\circlearrowleft}$ *to be zero. The Riccati operator* Π *of* Ψ *can be written in the following alternative forms:*

$$\Pi = \mathcal{C}^*J\mathcal{C} - \mathcal{K}^*S\mathcal{K} = \mathcal{C}^*J\mathcal{C}_{\circlearrowleft} = \mathcal{C}_{\circlearrowleft}{}^*J\mathcal{C}_{\circlearrowleft} = \mathcal{C}_{\circlearrowleft}{}^*J\mathcal{C}.$$

(ii) Conversely, suppose that $\left[\begin{smallmatrix}y^{\mathrm{crit}}(x_0)\\ u^{\mathrm{crit}}(x_0)\\ w^{\mathrm{crit}}(x_0)\end{smallmatrix}\right]$ *is equal to the output of some stable state feedback perturbation* $\Psi_{\circlearrowleft}$ *of* Ψ *with initial value* x_0*, initial time* 0*, zero control, zero disturbance, and some admissible stable state feedback pair* $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix}$*. Then there exists an operator* $S = S^* \in \mathcal{L}(U \times W)$ *such that* $\mathcal{N}\mathcal{X}$ *is a* (J,S)*-inner-outer factorization of* $\mathcal{D}$*, where* $\mathcal{N} = \mathcal{D}(I-\mathcal{F})^{-1}$ *and* $\mathcal{X} = (I-\mathcal{F})$*. Moreover,* $\mathcal{K}$ *is given by* $\mathcal{K} = -S^{-1}\pi_+\mathcal{N}^*J\mathcal{C}$*.*

(iii) Let the two equivalent conditions (i) and (ii) hold. If $y = \mathcal{C}_{\circlearrowleft}x_0 + \mathcal{D}_{\circlearrowleft 1}\pi_+u_{\circlearrowleft} + \mathcal{D}_{\circlearrowleft 2}\pi_+w_{\circlearrowleft}$ is the first output of the critical closed loop system $\Psi_{\circlearrowleft}$ with initial state $x_0 \in H$, control $u_{\circlearrowleft} \in L^2(\mathbf{R}^+;U)$, and disturbance $w_{\circlearrowleft} \in L^2(\mathbf{R}^+;W)$, then the closed loop cost $Q_{\circlearrowleft}(x_0,u_{\circlearrowleft},w_{\circlearrowleft})$ is given by

$$\begin{aligned} Q_{\circlearrowleft}(x_0,u_{\circlearrowleft},w_{\circlearrowleft}) &= \int_{\mathbf{R}^+} \langle y(s), Jy(s)\rangle_Y \; ds \\ &= \langle x_0, \Pi x_0\rangle_H + \int_{\mathbf{R}^+} \left\langle \begin{bmatrix} u_{\circlearrowleft}(s) \\ w_{\circlearrowleft}(s) \end{bmatrix}, S \begin{bmatrix} u_{\circlearrowleft}(s) \\ w_{\circlearrowleft}(s) \end{bmatrix} \right\rangle_U ds. \end{aligned} \tag{4.1}$$

To get any further in our proof of the implication (II) $\Rightarrow$ (III) we need a feedback representation of the critical minimax solution of the type described in Theorem 4.5, and we therefore have to invoke the following "regularity" hypothesis.

Hypothesis 4.6 *The input/output map $\mathcal{D}$ has a (J,S)-inner-outer factorization.*

By Lemma 4.4, this hypothesis is redundant if U and W are finite-dimensional and Ψ is J-coercive and has an L^1 impulse response. It is still an open problem to what extent this hypothesis can be weakened.

The reason for calling S the *sensitivity operator* associated with the given factorization is found in the final formula for the cost given in part (iii) of Theorem 4.5. Observe that this formula rewrites the cost in terms of the initial state x_0 and the two closed loop inputs $u_{\circlearrowleft}$ and $w_{\circlearrowleft}$ in Figure 3. This formula plays a key role in the subsequent development.

Several times in the sequel we need to pass back and forth between the open loop system Ψ and the corresponding closed loop system $\Psi_{\circlearrowleft}$. This passage is greatly simplified by the following remarks.

Remark 4.7 ([Staffans 1998a, Remark 3.9]) *It is possible, and in many cases more convenient, to replace the feedback output z in Figure 4 by the output $v = \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, which is equal to the input of the original system. (In this figure we have combined the two inputs $\left[\begin{smallmatrix} u_{\circlearrowleft} \\ w_{\circlearrowleft} \end{smallmatrix}\right]$ into one input denoted by $v_{\circlearrowleft}$, and also combined the two feedback outputs signals into one signal called z; cf. Figure 3.) This only amounts to the addition of a identity feedforward term to the input/output map from $v_{\circlearrowleft}$ to z, so the new input/output map from z to v is $\mathcal{M}$ instead of $\mathcal{M} - I$ that appears in the bottom right corner in the definition of $\Psi_{\circlearrowleft}$. All the other elements of $\Psi_{\circlearrowleft}$ remain unchanged in this setting.*

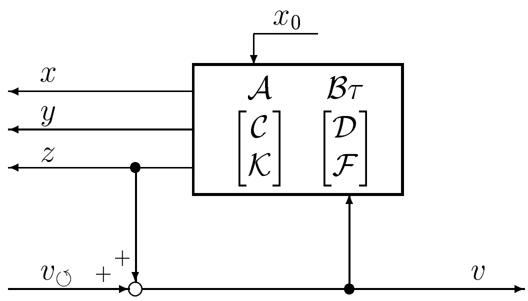

Figure 4: Closed loop version of state feedback connection

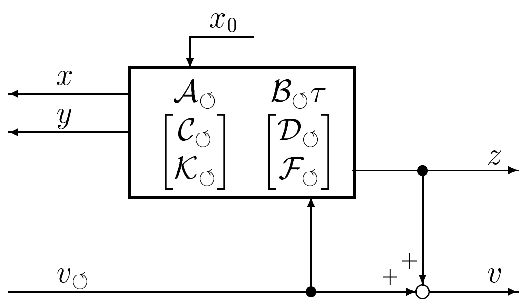

Figure 5: Closed loop system written in open loop form

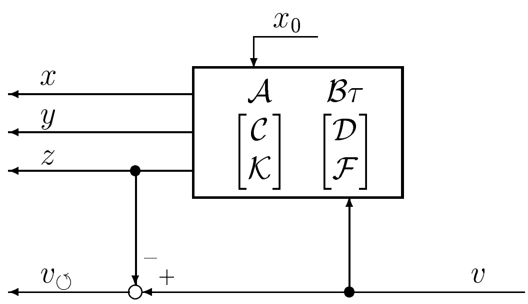

Figure 6: Open loop version of state feedback connection

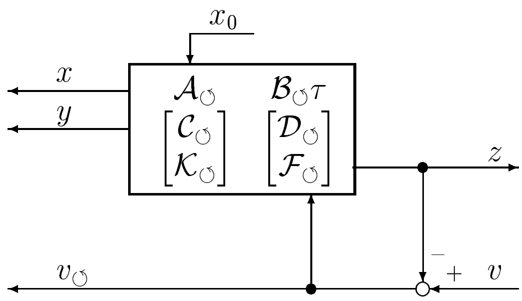

Figure 7: Use negative feedback to recover the open loop system

Lemma 4.8 *The state x, output y, and state feedback output z of the closed loop system $\Psi_\circlearrowleft$ with initial value x_0 and control/disturbance pair $v_\circlearrowleft = \left[\begin{smallmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{smallmatrix}\right]$ is equal to the state, output, and state feedback output of the open loop system*

$$\Psi^{\text{ext}} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \end{bmatrix} \end{bmatrix}$$

with initial value x_0 and control/disturbance pair

$$\begin{aligned}
\begin{bmatrix} u \\ w \end{bmatrix} &= v = z + \pi_+ v_\circlearrowleft \\
&= \mathcal{K}x_0 + \mathcal{F}v + \pi_+ v_\circlearrowleft \\
&= \mathcal{K}_\circlearrowleft x_0 + \mathcal{F}_\circlearrowleft \pi_+ v_\circlearrowleft + \pi_+ v_\circlearrowleft \\
&= \begin{bmatrix} \mathcal{K}_1 x_0 + \mathcal{F}_{11} u + \mathcal{F}_{12} w + \pi_+ u_\circlearrowleft \\ \mathcal{K}_2 x_0 + \mathcal{F}_{21} u + \mathcal{F}_{22} w + \pi_+ w_\circlearrowleft \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{K}_{\circlearrowleft 1} x_0 + \mathcal{M}_{11} \pi_+ u_\circlearrowleft + \mathcal{M}_{12} \pi_+ w_\circlearrowleft \\ \mathcal{K}_{\circlearrowleft 2} x_0 + \mathcal{M}_{21} \pi_+ u_\circlearrowleft + \mathcal{M}_{22} \pi_+ w_\circlearrowleft \end{bmatrix},
\end{aligned}$$

where $\mathcal{M} = (I - \mathcal{F})^{-1}$. Conversely, the state x, output y, and state feedback output z of the open loop system Ψ^{ext} with initial value x_0 and control/disturbance pair $v = \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ is equal to the state, output, and state feedback output of the closed loop system $\Psi_\circlearrowleft$ with initial value x_0 and con-

trol/disturbance pair

$$\begin{aligned}\begin{bmatrix} u_{\circlearrowleft} \\ w_{\circlearrowleft} \end{bmatrix} &= v_{\circlearrowleft} = -z + \pi_+ v \\ &= -\mathcal{K}_{\circlearrowleft} x_0 - \mathcal{F}_{\circlearrowleft} v_{\circlearrowleft} + \pi_+ v \\ &= -\mathcal{K} x_0 - \mathcal{F} \pi_+ v + \pi_+ v \\ &= \begin{bmatrix} -\mathcal{K}_{\circlearrowleft 1} x_0 - \mathcal{F}_{\circlearrowleft 11} u_{\circlearrowleft} - \mathcal{F}_{\circlearrowleft 12} w_{\circlearrowleft} + \pi_+ u \\ -\mathcal{K}_{\circlearrowleft 2} x_0 - \mathcal{F}_{\circlearrowleft 21} u_{\circlearrowleft} - \mathcal{F}_{\circlearrowleft 22} w_{\circlearrowleft} + \pi_+ w \end{bmatrix} \\ &= \begin{bmatrix} -\mathcal{K}_1 x_0 + \mathcal{X}_{11} \pi_+ u + \mathcal{X}_{12} \pi_+ w \\ -\mathcal{K}_2 x_0 + \mathcal{X}_{21} \pi_+ u + \mathcal{X}_{22} \pi_+ w \end{bmatrix},\end{aligned}$$

where $\mathcal{X} = I - \mathcal{F} = \mathcal{M}^{-1}$.

Proof. The first half of this lemma is obvious; see the equivalent Figures 4 and 5. The second half of the lemma is equally obvious since the connections drawn in Figures 6 and 7 are equivalent to those in Figures 4 and 5. $\square$

Corollary 4.9 *The open and closed loop cost functions defined in (1.1) and (4.1) satisfy*

$$Q(x_0, u, w) = Q_{\circlearrowleft}(x_0, u_{\circlearrowleft}, w_{\circlearrowleft})$$

if we choose the control and disturbance signals u, w, $u_{\circlearrowleft}$, *and* $w_{\circlearrowleft}$ *to satisfy any one of the following four equivalent sets of equations*

$$\begin{aligned}\begin{bmatrix} u_{\circlearrowleft} \\ w_{\circlearrowleft} \end{bmatrix} &= \begin{bmatrix} -\mathcal{K}_{\circlearrowleft 1} x_0 - \mathcal{F}_{\circlearrowleft 11} u_{\circlearrowleft} - \mathcal{F}_{\circlearrowleft 12} w_{\circlearrowleft} + \pi_+ u \\ -\mathcal{K}_{\circlearrowleft 2} x_0 - \mathcal{F}_{\circlearrowleft 21} u_{\circlearrowleft} - \mathcal{F}_{\circlearrowleft 22} w_{\circlearrowleft} + \pi_+ w \end{bmatrix}, \\ \begin{bmatrix} u_{\circlearrowleft} \\ w_{\circlearrowleft} \end{bmatrix} &= \begin{bmatrix} -\mathcal{K}_1 x_0 + \mathcal{X}_{11} \pi_+ u + \mathcal{X}_{12} \pi_+ w \\ -\mathcal{K}_2 x_0 + \mathcal{X}_{21} \pi_+ u + \mathcal{X}_{22} \pi_+ w \end{bmatrix}, \\ \begin{bmatrix} u \\ w \end{bmatrix} &= \begin{bmatrix} \mathcal{K}_1 x_0 + \mathcal{F}_{11} u + \mathcal{F}_{12} w + \pi_+ u_{\circlearrowleft} \\ \mathcal{K}_2 x_0 + \mathcal{F}_{21} u + \mathcal{F}_{22} w + \pi_+ w_{\circlearrowleft} \end{bmatrix}, \\ \begin{bmatrix} u \\ w \end{bmatrix} &= \begin{bmatrix} \mathcal{K}_{\circlearrowleft 1} x_0 + \mathcal{M}_{11} \pi_+ u_{\circlearrowleft} + \mathcal{M}_{12} \pi_+ w_{\circlearrowleft} \\ \mathcal{K}_{\circlearrowleft 2} x_0 + \mathcal{M}_{21} \pi_+ u_{\circlearrowleft} + \mathcal{M}_{22} \pi_+ w_{\circlearrowleft} \end{bmatrix}.\end{aligned}$$

Here $\mathcal{F} = I - \mathcal{X}$ *and* $\mathcal{M} = \mathcal{X}^{-1}$.

Let us remark that the first and third equation are written in feedback form corresponding to Figures 7 and 4, and that the second and forth equation above are written in explicit input/output form, corresponding Figures 6 and 5.

For the convenience of the reader, let us recall still a few more results from Staffans [1998b] and Staffans [1998c] concerning the feedback representation of the critical solution.

Lemma 4.10 *Let $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ be a particular $\widetilde{S}$-inner-outer factorization of $\mathcal{D}$, and define $\widetilde{\mathcal{M}} = \widetilde{\mathcal{X}}^{-1}$. Then the set of all possible sensitivity operators S, the set of all possible (J,S)-inner-outer factorizations of $\mathcal{D}$, and the set of all possible operators $\mathcal{M}$ in Theorem 4.5 can be parameterized as*

$$S = E^*\widetilde{S}E, \qquad \mathcal{X} = E^{-1}\widetilde{\mathcal{X}}, \qquad \mathcal{N} = \widetilde{\mathcal{N}}E, \qquad \mathcal{M} = \widetilde{\mathcal{M}}E,$$

where E varies over the set of all invertible operators in $\mathcal{L}(U \times W)$. The corresponding feedback pair $\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix}$ in Theorem 4.5 is given by

$$\mathcal{K} = E^{-1}\widetilde{\mathcal{K}}, \qquad (I - \mathcal{F}) = E^{-1}(I - \widetilde{\mathcal{F}}),$$

where $\widetilde{\mathcal{K}} = -\widetilde{S}\pi_+\widetilde{\mathcal{N}}^*J\mathcal{C}$ and $\widetilde{\mathcal{F}} = \left(I - \widetilde{\mathcal{M}}^{-1}\right)$, i.e., $\begin{bmatrix}\widetilde{\mathcal{K}} & \widetilde{\mathcal{F}}\end{bmatrix}$ is the feedback pair in Theorem 4.5 corresponding to the factorization $\mathcal{D} = \widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$. The parameterized version of the formula for the closed loop system in Theorem 4.5 is

$$\Psi_{\circlearrowleft} = \begin{bmatrix} \mathcal{A}_{\circlearrowleft} & \mathcal{B}_{\circlearrowleft} \\ \begin{bmatrix}\mathcal{C}_{\circlearrowleft} \\ \mathcal{K}_{\circlearrowleft}\end{bmatrix} & \begin{bmatrix}\mathcal{D}_{\circlearrowleft} \\ \mathcal{F}_{\circlearrowleft}\end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}\widetilde{\mathcal{M}}\tau\widetilde{\mathcal{K}} & \mathcal{B}\widetilde{\mathcal{M}}E \\ \begin{bmatrix}\mathcal{C} + \widetilde{\mathcal{N}}\widetilde{\mathcal{K}} \\ \widetilde{\mathcal{M}}\widetilde{\mathcal{K}}\end{bmatrix} & \begin{bmatrix}\widetilde{\mathcal{N}}E \\ \widetilde{\mathcal{M}}E - I\end{bmatrix} \end{bmatrix}.$$

The first column is independent of E (but the second is not).

Proof. This follows from [Staffans 1998b, Proposition 4.7] and [Staffans 1998c, Remark 5.2]. $\square$

The operator E has a very simple interpretation: it represents a coordinate change in the input space for the closed loop system.

Proposition 4.11 *In addition to the notation introduced in Lemma 4.10, denote the vectors $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u_{\circlearrowleft} \\ w_{\circlearrowleft} \end{smallmatrix}\right]$ by v and $v_{\circlearrowleft}$, respectively. Then the two diagrams drawn in Figures 8 and 9 are equivalent in the sense that the relationships between all the signals with identical names are identical in the two diagrams (but z differs in general from $\tilde{z}$.)*

Proof. This follows from [Staffans 1998b, Proposition 4.8] and [Staffans 1998c, Remark 5.2]. $\square$

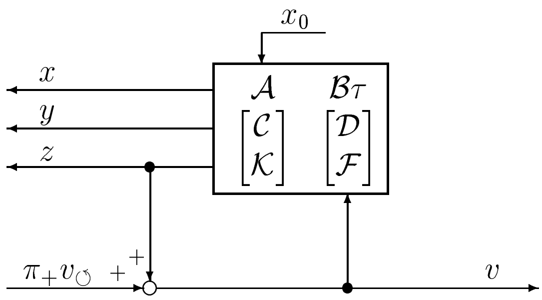

Figure 8: Internal parameterization of the feedback equilibrium

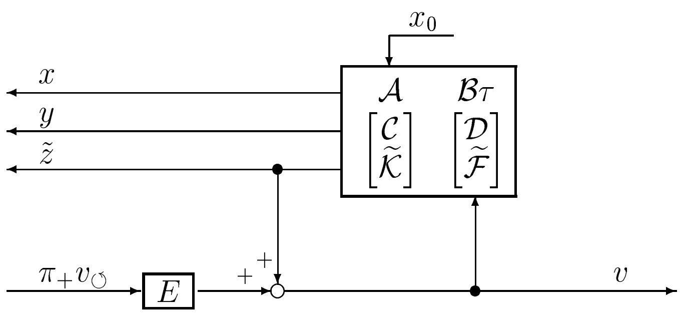

Figure 9: External parameterization of the feedback equilibrium

5 Minimax Properties of the Closed Loop System

Lemma 4.10 contains the free parameter E. According to Theorem 4.5 and Lemma 4.10, all possible choices of E lead to equivalent control strategies in the sense that as long as neither of the two players deviate from their "critical" strategy (i.e., $v_{\circlearrowleft} = 0$ in Figures 8 and 9) the actual control $u^{\mathrm{crit}}(x_0)$ and disturbance $w^{\mathrm{crit}}(x_0)$ will remain the same, and they are equal to the minimax pair defined in Lemma 3.5. However, if either of the players deviates from the critical strategy, then the behavior of the closed loop system depends strongly on the parameter E. This parameter must be chosen in such a way that the closed loop system has the appropriate minimax property.

Definition 5.1 *Let $Q\colon H \times L^2(\mathbf{R}^+;U) \times L^2(\mathbf{R}^+;W) \to \mathbf{R}$. We call the point $(x_0, u_0, w_0) \in H \times L^2(\mathbf{R}^+;U) \times L^2(\mathbf{R}^+;W)$*

(i) a saddle point of Q (with respect to the last two variables) if

$$Q(x_0, u, w_0) \geq Q(x_0, u_0, w_0) \text{ and}$$
$$Q(x_0, u_0, w) \leq Q(x_0, u_0, w_0)$$

for all $u \in L^2(\mathbf{R}^+;U)$ and $w \in L^2(\mathbf{R}^+;W)$;

(ii) a uniform saddle point of Q if

$$Q(x_0, u, w_0) \geq Q(x_0, u_0, w_0) + \epsilon \|u - u_0\|_{L^2(\mathbf{R}^+;U)} \text{ and}$$
$$Q(x_0, u_0, w) \leq Q(x_0, u_0, w_0) - \epsilon \|w - w_0\|_{L^2(\mathbf{R}^+;W)}$$

for some $\epsilon > 0$ and all $u \in L^2(\mathbf{R}^+;U)$ and $w \in L^2(\mathbf{R}^+;W)$;

(iii) a saddle point of Q with principal axes U and W if

$$Q(x_0, u, w) \geq Q(x_0, u_0, w) \text{ and}$$
$$Q(x_0, u, w) \leq Q(x_0, u, w_0)$$

for all $u \in L^2(\mathbf{R}^+;U)$ and $w \in L^2(\mathbf{R}^+;W)$;

(iv) a uniform saddle point of Q with principal axes U and W if

$$Q(x_0, u, w) \geq Q(x_0, u_0, w) + \epsilon \|u - u_0\|_{L^2(\mathbf{R}^+;U)} \text{ and}$$
$$Q(x_0, u, w) \leq Q(x_0, u, w_0) - \epsilon \|w - w_0\|_{L^2(\mathbf{R}^+;W)}$$

for some $\epsilon > 0$ and all $u \in L^2(\mathbf{R}^+;U)$ and $w \in L^2(\mathbf{R}^+;W)$.

Theorem 5.2 *Let* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ *be a stable J-coercive well-posed linear system on $(U \times W, H, Y)$, and suppose that $\mathcal{D}$ has a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. Write S in block form $S = \left[\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right]$ corresponding to the natural splitting of the space $U \times W$ into its components U and W. Let $Q_\circlearrowleft$ be the closed loop cost function defined in (4.1).*

(i) *$Q_\circlearrowleft(x_0, u_\circlearrowleft, w_\circlearrowleft)$ is [uniformly] convex with respect to $u_\circlearrowleft$ if and only if S_{11} is [uniformly] positive.*

(ii) *$Q_\circlearrowleft(x_0, u_\circlearrowleft, w_\circlearrowleft)$ is [uniformly] concave with respect to $w_\circlearrowleft$ if and only if S_{22} is [uniformly] negative.*

(iii) *For each $x_0 \in H$, the point $(x_0, 0, 0)$ is a [uniform] saddle point of $Q_\circlearrowleft$ if and only if S_{11} is [uniformly] positive and S_{22} is [uniformly] negative.*

(iv) *For each $x_0 \in H$, the following conditions are equivalent:*

(a) *$(x_0, 0, 0)$ is a saddle point of $Q_\circlearrowleft$ with principal axes U and W;*

(b) *$(x_0, 0, 0)$ is a uniform saddle point of $Q_\circlearrowleft$ with principal axes U and W;*

(c) *$S_{11} \geq 0$, $S_{22} \leq 0$, $S_{12} = 0$, and $S_{21} = 0$;*

(d) *$S_{11} >> 0$, $S_{22} << 0$, $S_{12} = 0$, and $S_{21} = 0$.*

(v) *If $S_{11} >> 0$ then, for each fixed $w_\circlearrowleft \in L^2(\mathbf{R}^+; W)$, the minimum of $Q_\circlearrowleft(x_0, u_\circlearrowleft, w_\circlearrowleft)$ with respect to $u \in L^2(\mathbf{R}^+; U)$ is achieved for $u_\circlearrowleft = -S_{11}^{-1} S_{12} w_\circlearrowleft$. If we replace the variable $u_\circlearrowleft$ by the new independent variable $\tilde{u} = u_\circlearrowleft + S_{11}^{-1} S_{12} w_\circlearrowleft$, then*

$$\begin{bmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{bmatrix} = E_1 \begin{bmatrix} \tilde{u} \\ w_\circlearrowleft \end{bmatrix} = \begin{bmatrix} I & -S_{11}^{-1} S_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{u} \\ w_\circlearrowleft \end{bmatrix},$$

and we get a new closed loop system of the type described Theorem 4.5 with S replaced by the congruent operator

$$\begin{aligned} E_1^* S E_1 &= \begin{bmatrix} I & 0 \\ -S_{21} S_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} I & -S_{11}^{-1} S_{12} \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} - S_{21} S_{11}^{-1} S_{12} \end{bmatrix}, \end{aligned}$$

and $\mathcal{N}$ and $\mathcal{M}$ replaced by $\mathcal{N}E_1$ and $\mathcal{M}E_1$, respectively. If $S_{22} - S_{21}S_{11}^{-1}S_{12} \le 0$, or equivalently, if $S_{22} - S_{21}S_{11}^{-1}S_{12} << 0$, then this results in a new closed loop system with a uniform saddle point with principal axes U and W.

(vi) If $S_{22} << 0$ then, for each fixed $u_\circlearrowleft \in L^2(\mathbf{R}^+;U)$, the maximum of $Q_\circlearrowleft(x_0, u_\circlearrowleft, w_\circlearrowleft)$ with respect to $w_\circlearrowleft \in L^2(\mathbf{R}^+;W)$ is achieved for $w_\circlearrowleft = -S_{22}^{-1}S_{21}u_\circlearrowleft$. If we replace the variable $w_\circlearrowleft$ by the new independent variable $\tilde{w} = w_\circlearrowleft + S_{22}^{-1}S_{21}u_\circlearrowleft$, then

$$\begin{bmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{bmatrix} = E_2 \begin{bmatrix} u_\circlearrowleft \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -S_{22}^{-1}S_{21} & I \end{bmatrix} \begin{bmatrix} u_\circlearrowleft \\ \tilde{w} \end{bmatrix},$$

and we get a new closed loop system of the type described in Theorem 4.5 with S replaced by the congruent operator

$$\begin{aligned} E_2^* S E_2 &= \begin{bmatrix} I & -S_{12}S_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -S_{22}^{-1}S_{21} & I \end{bmatrix} \\ &= \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & 0 \\ 0 & S_{22} \end{bmatrix}, \end{aligned}$$

and $\mathcal{N}$ and $\mathcal{M}$ replaced by $\mathcal{N}E_2$ and $\mathcal{M}E_2$, respectively. If $S_{11} - S_{12}S_{22}^{-1}S_{21} \ge 0$, or equivalently, $S_{11} - S_{12}S_{22}^{-1}S_{21} >> 0$, then this results in a new closed loop system with a uniform saddle point with principal axes U and W.

(vii) If $(x_0, 0, 0)$ is a uniform saddle point of $Q_\circlearrowleft$, then both part (v) and part (vi) apply, and both the resulting closed loop systems have a uniform saddle point with principal axes U and W.

Proof. (i)–(iv) These four claims follow directly from part (iii) of Theorem 4.5. (Observe that parts (c) and (d) of (iv) are equivalent because of the invertibility of S.)

(v) To prove (v) we use part (iii) of Theorem 4.5, and rearrange the terms (complete the square with respect to $u_\circlearrowleft$) to get

$$\begin{aligned} Q_\circlearrowleft(x_0, u_\circlearrowleft, w_\circlearrowleft) &= \langle x_0, \Pi x_0 \rangle_H \\ &\qquad + \langle u_\circlearrowleft, S_{11}u_\circlearrowleft + S_{12}w_\circlearrowleft \rangle_{L^2(\mathbf{R}^+;U)} + \langle w_\circlearrowleft, S_{21}u_\circlearrowleft + S_{22}w_\circlearrowleft \rangle_{L^2(\mathbf{R}^+;W)} \\ &= \langle x_0, \Pi x_0 \rangle_H \\ &\qquad + \left\langle \left(u_\circlearrowleft + S_{11}^{-1}S_{12}w_\circlearrowleft\right), S_{11}\left(u_\circlearrowleft + S_{11}^{-1}S_{12}w_\circlearrowleft\right) \right\rangle_{L^2(\mathbf{R}^+;U)} \\ &\qquad + \left\langle w_\circlearrowleft, \left(S_{22} - S_{21}S_{11}^{-1}S_{12}\right) w_\circlearrowleft \right\rangle_{L^2(\mathbf{R}^+;W)}. \end{aligned}$$

This expression is minimized by the function $u_{\circlearrowleft} = -S_{11}^{-1}S_{12}w_{\circlearrowleft}$. The proof of the fact that the change of variable $\tilde{u} = u_{\circlearrowleft} + S_{11}^{-1}S_{12}w_{\circlearrowleft}$ leads to a new closed loop system with the listed properties is the same as the proof of Proposition 4.11. If $S_{22} - S_{21}S_{11}^{-1}S_{12} \leq 0$, then it must be uniformly negative since S is invertible.

(vi) The proof of (vi) is similar to the proof of (v).

(vii) This follows from (iii). $\square$

Remark 5.3 *It follows from Theorem 5.2 that not all choices of the parameter E in Lemma 4.10 lead to an acceptable closed loop system. In particular, we observe the following facts:*

(i) It is not in the interest of the minimizing player (the control engineer) to take part in a feedback/feedforward scheme where S_{22} is anything but negative, because in such a scheme the maximizing player can make the closed loop cost arbitrarily large by choosing the closed loop disturbance $w_{\circlearrowleft}$ appropriately.

(ii) It is not in the interest of the maximizing player (nature) to take part in a feedback/feedforward scheme where S_{11} is anything but positive, because in such a scheme the minimizing player can make the cost of the closed loop system arbitrarily negative by choosing the closed loop control $u_{\circlearrowleft}$ appropriately. In this case the feedback/feedforward policy for the second player even leads to a worse result than the open loop policy, because for each fixed open loop disturbance w the cost is bounded from below; cf. (2.5).

(iii) If $S_{11} >> 0$ and $S_{12} \neq 0$, then the minimizing player can improve the outcome of the game (i.e., decrease the value of the cost function) for nonzero closed loop disturbances $w_{\circlearrowleft}$ by using the policy described in part (v) of Theorem 5.2. This change does not affect the disturbance feedback/feedforward equation

$$w = \mathcal{K}_2 x_0 + \mathcal{F}_{21}u + \mathcal{F}_{22}w + \pi_+ w_{\circlearrowleft},$$

but it does change the control feedback/feedforward equation from

$$u = \mathcal{K}_1 x_0 + \mathcal{F}_{11}u + \mathcal{F}_{12}w + \pi_+ u_{\circlearrowleft}$$

into

$$u = \left(\mathcal{K}_1 + S_{11}^{-1} S_{12} \mathcal{K}_2\right) x_0 + \left(\mathcal{F}_{11} + S_{11}^{-1} S_{12} \mathcal{F}_{21}\right) u \\ + \left(\mathcal{F}_{12} + S_{11}^{-1} S_{12} \left(\mathcal{F}_{22} - I\right)\right) w + \pi_+ u_{\circlearrowleft}.$$

(iv) *If $S_{22} << 0$ and $S_{21} \neq 0$, then the maximizing player can improve the outcome of the game (i.e., increase the value of the cost function) for nonzero closed loop controls $u_{\circlearrowleft}$ by using the policy described in part (vi) of Theorem 5.2. This change does not affect the control feedback/feedforward equation*

$$u = \mathcal{K}_1 x_0 + \mathcal{F}_{11} u + \mathcal{F}_{12} w + \pi_+ u_{\circlearrowleft}$$

but it does change the disturbance feedback/feedforward equation from

$$w = \mathcal{K}_2 x_0 + \mathcal{F}_{21} u + \mathcal{F}_{22} w + \pi_+ w_{\circlearrowleft}$$

into

$$u = \left(\mathcal{K}_2 + S_{22}^{-1} S_{21} \mathcal{K}_1\right) x_0 + \left(\mathcal{F}_{21} + S_{22}^{-1} S_{21} \left(\mathcal{F}_{11} - I\right)\right) u \\ + \left(\mathcal{F}_{22} + S_{22}^{-1} S_{21} \mathcal{F}_{12}\right) w + \pi_+ w_{\circlearrowleft}.$$

Because of the facts listed in Remark 5.3, we shall in the sequel only study factorizations for which $S_{22} << 0$ and $S_{11} - S_{12} S_{22}^{-1} S_{21} >> 0$. Much of the time we shall, in addition, assume that the off-diagonal terms S_{12} and $S_{21} = S_{12}^*$ vanish and that $S_{11} >> 0$. In the latter case the closed loop cost function $Q_{\circlearrowleft}$ has a uniform saddle point with principal axes U and W.

Is it then always possible to take $S_{11} >> 0$ and $S_{22} << 0$? The answer is "yes" if Ψ is minimax J-coercive (rather than just plain J-coercive).

Lemma 5.4 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be minimax J-coercive, and suppose that $\mathcal{D}^* J \mathcal{D}$ has an S-spectral factor $\mathcal{X}$. Then S is a congruence transformation of the operator $T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ (where the block form is the one induced by the natural splitting of $U \times W$ into its components U and W), i.e., $S = E^* T E$ for some invertible operator $E \in \mathcal{L}(U \times W)$. Moreover, it is possible to find a T-spectral factor $\mathcal{X}_T$, and the set of all possible S-spectral factorizations of $\mathcal{D}^* J \mathcal{D}$ can be parameterized as $S = E^* T E$ and $\mathcal{X} = E^{-1} \mathcal{X}_T$, where E varies over the set of all invertible operators in $\mathcal{L}(U \times W)$. In all cases the dimension of the positive eigenspace of S is equal to the dimension of U, and the dimension of the negative eigenspace of S is equal to the dimension of W. (These dimensions are called the* inertia *of S.)*

Proof. (i) Use the Schur decomposition (cf. the proof of Lemma 3.3) to rewrite S in the form

$$\begin{aligned} S &= (\mathcal{X}^*)^{-1} \begin{bmatrix} \mathcal{D}_1^* \\ \mathcal{D}_2^* \end{bmatrix} J \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \end{bmatrix} \mathcal{X}^{-1} \\ &= (\mathcal{X}^*)^{-1} \begin{bmatrix} I & 0 \\ \mathcal{D}_2^* J \mathcal{D}_1 (\mathcal{D}_1^* J \mathcal{D}_1)^{-1} & I \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathcal{D}_1^* J \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2^* \left(J - J\mathcal{D}_1 (\mathcal{D}_1^* J \mathcal{D}_1)^{-1} \mathcal{D}_1^* J \right) \mathcal{D}_2 \end{bmatrix} \\ &\quad \times \begin{bmatrix} I & (\mathcal{D}_1^* J \mathcal{D}_1)^{-1} \mathcal{D}_1^* J \mathcal{D}_2 \\ 0 & I \end{bmatrix} \mathcal{X}^{-1}, \end{aligned}$$

where, according to Lemma 3.3(iii), $\mathcal{D}_1^* J \mathcal{D}_1 >> 0$ on $L^2(\mathbf{R}; U)$, and

$$\mathcal{D}_2^* \left(J - J\mathcal{D}_1 (\mathcal{D}_1^* J \mathcal{D}_1)^{-1} \mathcal{D}_1^* J \right) \mathcal{D}_2 << 0$$

on $L^2(\mathbf{R}; Y)$. Use [Staffans 1998c, Lemma 4.3(iv)] to factor these two operators as

$$\mathcal{D}_1^* J \mathcal{D}_1 = \mathcal{X}_1^* \mathcal{X}_1, \qquad \mathcal{D}_2^* \left(J - J\mathcal{D}_1 (\mathcal{D}_1^* J \mathcal{D}_1)^{-1} \mathcal{D}_1^* J \right) \mathcal{D}_2 = -\mathcal{X}_2^* \mathcal{X}_2,$$

and to conclude that S can be written in the form

$$S = \mathcal{U}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \mathcal{U},$$

where

$$\mathcal{U} = \begin{bmatrix} \mathcal{X}_1 & 0 \\ 0 & \mathcal{X}_2 \end{bmatrix} \begin{bmatrix} I & (\mathcal{D}_1^* J \mathcal{D}_1)^{-1} \mathcal{D}_1^* J \mathcal{D}_2 \\ 0 & I \end{bmatrix} \mathcal{X}^{-1}$$

has a bounded inverse in $TI(U \times W)$.

The equation above induces an analogous equation in the frequency domain. The operator $\mathcal{U}$ induces a strongly measurable $\mathcal{L}(U \times W)$-valued L^∞ function $\widehat{\mathcal{U}}$ defined on the imaginary axis $j\mathbf{R}$, called the *symbol* of $\mathcal{U}$; see, e.g., [Fourès and Segal 1955, Theorem 1] or [Thomas 1997, Theorem 5.2] (this is where we need the assumption that U and W are separable). This symbol is determined by the fact that if we use a hat $\widehat{\ }$ to represent the bilateral Laplace (Plancherel) transform, then

$$\widehat{\left[\mathcal{U} \begin{bmatrix} u \\ w \end{bmatrix} \right]}(j\omega) = \widehat{\mathcal{U}}(j\omega) \begin{bmatrix} \hat{u}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}$$

for all $u \in L^2(\mathbf{R}; U)$, $w \in L^2(\mathbf{R}; W)$, and for almost all $\omega \in \mathbf{R}$. The algebraic structure of $TI(U \times W)$ is preserved under the passage to the symbols, so we find that

$$S = \widehat{\mathcal{U}}^*(j\omega) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \widehat{\mathcal{U}}(j\omega),$$

for almost all $\omega \in \mathbf{R}$. Fix any ω for which this equation holds. We conclude that S is a congruence transformation of the operator $T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$. The claims about the dimensions of the positive and negative eigenspaces of S then follow from the standard properties of congruence transformations, and the remaining claims from Lemma 4.2 and Lemma 4.10. $\square$

Lemma 5.4 implies the following result.

Lemma 5.5 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be minimax J-coercive, and suppose that $\mathcal{D}^* J \mathcal{D}$ has an S-spectral factor $\mathcal{X}$. Split S into $S = \left[\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right]$, and suppose that the off diagonal terms S_{12} and S_{21} vanish. Then*

(i) if either U or W is finite-dimensional and $S_{11} \geq 0$, then $S_{11} >> 0$ and $S_{22} << 0$,

(ii) if either U or W is finite-dimensional and $S_{22} \leq 0$, then $S_{11} >> 0$ and $S_{22} << 0$.

Proof. This follows from the fact that the dimension of the positive eigenspace of S equals the dimension of U, and that the dimension of the negative eigenspace of S equals the dimension of W. $\square$

Remark 5.6 *The claims (i) and (ii) in Lemma 5.5 are not true if both U and W are infinite-dimensional, as the following counterexample shows. Suppose that $\mathcal{D}^* J \mathcal{D}$ has a T-spectral factor $\mathcal{X}_T$, where $T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$; cf. Lemma 5.4. Choose an arbitrary orthonormal basis $\{e_n\}_{n=-\infty}^{0}$ in U, and another arbitrary orthonormal basis $\{e_n\}_{n=1}^{\infty}$ in W. Let $E \in \mathcal{L}(U \times W)$ be the shift operator that maps e_n into e_{n-1} for all n. This operator is invertible in $\mathcal{L}(U \times W)$, so if we define $\mathcal{X} = E^{-1}\mathcal{X}_T$ and $S = E^* T E$, then $\mathcal{X}$ is a S-spectral factor of $\mathcal{D}^* J \mathcal{D}$; cf. Lemma 5.4. Moreover, $S_{11} = I >> 0$, $S_{12} = 0$, $S_{21} = 0$, but it is not true that $S_{22} << 0$, because $\langle e_1, S e_1 \rangle = 1 > 0$.*

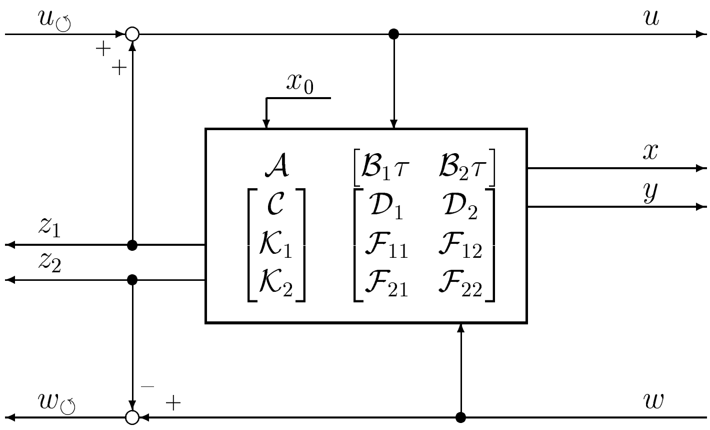

Figure 10: Final semi-closed feedback connection

6 Cutting the Disturbance Feedback Loop

Up to now we have considered feedback representations of the minimax equilibrium where the feedback enters both through the control variable u and through the disturbance variable w. We have also investigated the saddle point properties of the closed loop system. We have found that if the parameter E is chosen appropriately, then the closed loop system has a saddle point with principal axes U and W, and the optimal strategies for both the minimizing player (the control engineer) and the maximizing player (nature) is to take the inputs $u_\circlearrowleft$ and $w_\circlearrowleft$ to the closed loop system $\Psi_\circlearrowleft$ to be zero.

The statement above applies as long as we impose the given "double" feedback structure on the solution. However, in the original formulation the maximizing player is allowed to choose the original disturbance variable $w \in L^2(\mathbf{R}^+; W)$ in an arbitrary manner. In particular, nothing forces him to employ the given feedback formula; he may choose to cut the disturbance feedback loop and to apply an arbitrary open loop disturbance w instead, as drawn in Figure 10. If this is done, then we get a system where the control u is generated by a feedback loop, whereas the disturbance is open loop. This is the structure of feedback/feedforward solution that we were looking for in the first place; cf. Definition 2.2 and conditions (III)–(IV) in Theorem 1.3.

The following theorem describes what happens when the disturbance feedback loop is opened:

Theorem 6.1 *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be a stable J-coercive well-posed linear system on $(U \times W, H, Y)$, and suppose that $\mathcal{D}$ has a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. Introduce the same notations as in Theorem 4.5. Then the following conditions are equivalent:*

(i) The operator $\left[\begin{smallmatrix} 0 & I & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ is an admissible stable output feedback operator for the extended open loop system (see [Staffans 1997, Definition 19])

$$\Psi^{\text{ext}} = \begin{bmatrix} \mathcal{A} & \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \end{bmatrix} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K}_1 \\ \mathcal{K}_2 \end{bmatrix} & \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \\ I - \mathcal{X}_{11} & -\mathcal{X}_{12} \\ -\mathcal{X}_{21} & I - \mathcal{X}_{22} \end{bmatrix} \end{bmatrix}.$$

(ii) The operator $\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & -I \end{smallmatrix}\right]$ is an admissible stable output feedback operator for the extended closed loop system

$$\Psi_{\circlearrowleft} = \begin{bmatrix} \mathcal{A}_{\circlearrowleft} & \begin{bmatrix} \mathcal{B}_{\circlearrowleft 1} & \mathcal{B}_{\circlearrowleft 2} \end{bmatrix} \\ \begin{bmatrix} \mathcal{C}_{\circlearrowleft} \\ \mathcal{K}_{\circlearrowleft 1} \\ \mathcal{K}_{\circlearrowleft 2} \end{bmatrix} & \begin{bmatrix} \mathcal{N}_1 & \mathcal{N}_2 \\ \mathcal{M}_{11} - I & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} - I \end{bmatrix} \end{bmatrix}.$$

(iii) $\mathcal{X}_{11}$ has an inverse in $TIC(U)$.

(iv) $\mathcal{M}_{22}$ has an inverse in $TIC(W)$.

Moreover, the two closed loop systems that we get in (i) and (ii) are the same, and they are given by

$$\begin{aligned}
\Psi^{\curvearrowleft} &= \begin{bmatrix} \mathcal{A}^{\curvearrowleft} & \begin{bmatrix} \mathcal{B}_1^{\curvearrowleft}\tau & \mathcal{B}_2^{\curvearrowleft}\tau \end{bmatrix} \\ \begin{bmatrix} \mathcal{C}^{\curvearrowleft} \\ \mathcal{K}_1^{\curvearrowleft} \\ \mathcal{K}_2^{\curvearrowleft} \end{bmatrix} & \begin{bmatrix} \mathcal{D}_1^{\curvearrowleft} & \mathcal{D}_2^{\curvearrowleft} \\ \mathcal{F}_{11}^{\curvearrowleft} & \mathcal{F}_{12}^{\curvearrowleft} \\ \mathcal{F}_{21}^{\curvearrowleft} & \mathcal{F}_{22}^{\curvearrowleft} \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A} + \mathcal{B}_1\tau\mathcal{X}_{11}^{-1}\mathcal{K}_1 & \begin{bmatrix} \mathcal{B}_1\mathcal{X}_{11}^{-1}\tau & \mathcal{B}_2\tau - \mathcal{B}_1\mathcal{X}_{11}^{-1}\mathcal{X}_{12}\tau \end{bmatrix} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}_1\mathcal{X}_{11}^{-1}\mathcal{K}_1 \\ \mathcal{X}_{11}^{-1}\mathcal{K}_1 \\ \mathcal{K}_2 - \mathcal{X}_{21}\mathcal{X}_{11}^{-1}\mathcal{K}_1 \end{bmatrix} & \begin{bmatrix} \mathcal{D}_1\mathcal{X}_{11}^{-1} & \mathcal{D}_2 - \mathcal{D}_1\mathcal{X}_{11}^{-1}\mathcal{X}_{12} \\ \mathcal{X}_{11}^{-1} - I & -\mathcal{X}_{11}^{-1}\mathcal{X}_{12} \\ -\mathcal{X}_{21}\mathcal{X}_{11}^{-1} & I - \mathcal{X}_{22} + \mathcal{X}_{21}\mathcal{X}_{11}^{-1}\mathcal{X}_{12} \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{A}_{\circlearrowleft} - \mathcal{B}_{\circlearrowleft 2}\tau\mathcal{M}_{22}^{-1}\mathcal{K}_{\circlearrowleft 2} & \begin{bmatrix} \mathcal{B}_{\circlearrowleft 1}\tau - \mathcal{B}_{\circlearrowleft 2}\mathcal{M}_{22}^{-1}\mathcal{M}_{21}\tau & \mathcal{B}_{\circlearrowleft 2}\mathcal{M}_{22}^{-1}\tau \end{bmatrix} \\ \begin{bmatrix} \mathcal{C}_{\circlearrowleft} - \mathcal{N}_2\mathcal{M}_{22}^{-1}\mathcal{K}_{\circlearrowleft 2} \\ \mathcal{K}_{\circlearrowleft 1} - \mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{K}_{\circlearrowleft 2} \\ \mathcal{M}_{22}^{-1}\mathcal{K}_{\circlearrowleft 2} \end{bmatrix} & \begin{bmatrix} \mathcal{N}_1 - \mathcal{N}_2\mathcal{M}_{22}^{-1}\mathcal{M}_{21} & \mathcal{N}_2\mathcal{M}_{22}^{-1} \\ \mathcal{M}_{11} - I - \mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{M}_{21} & \mathcal{M}_{12}\mathcal{M}_{22}^{-1} \\ \mathcal{M}_{22}^{-1}\mathcal{M}_{21} & I - \mathcal{M}_{22}^{-1} \end{bmatrix} \end{bmatrix}.
\end{aligned}$$

Proof. This follows from a standard result on repeated feedbacks; see [Staffans 1998a, Definition 3.1 and Propositions 3.2–3.3]. $\square$

Lemma 6.2 *The four equivalent conditions in Theorem 6.1 are satisfied if and only if $\mathcal{X}$ and $\mathcal{M}$ can be factored (boundedly) as*

$$\mathcal{X} = \underline{\mathcal{M}}^{-1}\overline{\mathcal{X}}, \qquad \mathcal{M} = \overline{\mathcal{X}}^{-1}\underline{\mathcal{M}},$$

in $TIC(U \times W)$; here

$$\begin{aligned}
\overline{\mathcal{X}} &= \begin{bmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ 0 & I \end{bmatrix}, \\
\underline{\mathcal{M}} &= \begin{bmatrix} I & 0 \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}, \\
\overline{\mathcal{X}}^{-1} &= \begin{bmatrix} I + \mathcal{F}_{11}^{\curvearrowleft} & \mathcal{F}_{12}^{\curvearrowleft} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{X}_{11}^{-1} & -\mathcal{X}_{11}^{-1}\mathcal{X}_{12} \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{M}_{11} - \mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{M}_{21} & \mathcal{M}_{12}\mathcal{M}_{22}^{-1} \\ 0 & I \end{bmatrix}, \\
\underline{\mathcal{M}}^{-1} &= \begin{bmatrix} I & 0 \\ -\mathcal{F}_{21}^{\curvearrowleft} & I - \mathcal{F}_{22}^{\curvearrowleft} \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ -\mathcal{M}_{22}^{-1}\mathcal{M}_{21} & \mathcal{M}_{22}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ \mathcal{X}_{21}\mathcal{X}_{11}^{-1} & \mathcal{X}_{22} - \mathcal{X}_{21}\mathcal{X}_{11}^{-1}\mathcal{X}_{12} \end{bmatrix}.
\end{aligned}$$

Moreover, if we replace either the last row or the second last row of the system $\Psi^{\curvearrowleft}$ by a zero row, then the two systems that we get in this way are given by

$$\begin{bmatrix} \mathcal{A}^{\curvearrowleft} & \begin{bmatrix} \mathcal{B}_1^{\curvearrowleft}\tau & \mathcal{B}_2^{\curvearrowleft}\tau \end{bmatrix} \\ \begin{bmatrix} \mathcal{C}^{\curvearrowleft} \\ \mathcal{K}_1^{\curvearrowleft} \\ 0 \end{bmatrix} & \begin{bmatrix} \mathcal{D}_1^{\curvearrowleft} & \mathcal{D}_2^{\curvearrowleft} \\ \mathcal{F}_{11}^{\curvearrowleft} & \mathcal{F}_{12}^{\curvearrowleft} \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}\tau\overline{\mathcal{X}}^{-1}\overline{\mathcal{K}} & \mathcal{B}\overline{\mathcal{X}}^{-1}\tau \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}\overline{\mathcal{X}}^{-1}\overline{\mathcal{K}} \\ \overline{\mathcal{X}}^{-1}\overline{\mathcal{K}} \end{bmatrix} & \begin{bmatrix} \mathcal{D}\overline{\mathcal{X}}^{-1} \\ \overline{\mathcal{X}}^{-1} - I \end{bmatrix} \end{bmatrix},$$

$$\begin{bmatrix} \mathcal{A}^{\curvearrowleft} & \begin{bmatrix} \mathcal{B}_1^{\curvearrowleft}\tau & \mathcal{B}_2^{\curvearrowleft}\tau \end{bmatrix} \\ \begin{bmatrix} \mathcal{C}^{\curvearrowleft} \\ 0 \\ \mathcal{K}_2^{\curvearrowleft} \end{bmatrix} & \begin{bmatrix} \mathcal{D}_1^{\curvearrowleft} & \mathcal{D}_2^{\curvearrowleft} \\ 0 & 0 \\ \mathcal{F}_{21}^{\curvearrowleft} & \mathcal{F}_{22}^{\curvearrowleft} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{\circlearrowleft} - \mathcal{B}_{\circlearrowleft}\tau\underline{\mathcal{M}}^{-1}\underline{\mathcal{K}}_{\circlearrowleft} & \mathcal{B}_{\circlearrowleft}\underline{\mathcal{M}}^{-1}\tau \\ \begin{bmatrix} \mathcal{C}_{\circlearrowleft} - \mathcal{N}\underline{\mathcal{M}}^{-1}\underline{\mathcal{K}}_{\circlearrowleft} \\ \underline{\mathcal{M}}^{-1}\underline{\mathcal{K}}_{\circlearrowleft} \end{bmatrix} & \begin{bmatrix} \mathcal{N}\underline{\mathcal{M}}^{-1} \\ I - \underline{\mathcal{M}}^{-1} \end{bmatrix} \end{bmatrix},$$

where

$$\overline{\mathcal{K}} = \begin{bmatrix} \mathcal{K}_1 \\ 0 \end{bmatrix}, \qquad \underline{\mathcal{K}}_{\circlearrowleft} = \begin{bmatrix} 0 \\ \mathcal{K}_{\circlearrowleft 2} \end{bmatrix}.$$

In particular,

$$\begin{aligned}
\mathcal{C}^{\curvearrowright} &= \mathcal{C} + \mathcal{D}\overline{\mathcal{X}}^{-1}\overline{\mathcal{K}} = \mathcal{C}_{\circlearrowleft} - \mathcal{N}\underline{\mathcal{M}}^{-1}\underline{\mathcal{K}}_{\circlearrowleft}, \\
\mathcal{D}^{\curvearrowright} &= \mathcal{D}\overline{\mathcal{X}}^{-1} = \mathcal{N}\underline{\mathcal{M}}^{-1}, \\
\mathcal{K}^{\curvearrowright} &= \overline{\mathcal{X}}^{-1}\overline{\mathcal{K}} + \underline{\mathcal{M}}^{-1}\underline{\mathcal{K}}_{\circlearrowleft}, \\
\mathcal{F}^{\curvearrowright} &= \overline{\mathcal{X}}^{-1} - \underline{\mathcal{M}}^{-1}.
\end{aligned}$$

Proof. We leave the straightforward algebraic proof to the reader. $\square$

Definition 6.3 *A (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ of $\mathcal{D} \in TIC(U \times W; Y)$ is feasible if $S = S^*$ is invertible in $\mathcal{L}(U \times W)$, $\mathcal{X} = \left[\begin{smallmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{smallmatrix}\right]$ is invertible in $TIC(U \times W)$ and $\mathcal{X}_{11}$ is invertible in $TIC(U)$.*

Compare this definition to conditions(III) in 1.3 and Theorems 4.5(i) and 6.1(iii).

Theorem 6.4 *Let $J \in \mathcal{L}(Y)$ and let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be a stable minimax J-coercive well-posed linear system on $(U \times W, H, Y)$, and suppose that $\mathcal{D}$ has a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. If we choose this factorization in such a way that $S = T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ (this is possible according to Lemma 5.4), then it is feasible.*

This theorem shows that condition (II) in Theorem 1.3 together with Hypothesis 4.6 implies (III).

Proof of Theorem 6.4. According to Lemma 3.3(i) and Theorems 4.5(i) and 6.1, to prove Theorem 6.4 it suffices to show that $\mathcal{M}_{22}$ has a causal inverse, where $\mathcal{M} = \left[\begin{smallmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{smallmatrix}\right] = \mathcal{X}^{-1}$. We shall do this by applying [Staffans 1998c, Lemma 4.11(iii)] to $\mathcal{M}_{22}$. The main assumption of that lemma consists of the two conditions

$$\mathcal{M}_{22}^*\mathcal{M}_{22} >> 0 \text{ on } L^2(\mathbf{R}, W), \tag{6.1}$$

$$\mathcal{M}_{22}\pi_+\mathcal{M}_{22}^* >> 0 \text{ on } L^2(\mathbf{R}^+, W). \tag{6.2}$$

Let us start with the verification of condition (6.1). By Theorem 4.5(iii), the closed loop cost with initial value $x_0 = 0$ and control $u_{\circlearrowleft} = 0$ is

$$Q_{\circlearrowleft}(0, 0, w_{\circlearrowleft}) = \langle w_{\circlearrowleft}, S_{22} w_{\circlearrowleft} \rangle \le -\epsilon \|w_{\circlearrowleft}\|^2_{L^2(\mathbf{R}^+;W)}$$

for some $\epsilon > 0$, since we assume $S_{22} << 0$. On the other hand, by Corollary 4.9 and Lemma 2.6 (see, in particular, (2.5)),

$$\begin{aligned} Q_{\circlearrowleft}(0, 0, w_{\circlearrowleft}) &= Q(0, \mathcal{M}_{12}\pi_+ w_{\circlearrowleft}, \mathcal{M}_{22}\pi_+ w_{\circlearrowleft}) \\ &\ge Q^{\min}(0, \mathcal{M}_{22}\pi_+ w_{\circlearrowleft}) \\ &\ge -K \, \|\mathcal{M}_{22}\pi_+ w_{\circlearrowleft}\|^2_{L^2(\mathbf{R}^+;Y)}, \end{aligned}$$

for some $K < \infty$. Thus,

$$\pi_+ \mathcal{M}_{22}^* \mathcal{M}_{22} \pi_+ >> 0 \text{ on } L^2(\mathbf{R}^+, W),$$

which by [Staffans 1998c, Lemma 4.4(ii)] is equivalent to (6.1). (So far we have used only the assumptions $S_{22} << 0$ and $\pi_+ \mathcal{D}_1^* J \mathcal{D}_1 \pi_+ >> 0$.)

Next we verify condition (6.2). Since $\mathcal{X}$ is an T-spectral factor of $\mathcal{D}^* J \mathcal{D}$, by [Staffans 1998c, Lemma 4.3(iii)], the inverse of the Toeplitz operator $\pi_+ \mathcal{D}^* J \mathcal{D} \pi_+$ is $\mathcal{X}^{-1}\pi_+ T^{-1} (\mathcal{X}^*)^{-1} = \mathcal{M}\pi_+ T \mathcal{M}^*$ (note that $T^{-1} = T$). Thus, by Lemma 3.4(iii),

$$-\mathcal{M}_{22}\pi_+\mathcal{M}_{22}^* << -\mathcal{M}_{21}\pi_+\mathcal{M}_{21}^* \le 0,$$

which proves (6.2).

By [Staffans 1998c, Lemma 4.11(iii)], $\mathcal{M}_{22}$ is invertible in $TIC(W)$, hence $\mathcal{X}_{11}$ is invertible in $TIC(U)$ (cf. Theorem 6.1). $\square$

Let us end this section with some remarks which simplify the passage between between the open loop system Ψ, the semi-closed loop system $\Psi^{\frown}$, and the closed loop system $\Psi_{\circlearrowleft}$.

Remark 6.5 *It is possible to pass between the open loop system Ψ, the semi-closed loop system $\Psi^{\frown}$, and the closed loop system $\Psi_{\circlearrowleft}$ essentially in the same way as we passed from Ψ to $\Psi_{\circlearrowleft}$ and back in Lemma 4.8. It involves three diagrams built around Ψ with different directions of the lines on the top and the bottom, three diagrams built around $\Psi^{\frown}$ (out of which Figure 11 is one), and three more diagrams built around $\Psi_{\circlearrowleft}$. We leave the exact formulation of*

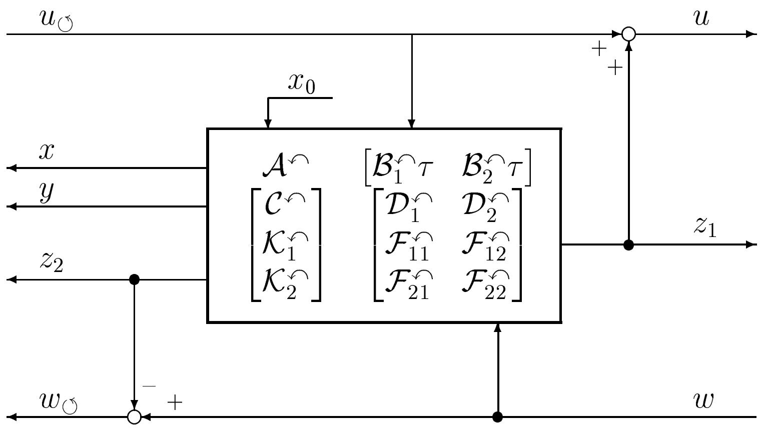

Figure 11: Semi-closed system written in open loop form

this result to the reader. The key observation is that we have three equivalent expressions for z_1*, namely*

$$\begin{aligned} z_1 &= \mathcal{K}_1 x_0 + \mathcal{F}_{11}\pi_+ u + \mathcal{F}_{12}\pi_+ w \\ &= \mathcal{K}_1^\curvearrowleft x_0 + \mathcal{F}_{11}^\curvearrowleft \pi_+ u_\circlearrowleft + \mathcal{F}_{12}^\curvearrowleft \pi_+ w \\ &= \mathcal{K}_{\circlearrowleft 1} x_0 + \mathcal{F}_{\circlearrowleft 11}\pi_+ u_\circlearrowleft + \mathcal{F}_{\circlearrowleft 12}\pi_+ w_\circlearrowleft, \end{aligned}$$

and likewise, there are three equivalent expressions for z_2*, namely*

$$\begin{aligned} z_2 &= \mathcal{K}_2 x_0 + \mathcal{F}_{21}\pi_+ u + \mathcal{F}_{22}\pi_+ w \\ &= \mathcal{K}_2^\curvearrowleft x_0 + \mathcal{F}_{21}^\curvearrowleft \pi_+ u_\circlearrowleft + \mathcal{F}_{22}^\curvearrowleft \pi_+ w \\ &= \mathcal{K}_{\circlearrowleft 2} x_0 + \mathcal{F}_{\circlearrowleft 21}\pi_+ u_\circlearrowleft + \mathcal{F}_{\circlearrowleft 22}\pi_+ w_\circlearrowleft. \end{aligned}$$

In particular, it follows from Figure 11 that we can pass from the semi-closed loop system $\Psi^\curvearrowleft$ *to the open loop system* Ψ *by keeping the open loop disturbance* w *intact and replacing the closed loop control* $u_\circlearrowleft$ *by the open loop control*

$$\begin{aligned} u = z_1 + \pi_+ u_\circlearrowleft &= \mathcal{K}_1^\curvearrowleft x_0 + (I + \mathcal{F}_{11}^\curvearrowleft)\pi_+ u_\circlearrowleft + \mathcal{F}_{12}^\curvearrowleft \pi_+ w, \\ &= \mathcal{K}_1^\curvearrowleft x_0 + \mathcal{X}_{11}^{-1}\pi_+ u_\circlearrowleft - \mathcal{X}_{11}^{-1}\mathcal{X}_{12}^{-1}\pi_+ w, \end{aligned}$$

and that we can pass from the same system to the fully closed loop system $\Psi_\circlearrowleft$ *by keeping the closed loop control* $u_\circlearrowleft$ *intact, but replacing the open loop* *disturbance w by the closed loop disturbance*

$$
\begin{aligned}
w_{\circlearrowleft} = -z_2 + \pi_+ w &= -\mathcal{K}_2^{\frown} x_0 - \mathcal{F}_{21}^{\frown} \pi_+ u_{\circlearrowleft} + (I - \mathcal{F}_{22}^{\frown}) \pi_+ w \\
&= -\mathcal{K}_2^{\frown} x_0 - \mathcal{M}_{22}^{-1} \mathcal{M}_{21} \pi_+ u_{\circlearrowleft} + \mathcal{M}_{22}^{-1} \pi_+ w.
\end{aligned}
$$

7 Central Suboptimal Controllers

Only two of the implications in Theorem 1.3 remains to be proved, namely (III) $\Rightarrow$ (IV) $\Rightarrow$ (V). Out of these the latter implication is trivial, once we have defined what we mean by a central state feedback/feedforward controller.

Let us recall the definition of a stabilizing state feedback/feedforward controller for Ψ presented in Definition 2.2. Clearly, if we ignore the bottom part of Figure 10 which produces the signals z_2 and $w_{\circlearrowleft}$, then this figure becomes functionally equivalent to Figure 2 with $(\mathcal{K}, \mathcal{F}_1, \mathcal{F}_2)$ replaced by $(\mathcal{K}_1, \mathcal{F}_{11}, \mathcal{F}_{12})$. This means that Definition 2.2 can be applied to the semi-closed loop system $\Psi^{\frown}$. Recall that $\mathcal{F}$ in Figure 10 is given by $\mathcal{F} = I - \mathcal{X}$, hence $\mathcal{F}_1$ and $\mathcal{F}_2$ can alternatively be written as $\mathcal{F}_1 = I - \mathcal{X}_{11}$ and $\mathcal{F}_2 = -\mathcal{X}_{12}$.

Definition 7.1 *A central state feedback/feedforward controller $(\mathcal{K}, \mathcal{F}_1, \mathcal{F}_2)$ for $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ is a stabilizing state feedback/feedforward controller which can be obtained from a semi-closed loop system $\Psi^{\frown}$ of the type drawn in Figure 10, i.e., $\mathcal{D}$ has a feasible (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X} = \left[\begin{smallmatrix} \mathcal{N}_1 \\ \mathcal{N}_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \end{smallmatrix}\right]$ such that $\mathcal{F}_1 = I - \mathcal{X}_{11}$, $\mathcal{F}_2 = -\mathcal{X}_{12}$, and $\mathcal{K}$ is the first component of $-S^{-1}\pi_+ \mathcal{N}^* J \mathcal{C}$.*

Indeed, comparing this definition to Definition 2.2 we realize that the implication (IV) $\Rightarrow$ (V) in Theorem 1.3 is trivial. See Theorem 9.1 for an explanation of in which sense this controller is “central”.

To prove the final implication (III) $\Rightarrow$ (IV) in Theorem 1.3 we need to study the minimax properties of the semi-closed loop system $\Psi^{\frown}$.

Theorem 7.2 *Let $J \in \mathcal{L}(Y)$ and let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathcal{A} & [\mathcal{B}_1 \ \mathcal{B}_2] \\ \mathcal{C} & [\mathcal{D}_1 \ \mathcal{D}_2] \end{smallmatrix}\right]$ be a stable well-posed linear system on $(U \times W, H, Y)$, and suppose that $\mathcal{D}$ has a feasible (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. Let $Q^{\frown}(x_0, u_{\circlearrowleft}, w)$ be the cost function associated with the semi-closed loop system $\Psi^{\frown}$, i.e,*

$$
Q^{\frown}(x_0, u_{\circlearrowleft}, w) = \int_{\mathbf{R}^+} \langle y(s), Jy(s) \rangle_Y \, ds,
$$

where $y = \mathcal{C}^{\frown} x_0 + \mathcal{D}_1^{\frown} \pi_+ u_{\circlearrowleft} + \mathcal{D}_2^{\frown} \pi_+ w$ is the output of the semi-closed loop system $\Psi^{\frown}$ with initial state $x_0 \in H$, control $u_{\circlearrowleft} \in L^2(\mathbf{R}^+; U)$, and disturbance $w \in L^2(\mathbf{R}^+; W)$. Then the following claims are true.

(i) The semi-closed loop cost function $Q^{\frown}(x_0, u_{\circlearrowleft}, w)$ can be written in the form

$$
\begin{aligned}
&Q^{\frown}(x_0, u_{\circlearrowleft}, w) - \langle x_0, \Pi x_0 \rangle_H \\
&\quad = \left\langle \mathcal{D}\overline{\mathcal{X}} \begin{bmatrix} \pi_+ u_{\circlearrowleft} \\ \pi_+ w - \mathcal{K}_{\circlearrowleft 2} x_0 \end{bmatrix}, J\mathcal{D}\overline{\mathcal{X}} \begin{bmatrix} \pi_+ u_{\circlearrowleft} \\ \pi_+ w - \mathcal{K}_{\circlearrowleft 2} x_0 \end{bmatrix} \right\rangle_{L^2(\mathbf{R}^+;Y)} \\
&\quad = \left\langle \underline{\mathcal{M}}^{-1} \begin{bmatrix} \pi_+ u_{\circlearrowleft} \\ \pi_+ w - \mathcal{K}_{\circlearrowleft 2} x_0 \end{bmatrix}, S\underline{\mathcal{M}}^{-1} \begin{bmatrix} \pi_+ u_{\circlearrowleft} \\ \pi_+ w - \mathcal{K}_{\circlearrowleft 2} x_0 \end{bmatrix} \right\rangle_{L^2(\mathbf{R}^+;Y)},
\end{aligned}
$$

where $\overline{\mathcal{X}}$ and $\underline{\mathcal{M}}$ are defined as in Lemma 6.2.

(ii) For each $x_0 \in H$ and $w \in L^2(\mathbf{R}^+; W)$, the function $u \mapsto Q^{\frown}(x_0, u, w)$ is convex on $L^2(\mathbf{R}^+; U)$ iff $\pi_+ \mathcal{D}_1^* J \mathcal{D}_1 \pi_+ \geq 0$ on $L^2(\mathbf{R}^+; U)$, and it is uniformly convex on $L^2(\mathbf{R}^+; U)$ iff $\pi_+ \mathcal{D}_1^* J \mathcal{D}_1 \pi_+ >> 0$ on $L^2(\mathbf{R}^+; U)$.

(iii) For each $x_0 \in H$ and $u \in L^2(\mathbf{R}^+; U)$, the function $w \mapsto Q^{\frown}(x_0, u, w)$ is concave on $L^2(\mathbf{R}^+; W)$ iff $S_{22} \leq 0$, and it is uniformly concave on $L^2(\mathbf{R}^+; W)$ iff $S_{22} << 0$.

(iv) For each $x_0 \in H$, $(x_0, 0, \mathcal{K}_{\circlearrowleft 2} x_0)$ is a saddle point of $Q^{\frown}$ iff $\pi_+ \mathcal{D}_1^* J \mathcal{D}_1 \pi_+ \geq 0$ and $S_{22} \leq 0$, and it is a uniform saddle point iff $\pi_+ \mathcal{D}_1^* J \mathcal{D}_1 \pi_+ >> 0$ and $S_{22} << 0$.

Proof. (i) We leave the straightforward proof of (i) to the reader. It is based on Theorem 6.1, Lemma 6.2, and (3.7).

(ii) By (i) and Lemma 6.2, the quadratic term of $Q^{\frown}(x_0, u_{\circlearrowleft}, w)$ with respect to $u_{\circlearrowleft}$ is

$$
\begin{aligned}
Q^{\frown}(0, u_{\circlearrowleft}, 0) &= \langle \mathcal{D}_1^{\frown} \pi_+ u_{\circlearrowleft}, J\mathcal{D}_1^{\frown} \pi_+ u_{\circlearrowleft} \rangle_{L^2(\mathbf{R}^+;Y)} \\
&= \left\langle \mathcal{X}_{11}^{-1} \pi_+ u_{\circlearrowleft}, \mathcal{D}_1^* J \mathcal{D}_1 \mathcal{X}_{11}^{-1} \pi_+ u_{\circlearrowleft} \right\rangle_{L^2(\mathbf{R}^+;Y)}.
\end{aligned}
$$

As $\mathcal{X}_{11}^{-1}$ is invertible in $TIC(U)$, this function if convex iff $\pi_+ \mathcal{D}_1^* J \mathcal{D}_1 \pi_+ \geq 0$ on $L^2(\mathbf{R}^+; U)$, and it is uniformly convex iff $\pi_+ \mathcal{D}_1^* J \mathcal{D}_1 \pi_+ >> 0$ on $L^2(\mathbf{R}^+; U)$.

(iii) By (i) and Lemma 6.2, the quadratic term of $Q^{\frown}(x_0, u_{\circlearrowleft}, w)$ with respect to w is

$$\begin{aligned} Q^{\frown}(0,0,w) &= \langle \mathcal{D}_2^{\frown}\pi_+ w, J\mathcal{D}_2^{\frown}\pi_+ w\rangle_{L^2(\mathbf{R}^+;Y)} \\ &= \left\langle \mathcal{M}_{22}^{-1}\pi_+ w, S_{22}\mathcal{M}_{22}^{-1}\pi_+ w\right\rangle_{L^2(\mathbf{R}^+;Y)}. \end{aligned}$$

As $\mathcal{M}_{22}^{-1}$ is invertible in $TIC(W)$, this function if concave iff $S_{22} \le 0$, and it is uniformly concave iff $S_{22} << 0$.

(iv) This follows from (i)–(iii). $\square$

Corollary 7.3 *The central state feedback/feedforward controller induced by a feasible (J, S)-inner-outer factorization $\mathcal{N}\mathcal{X}$ of $\mathcal{D}$ is [uniformly] suboptimal if and only if S_{22} is [uniformly] negative.*

Proof. This follows from Definitions 2.2 and 7.1 and Theorem 7.2. $\square$

Theorem 6.4 and Corollary 7.3 give us the final implication (III) $\Rightarrow$ (IV) in Theorem 1.3. Thus, our proof of Theorem 1.3 is now complete.

8 Parameterization of All Suboptimal Central Controllers

Our next task is to develop a parameterization of the set of all uniformly suboptimal compensators. As a first step in this parameterization we investigate the set of all central suboptimal state feedback/feedforward controllers. To simplify the discussion we introduce the following additional definition.

Definition 8.1 *A (J, S)-inner-outer factorization $\mathcal{N}\mathcal{X}$ of $\mathcal{D}$ is [uniformly] suboptimal if it is feasible (see Definition 6.3) and the induced central state feedback/feedforward controller (see Definition 7.1) is [uniformly] suboptimal (see Definition 2.2).*

Thus, a feasible (J, S)-inner-outer factorization is [uniformly] suboptimal iff $S_{22} \le 0$ [$S_{22} << 0$]; see Corollary 7.3.

We begin by giving some necessary and some sufficient conditions on an (J, S)-inner-outer factorization in order for this factorization to be uniformly suboptimal.

Lemma 8.2 *Suppose that $\mathcal{D}$ has a uniformly suboptimal (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ and that $\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ >> 0$ on $L^2(\mathbf{R}^+;U)$. Then the following claims hold:*

(i) $S_{22} << 0$ *and* $S_{11} - S_{12}S_{22}^{-1}S_{21} >> 0$.

(ii) If the cross terms S_{12} and $S_{21} = S_{12}^*$ vanish, then $S_{11} >> 0$, and $\mathcal{F}_{21}^{\frown}$ satisfies

$$\begin{aligned}\|(-S_{22})^{1/2}\mathcal{F}_{21}^{\frown}S_{11}^{-1/2}\| &= \|(-S_{22})^{1/2}\mathcal{X}_{21}\mathcal{X}_{11}^{-1}S_{11}^{-1/2}\| \\ &= \|(-S_{22})^{1/2}\mathcal{M}_{22}^{-1}\mathcal{M}_{21}S_{11}^{-1/2}\| \\ &< 1,\end{aligned}$$

where we use the same notations as in Theorem 6.1.

(iii) $\mathcal{D}$ has a uniformly suboptimal (J,T)-inner-outer factorization, where $T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I\end{smallmatrix}\right]$

(iv) An arbitrary $(J,\widetilde{S})$-inner-outer factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ of $\mathcal{D}$ is uniformly suboptimal (in particular, it is feasible) if and only if $\widetilde{S}_{22} << 0$ and $\widetilde{S}_{11} - \widetilde{S}_{12}\widetilde{S}_{22}^{-1}\widetilde{S}_{21} \geq 0$, or equivalently, if and only if $\widetilde{S}_{22} << 0$ and $\widetilde{S}_{11} - \widetilde{S}_{12}\widetilde{S}_{22}^{-1}\widetilde{S}_{21} >> 0$.

Proof. (i) By Corollary 7.3, $S_{22} << 0$. We claim that we may, without loss of generality, assume that the cross terms S_{12} and $S_{21} = S_{12}^*$ vanish. If not, then we use the construction described in part (iv) of Remark 5.3 to replace S_{12} by zero. This does not change $\mathcal{X}_{11} = I - \mathcal{F}_{11}$ and S_{22}, hence it does not affect the uniform suboptimality of the factorization. However, it does replace S_{11} by $S_{11} - S_{12}S_{22}^{-1}S_{21}$. Thus, the uniform positivity of $S_{11} - S_{12}S_{22}^{-1}S_{21}$ follows from (ii).

(ii) Recall that $\mathcal{X}$ is an S-spectral factor of $\mathcal{D}^*J\mathcal{D}$, i.e., $\mathcal{X}^*S\mathcal{X} = \mathcal{D}^*J\mathcal{D}$. If the cross terms vanish, then the top left corner of this equation gives

$$\mathcal{X}_{11}^*S_{11}\mathcal{X}_{11} + \mathcal{X}_{21}^*S_{22}\mathcal{X}_{21} = \mathcal{D}_1^*J\mathcal{D}_1.$$

Since $\mathcal{X}_{11}$ is invertible in $TIC(U)$, we may rewrite this equation as

$$S_{11} = (\mathcal{X}_{11}^*)^{-1}\mathcal{X}_{21}^*(-S_{22})\mathcal{X}_{21}\mathcal{X}_{11}^{-1} + (\mathcal{X}_{11}^*)^{-1}\mathcal{D}_1^*J\mathcal{D}_1\mathcal{X}_{11}^{-1}.$$

The assumptions $\pi_+\mathcal{D}_1^* J\mathcal{D}_1\pi_+ >> 0$ and $S_{22} << 0$ imply that the right-hand side is uniformly positive, hence $S_{11} >> 0$. Multiply the equation above by $S_{11}^{-1/2}$ both to the left and to the right to get

$$I = S_{11}^{-1/2}(\mathcal{X}_{11}^*)^{-1}\mathcal{X}_{21}^*(-S_{22})\mathcal{X}_{21}\mathcal{X}_{11}^{-1}S_{11}^{-1/2} + S_{11}^{-1/2}(\mathcal{X}_{11}^*)^{-1}\mathcal{D}_1^* J\mathcal{D}_1\mathcal{X}_{11}^{-1}S_{11}^{-1/2},$$

where the last term is uniformly positive. This implies that

$$\|(-S_{22})^{1/2}\mathcal{X}_{21}\mathcal{X}_{11}^{-1}S_{11}^{-1/2}\| < 1.$$

(iii) As in the proof of (i), we may assume without loss of generality that the off-diagonal terms of S vanish. If $S_{11} \neq I$ or $S_{22} \neq I$, then we factor S as

$$S = \begin{bmatrix} S_{11}^{1/2} & 0 \\ 0 & (-S_{22})^{1/2} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} S_{11}^{1/2} & 0 \\ 0 & (-S_{22})^{1/2} \end{bmatrix}$$

and absorb the two factors to the left and right into $\mathcal{X}^*$ and $\mathcal{X}$; see Lemma 4.10.

(iv) The necessity of the two conditions $\widetilde{S}_{22} << 0$ and $\widetilde{S}_{11} - \widetilde{S}_{12}\widetilde{S}_{22}^{-1}\widetilde{S}_{21} >> 0$ follows from (i). Conversely, suppose that $\widetilde{S}_{22} << 0$ and $\widetilde{S}_{11} - \widetilde{S}_{12}\widetilde{S}_{22}^{-1}\widetilde{S}_{21} \geq 0$. We may remove the cross terms $\widetilde{S}_{12}$ and $\widetilde{S}_{21} = \widetilde{S}_{12}^*$ in the same way as we did in the proof of (i), without affecting the feasibility of the factorization. After this transformation we must have $\widetilde{S}_{11} >> 0$ since $\widetilde{S}$ is invertible, or in terms of the original data, $\widetilde{S}_{11} - \widetilde{S}_{12}\widetilde{S}_{22}^{-1}\widetilde{S}_{21} >> 0$. Next we transform $\widetilde{S}$ into T as we did in the proof of (iii), still without affecting the feasibility and uniform suboptimality of the given factorization (that we still denote by $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$). Thus, at this point we know that $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ is a (J,T)-inner-outer factorization of $\mathcal{D}$, but we do know know if it if feasible, i.e., we do not know if $\widetilde{\mathcal{X}}_{11}$ is invertible.

By (iii), $\mathcal{D}$ has a uniformly suboptimal (J,T)-inner-outer factorization. Let us for simplicity denote this factorization by $\mathcal{N}\mathcal{X}$. By Lemma 4.10, $\widetilde{\mathcal{X}} = E\mathcal{X}$ for some operator E satisfying $E^*TE = T$. Inverting this equation we get $T^{-1} = E^{-1}T^{-1}(E^{-1})^*$, or equivalently, $ETE^* = T$. The top left component of the equation $E^*TE = T$ gives

$$E_{11}^*E_{11} - E_{21}^*E_{21} = I,$$

and the top left component of the equation $ETE^* = T$ gives

$$E_{11}E_{11}^* - E_{12}E_{12}^* = I.$$

Together these two equations show that E_{11} is invertible (both $E_{11}^*E_{11} >> 0$ and $E_{11}E_{11}^* >> 0$), and that $\|E_{11}^{-1}E_{12}\| < 1$ and $\|E_{21}E_{11}^{-1}\| < 1$.

Since $\widetilde{\mathcal{X}} = E\mathcal{X}$, we have

$$\widetilde{\mathcal{X}}_{11} = E_{11}\mathcal{X}_{11} + E_{12}\mathcal{X}_{21} = E_{11}\left(I + E_{11}^{-1}E_{12}\mathcal{X}_{21}\mathcal{X}_{11}^{-1}\right)\mathcal{X}_{11}.$$

This operator is invertible in $TIC(U)$ since

$$\|E_{11}^{-1}E_{12}\mathcal{X}_{21}\mathcal{X}_{11}^{-1}\| \le \|E_{11}^{-1}E_{12}\|\|\mathcal{X}_{21}\mathcal{X}_{11}^{-1}\| < 1;$$

cf. (ii). $\square$

Next we investigate the correspondence between a feasible (J,S)-inner-outer factorization and the corresponding central compensator:

Definition 8.3 *We call the operator*

$$\mathcal{F}_{\widehat{12}} = -\mathcal{X}_{11}^{-1}\mathcal{X}_{12} = \mathcal{M}_{12}\mathcal{M}_{22}^{-1}$$

defined in Theorem 6.1 the central compensator induced by the feasible (J,S)-inner-outer factorization $\mathcal{N}\mathcal{X}$, and we call the factorization $\mathcal{N}\mathcal{X}$ a representation of this compensator (cf. Definition 2.4).

Every central compensator has more than one representation:

Lemma 8.4 *Let $\mathcal{D} = \mathcal{N}\mathcal{X}$ be a feasible (J,S)-inner-outer factorization of $\mathcal{D}$, and let $\mathcal{D} = \widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ be another feasible $(J,\widetilde{S})$-inner-outer factorization of $\mathcal{D}$. These factorizations induce the same central compensator if and only if $\widetilde{\mathcal{X}} = E\mathcal{X}$ for some (invertible) $E \in \mathcal{L}(U \times W)$ of the form*

$$E = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix},$$

where E_{11} is invertible in $\mathcal{L}(U)$ and E_{22} is invertible in $\mathcal{L}(W)$.

Proof. If $\widetilde{\mathcal{X}} = E\mathcal{X}$ for some E of the form given above, then $\widetilde{\mathcal{X}}_{11} = E_{11}\mathcal{X}_{11}$ and $\widetilde{\mathcal{X}}_{12} = E_{11}\mathcal{X}_{12}$, hence $\widetilde{\mathcal{X}}_{11}^{-1}\widetilde{\mathcal{X}}_{12} = \mathcal{X}_{11}^{-1}\mathcal{X}_{12}$, and the two factorizations induce the same central compensator.

To prove the converse part we first observe that, by Lemma 4.10, $\widetilde{\mathcal{X}} = E\mathcal{X}$ for some invertible $E \in \mathcal{L}(U \times W)$. Thus,

$$\begin{aligned}\widetilde{\mathcal{X}}_{11} &= E_{11}\mathcal{X}_{11} + E_{12}\mathcal{X}_{21},\\ \widetilde{\mathcal{X}}_{12} &= E_{11}\mathcal{X}_{12} + E_{12}\mathcal{X}_{22}.\end{aligned}$$

The two factorizations induce the same central compensator iff $\widetilde{\mathcal{X}}_{12} = \widetilde{\mathcal{X}}_{11}\mathcal{X}_{11}^{-1}\mathcal{X}_{12}$, i.e., iff

$$\begin{aligned}
\widetilde{\mathcal{X}}_{12} &= E_{11}\mathcal{X}_{12} + E_{12}\mathcal{X}_{22} \\
&= \widetilde{\mathcal{X}}_{11}\mathcal{X}_{11}^{-1}\mathcal{X}_{12} \\
&= \left(E_{11}\mathcal{X}_{11} + E_{12}\mathcal{X}_{21}\right)\mathcal{X}_{11}^{-1}\mathcal{X}_{12} \\
&= E_{11}\mathcal{X}_{12} + E_{12}\mathcal{X}_{21}\mathcal{X}_{11}^{-1}\mathcal{X}_{12}.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
0 &= E_{12}\left(\mathcal{X}_{22} - \mathcal{X}_{21}\mathcal{X}_{11}^{-1}\mathcal{X}_{12}\right) \\
&= E_{12}\mathcal{M}_{22}^{-1},
\end{aligned}$$

where the last equality follows from the formula for $\mathcal{F}_{22}^{\frown}$ given in Theorem 6.1. However, this is equivalent to the condition $E_{12} = 0$.

Substitute $E_{12} = 0$ into the preceding formula for $\widetilde{\mathcal{X}}_{11}$ to get $\widetilde{\mathcal{X}}_{11} = E_{11}\mathcal{X}_{11}$. Since both $\widetilde{\mathcal{X}}_{11}$ and $\mathcal{X}_{11}$ are invertible, E_{11} must be invertible. This, together with the invertibility of E implies that E_{22} is invertible. $\square$

Motivated by the preceding lemma, we make the following definition:

Definition 8.5 *The (J, S)-inner-outer factorization $\mathcal{N}\mathcal{X}$ of $\mathcal{D}$ is equivalent to the $(J, \widetilde{S})$-inner-outer factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ of $\mathcal{D}$ if $\widetilde{\mathcal{X}} = E\mathcal{X}$ for some $E \in \mathcal{L}(U \times W)$ of the form*

$$E = \begin{bmatrix} E_{11} & 0 \\ E_{21} & E_{22} \end{bmatrix},$$

with E_{11} is invertible in $\mathcal{L}(U)$ and E_{22} is invertible in $\mathcal{L}(W)$. These factorizations are strictly equivalent if, in addition, $E_{11} = I$.

The following lemma lists some of the properties of equivalent factorizations.

Lemma 8.6 *Suppose that the (J, S)-inner-outer factorization $\mathcal{N}\mathcal{X}$ of $\mathcal{D}$ is equivalent to the $(J, \widetilde{S})$-inner-outer factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ of $\mathcal{D}$, and let $E = \widetilde{\mathcal{X}}\mathcal{X}^{-1}$ be the operator in Definition 8.5. Then the following statements are true:*

(i) The two sensitivity operators $\widetilde{S}$ and S satisfy

$$\begin{aligned}
S_{11} - S_{12}S_{22}^{-1}S_{21} &= E_{11}^{*}\left(\widetilde{S}_{11} - \widetilde{S}_{12}\widetilde{S}_{22}^{-1}\widetilde{S}_{21}\right)E_{11}, \\
S_{22} &= E_{22}^{*}\widetilde{S}_{22}E_{22}.
\end{aligned}$$

(ii) The first factorization is feasible iff the second factorization is feasible.

(iii) The first factorization is [uniformly] suboptimal iff the second factorization is [uniformly] suboptimal.

(iv) The two factorizations induce the same central compensator, and if they are strictly equivalent, then the two semi-closed loop systems induced by these factorization are identical, if we ignore their last row corresponding to the output labeled z_2 in Figure 11.

Proof. (i) This proof is a mechanical computation based on Lemma 4.10.

(ii) Clearly, $\widetilde{\mathcal{X}} = E\mathcal{X}$ is invertible iff $\mathcal{X}$ is invertible, $S = E^*\widetilde{S}E$ is invertible iff S if invertible, and $\widetilde{\mathcal{X}}_{11} = E_{11}\mathcal{X}_{11}$ is invertible iff $\mathcal{X}_{11}$ is invertible.

(iii) This follows from (i), (ii) and Corollary 7.3.

(iv) The first claim follows from Lemma 8.4. If $E_{11} = I$, then, $\widetilde{\mathcal{X}}_{11} = \mathcal{X}_{11}$, $\widetilde{\mathcal{X}}_{12} = \mathcal{X}_{12}$, and Theorem 6.1(iv) shows that the first three rows of the semi-closed system $\Psi^{\frown}$ are identical for the two factorizations. $\square$

Lemma 8.7 *Every uniformly suboptimal $(J, \widetilde{S})$-inner-outer factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ of $\mathcal{D}$ is strictly equivalent to a uniformly suboptimal (J, S)-inner-outer factorization for which the off-diagonal terms S_{12} and $S_{21} = S_{12}^*$ vanish, and, if $\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ >> 0$, then it is equivalent to a uniformly suboptimal (J,T)-inner-outer factorization with sensitivity operator $T = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$.*

Proof. The proof of this lemma is contained in the proof of Lemma 8.2. $\square$

Theorem 8.8 *Suppose that $\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ >> 0$ on $L^2(\mathbf{R}^+; U)$, and that $\mathcal{D} = \mathcal{N}\mathcal{X}$ is a uniformly suboptimal (J, S)-inner-outer factorization of $\mathcal{D}$ for which the cross terms S_{12} and $S_{21} = S_{12}^*$ vanish. If $E_{12} \in \mathcal{L}(W; U)$ satisfies*

$$\left\| S_{11}^{1/2} E_{12} (-S_{22})^{-1/2} \right\| < 1, \tag{8.1}$$

and if we define E by

$$E = \begin{bmatrix} I & E_{12} \\ 0 & I \end{bmatrix},$$

then the factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}} = (\mathcal{N}E)(E^{-1}\mathcal{X})$ is also uniformly suboptimal; hence it induces a uniformly suboptimal central compensator. Every possible uniformly suboptimal central compensator has a representation of this form,

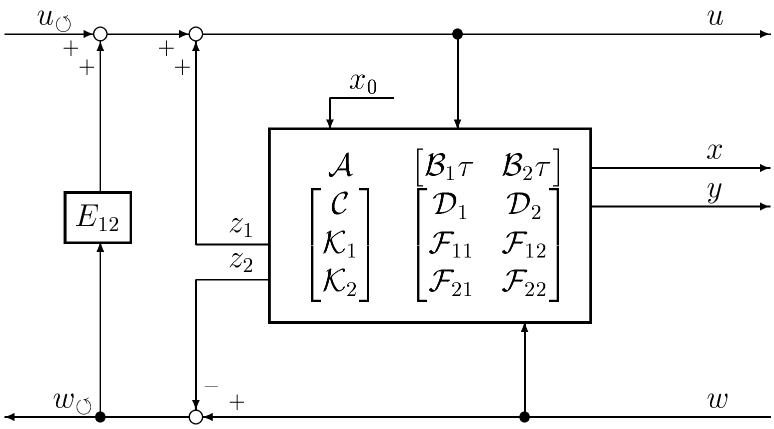

Figure 12: Parameterization of all suboptimal central compensators

i.e., by choosing the operator E_{12} appropriately we can generate every possible uniformly suboptimal central compensator. Moreover, different choices of E_{12} give rise to different central compensators, i.e., there is a one-to-one correspondence between the operator E_{12} and the corresponding central compensator.

Before proving this theorem, let us warn the reader that this parameterization does not generate all possible semi-closed loop systems $\Psi^{\frown}$. Instead it generates exactly one representative for each equivalence class, if we consider two semi-closed systems to be equivalent whenever they induced the same central compensator. (To get a parameterization of all possible feasible semi-closed loop systems it suffices to combine this theorem with Lemma 8.4.) Figure 12 contains a picture of the parameterization in Theorem 8.8.

Proof of Theorem 8.8. Choose some $E_{12} \in \mathcal{L}(W;U)$, and define E as in the theorem. A direct computation shows that the sensitivity operator $\widetilde{S}$ of the factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}} = (\mathcal{N}E)(E^{-1}\mathcal{X})$ is given by (cf. Lemma 4.10)

$$\widetilde{S} = \begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{21} & \widetilde{S}_{22} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{11}E_{12} \\ E_{12}^*S_{11} & S_{22} + E_{12}^*S_{11}E_{12} \end{bmatrix}.$$

In particular, $\widetilde{S}_{11}$ is always uniformly positive, and $\widetilde{S}_{22} << 0$ if and only (8.1) holds. This together with Lemma 8.2(iv) implies that the factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ is uniformly suboptimal iff (8.1) holds.

Conversely, suppose that $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ is a uniformly suboptimal $(J, \widetilde{S})$-inner-outer factorization of $\mathcal{D}$. Without loss of generality (see Lemma 8.7) we can suppose that the cross terms $\widetilde{S}_{12}$ and $\widetilde{S}_{21} = \widetilde{S}_{12}^*$ vanish. An inspection of the proof of Lemma 8.2(iv) shows that there is an invertible operator $F \in \mathcal{L}(U \times W)$ with invertible F_{11} such that

$$\widetilde{\mathcal{X}} = F\mathcal{X}.$$

Since F_{11} is invertible, F has the LU-decomposition

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} - F_{21}F_{11}^{-1}F_{12} \end{bmatrix} \begin{bmatrix} I & F_{11}^{-1}F_{12} \\ 0 & I \end{bmatrix}.$$

The first factor represents an equivalence transformation in the sense of Definition 8.5, so we can discard this factor if we at the same time replace the factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ by an equivalent one (for which we still use the same notation $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$). But this means that $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}} = (\mathcal{N}E)(E^{-1}\mathcal{X})$, where E is of the form given in Theorem 8.8 with $E = -F_{11}^{-1}F_{12}$.

The uniqueness claim follows from Lemma 8.4. $\square$

Remark 8.9 *A closer inspection of the proofs of Lemma 8.2(iv) and Theorem 8.8 show that if we relax* (8.1) *to* $\|S_{11}^{1/2}E_{12}(-S_{22})^{-1/2}\| \leq 1$, *then the resulting factorization is still suboptimal (but not uniformly). It is even possible to allow a norm slightly bigger than one without loosing feasibility, but in this case the suboptimality is lost. In particular, in this way we can construct an example of a feasible factorization that is not suboptimal.*

9 Parameterization of All Suboptimal Controllers

In the previous section we gave a parameterization of all uniformly suboptimal central compensators. The same parameterization can be used to generate all possible suboptimal or uniformly suboptimal compensators: it suffices to replace the static parameter E_{12} in Theorem 8.8 by a dynamic parameter $\mathcal{V}$, as indicated in Figure 13.

Theorem 9.1 *Suppose that* $\pi_+\mathcal{D}_1^*J\mathcal{D}_1\pi_+ >> 0$ *on* $L^2(\mathbf{R}^+; U)$, *and that* $\mathcal{D} = \mathcal{N}\mathcal{X}$ *is a uniformly suboptimal* (J, S)*-inner-outer factorization of* $\mathcal{D}$ *for which*

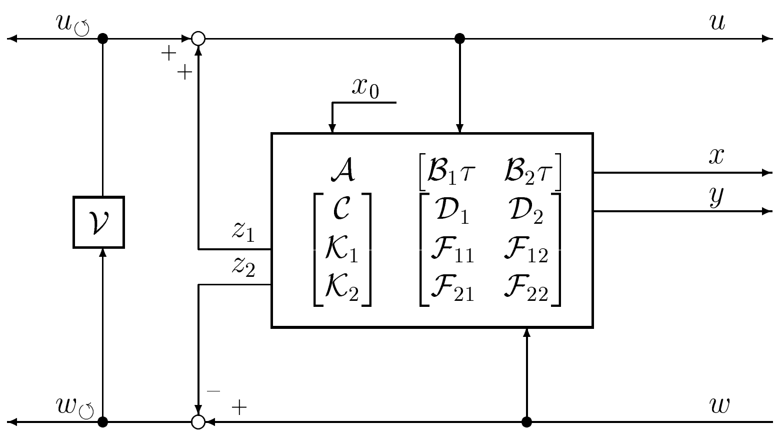

Figure 13: Parameterization of all suboptimal compensators

the cross terms S_{12} and $S_{21} = S_{12}^*$ vanish. Define $\mathcal{M} = \mathcal{X}^{-1}$. Then, for each $\mathcal{V} \in TIC(W;U)$ satisfying

$$\|S_{11}^{1/2}\mathcal{V}(-S_{22})^{-1/2}\| \le 1, \tag{9.1}$$

the operator $\mathcal{X}_{11} - \mathcal{V}\mathcal{X}_{21}$ has an inverse in $TIC(U)$, the operator $\mathcal{M}_{21}\mathcal{V} + \mathcal{M}_{22}$ has an inverse in $TIC(W)$, and the operator $\mathcal{U}$ defined by

$$\begin{aligned}
\mathcal{U} &= (I - \mathcal{F}_{11} + \mathcal{V}\mathcal{F}_{21})^{-1}(\mathcal{F}_{12} + \mathcal{V} - \mathcal{V}\mathcal{F}_{22})\\
&= (\mathcal{X}_{11} - \mathcal{V}\mathcal{X}_{21})^{-1}(-\mathcal{X}_{12} + \mathcal{V}\mathcal{X}_{22})\\
&= (\mathcal{V} + \mathcal{F}_{\circlearrowleft 11}\mathcal{V} + \mathcal{F}_{\circlearrowleft 12})(\mathcal{F}_{\circlearrowleft 21}\mathcal{V} + I + \mathcal{F}_{\circlearrowleft 22})^{-1}\\
&= (\mathcal{M}_{11}\mathcal{V} + \mathcal{M}_{12})(\mathcal{M}_{21}\mathcal{V} + \mathcal{M}_{22})^{-1}
\end{aligned} \tag{9.2}$$

is a suboptimal compensator for Ψ. The operator $\mathcal{U}$ is uniformly suboptimal for Ψ iff

$$\|S_{11}^{1/2}\mathcal{V}(-S_{22})^{-1/2}\| < 1. \tag{9.3}$$

Every possible suboptimal compensator $\mathcal{U}$ has a representation of the form (9.2) *for some $\mathcal{V}$ satisfying* (9.1)*, i.e., by appropriate choice of the parameter $\mathcal{V}$ we can generate every possible suboptimal compensator $\mathcal{U}$. Moreover, different choices of $\mathcal{V}$ give rise to different compensators, i.e., there is a one-to-one correspondence between the parameter $\mathcal{V}$ and the corresponding suboptimal compensator $\mathcal{U}$. (We get the central compensator by taking $\mathcal{V} = 0$.)*

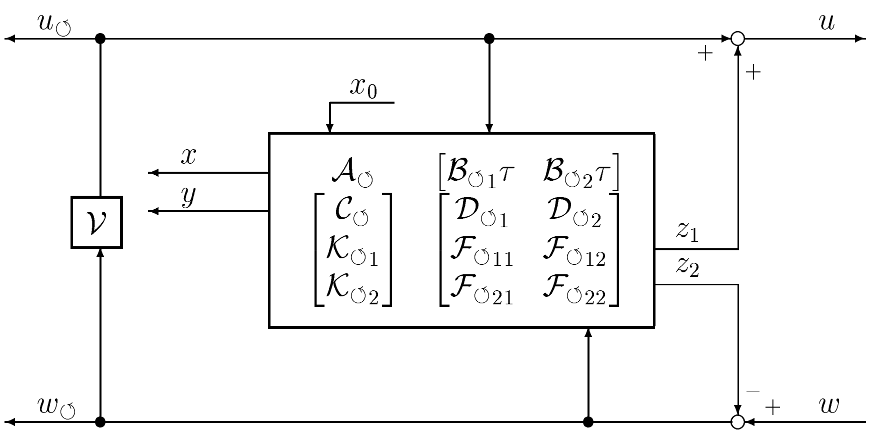

Figure 14: Suboptimal parameterization based on closed loop system

See Figures 13 and 14 for diagrams describing the parameterization in Theorem 9.1. If we in those diagram take $x_0 = 0$, then $\mathcal{U}$ is the mapping from w to u, and $\mathcal{V}$ is the mapping from $u_{\circlearrowleft}$ to $w_{\circlearrowleft}$.

Proof of Theorem 9.1. Consider the connection drawn in Figure 13. From that diagram we get

$$u = \mathcal{K}_1 x_0 + \mathcal{F}_{11} u + \mathcal{F}_{12}\pi_+ w + \mathcal{V}\left(\pi_+ w - \mathcal{K}_2 x_0 - \mathcal{F}_{21} u - \mathcal{F}_{22}\pi_+ w\right).$$

Formally, with $x_0 = 0$, this leads to the formula $u = \mathcal{U}\pi_+ w$, where $\mathcal{U}$ is the operator defined in the first two lines of the formula in Theorem 9.1.

Let us prove the claim about the invertibility of the operator $\mathcal{X}_{11} - \mathcal{V}\mathcal{X}_{21}$. The feasibility of the factorization implies that $\mathcal{X}_{11}$ is invertible in $TIC(U)$. Factor out $S_{11}^{1/2}\mathcal{X}_{11}$ to the right and $S_{11}^{-1/2}$ to the left to get

$$\begin{aligned}\mathcal{X}_{11} - \mathcal{V}\mathcal{X}_{21} &= \left(S_{11}^{-1/2} - \mathcal{V}\mathcal{X}_{21}\mathcal{X}_{11}^{-1}S_{11}^{-1/2}\right) S_{11}^{1/2}\mathcal{X}_{11}\\ &= S_{11}^{-1/2}\left(I - S_{11}^{1/2}\mathcal{V}(-S_{22})^{-1/2}(-S_{22})^{1/2}\mathcal{X}_{21}\mathcal{X}_{11}^{-1}S_{11}^{-1/2}\right) S_{11}^{1/2}\mathcal{X}_{11}.\end{aligned}$$

From this factorization and from Lemma 8.2 we find that $\mathcal{X}_{11} - \mathcal{V}\mathcal{X}_{21}$ is, indeed, invertible. Hence we can use the first part of (9.2) to define $\mathcal{U} \in TIC(U)$.

To derive the second part of (9.2) we rewrite Figure 13 into Figure 14, which is based on the closed loop system $\Psi_{\circlearrowleft}$ instead of on the extended open

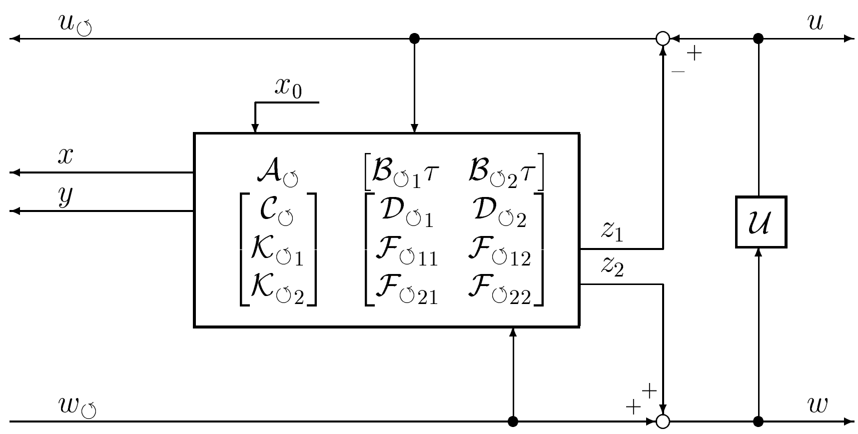

Figure 15: Recovery of the parameter of a suboptimal compensator

loop system Ψ. From this diagram we get

$$\begin{aligned} w_{\circlearrowleft} &= \pi_+ w - \mathcal{K}_{\circlearrowleft 2} x_0 - \mathcal{F}_{\circlearrowleft 21}\mathcal{V} w_{\circlearrowleft} - \mathcal{F}_{\circlearrowleft 22} w_{\circlearrowleft}, \\ u &= \mathcal{V} w_{\circlearrowleft} + \mathcal{K}_{\circlearrowleft 1} x_0 + \mathcal{F}_{\circlearrowleft 11}\mathcal{V} w_{\circlearrowleft} + \mathcal{F}_{\circlearrowleft 12} w_{\circlearrowleft}. \end{aligned}$$

Formally, with $x_0 = 0$, this leads to the definition of $\mathcal{U}$ to be given by the last two lines of (9.2). To show that the operator $\mathcal{M}_{21}\mathcal{V} + \mathcal{M}_{22}$ has an inverse in $TIC(W)$ one argues essentially in the same way as above, using Lemma 8.2 and the invertibility of $\mathcal{M}_{22}$.

The cost of $\pi_+ w$ in Figures 13 and 14 with $x_0 = 0$ can be written alternatively as (cf. Theorem 4.5 and Corollary 4.9)

$$\begin{aligned} Q(0, \mathcal{U}\pi_+ w, w) &= Q_{\circlearrowleft}(0, \mathcal{V}\pi_+ w_{\circlearrowleft}, w_{\circlearrowleft}) \\ &= \langle \mathcal{V}\pi_+ w_{\circlearrowleft}, S_{11}\mathcal{V}\pi_+ w_{\circlearrowleft}\rangle_{L^2(\mathbf{R}^+;U)} + \langle w_{\circlearrowleft}, S_{22} w_{\circlearrowleft}\rangle_{L^2(\mathbf{R}^+;W)}. \end{aligned}$$

From this the claims about the suboptimality and uniform suboptimality follow easily (cf. the proof of Lemma 8.2(iv)).

That the parameterization given in Theorem 9.1 generates all possible suboptimal compensators follows from Theorem 9.2 below. $\square$

The proof of the fact that the parameterization in Theorem 9.1 captures all possible suboptimal compensators and that there is a one-to-one correspondence between the compensator $\mathcal{U}$ and the parameter $\mathcal{V}$ requires some preliminary considerations:

Theorem 9.2 *Make the same assumptions as in Theorem 9.1.*

(i) The suboptimal compensators $\mathcal{U}$ obtained in Theorem 9.1 satisfy the following two invertibility conditions:

(a) $\mathcal{M}_{11} - \mathcal{U}\mathcal{M}_{21}$ has an inverse in $TIC(U)$,

(b) $\mathcal{X}_{22} + \mathcal{X}_{21}\mathcal{U}$ has an inverse in $TIC(W)$.

Moreover, the inverses above are given by

$$\begin{aligned} (\mathcal{M}_{11} - \mathcal{U}\mathcal{M}_{21})^{-1} &= \mathcal{X}_{11} - \mathcal{V}\mathcal{X}_{21}, \\ (\mathcal{X}_{22} + \mathcal{X}_{21}\mathcal{U})^{-1} &= \mathcal{M}_{21}\mathcal{V} + \mathcal{M}_{22}. \end{aligned} \tag{9.4}$$

(ii) The two invertibility conditions in part (i) are equivalent, and every suboptimal compensator $\mathcal{U}$ that satisfies (one of) these conditions is of the type described in Theorem 9.1. The corresponding operator $\mathcal{V}$ is given by

$$\begin{aligned} \mathcal{V} &= (I + \mathcal{F}_{\circlearrowleft 11} - \mathcal{U}\mathcal{F}_{\circlearrowleft 21})^{-1} (-\mathcal{F}_{\circlearrowleft 12} + \mathcal{U} + \mathcal{U}\mathcal{F}_{\circlearrowleft 22}) \\ &= (\mathcal{M}_{11} - \mathcal{U}\mathcal{M}_{21})^{-1} (-\mathcal{M}_{12} + \mathcal{U}\mathcal{M}_{22}) \\ &= (\mathcal{U} - \mathcal{F}_{11}\mathcal{U} - \mathcal{F}_{12}) (-\mathcal{F}_{21}\mathcal{U} + I - \mathcal{F}_{22})^{-1} \\ &= (\mathcal{X}_{11}\mathcal{U} + \mathcal{X}_{12}) (\mathcal{X}_{21}\mathcal{U} + \mathcal{X}_{22})^{-1}. \end{aligned} \tag{9.5}$$

(iii) The set of compensators $\mathcal{U}$ obtained in Theorem 9.1 is bounded in $TIC(W;U)$.

(iv) Every suboptimal compensator $\mathcal{U}$ has a parameterization of the type given by Theorem 9.1. Thus, all suboptimal compensators satisfy (i)–(iii).

Proof. The proof of (i) is a direct computation based on the fact that $\mathcal{M} = \mathcal{X}^{-1}$. We leave this computation to the reader.

(ii) Suppose that $\mathcal{U}$ satisfies condition (a) in part (i), i.e., that $\mathcal{M}_{11} - \mathcal{U}\mathcal{M}_{21}$ has an inverse in $TIC(U)$. Then the diagram drawn in Figure 15 defines a well-posed system (in the L^2-sense), and

$$\begin{aligned} u_{\circlearrowleft} &= u - \mathcal{K}_{\circlearrowleft 1}x_0 - \mathcal{F}_{\circlearrowleft 11}u_{\circlearrowleft} - \mathcal{F}_{\circlearrowleft 12}\pi_+ w_{\circlearrowleft} \\ &= \mathcal{U}(w_{\circlearrowleft} + \mathcal{K}_{\circlearrowleft 2}x_0 + \mathcal{F}_{\circlearrowleft 21}u_{\circlearrowleft} + \mathcal{F}_{\circlearrowleft 22}\pi_+ w_{\circlearrowleft}) - \mathcal{K}_{\circlearrowleft 1}x_0 - \mathcal{F}_{\circlearrowleft 11}u_{\circlearrowleft} - \mathcal{F}_{\circlearrowleft 12}\pi_+ w_{\circlearrowleft}. \end{aligned}$$

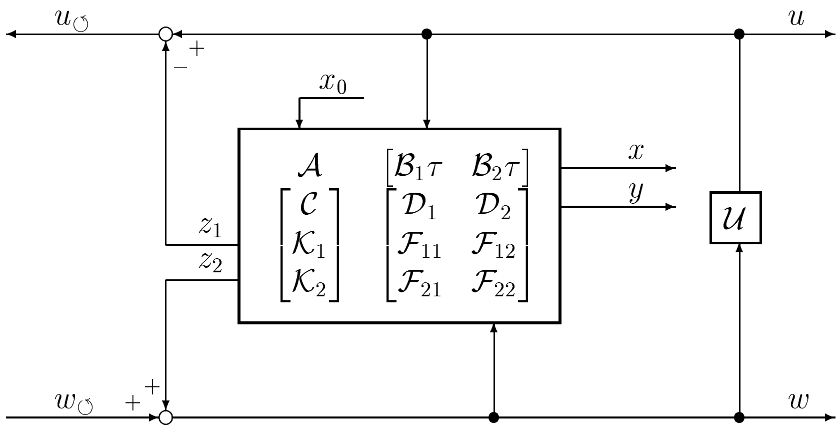

Figure 16: Alternative recovery of the parameter

If we here take $x_0 = 0$ and solve for $u_\circlearrowleft$, then we get the first two formulas in (9.5). The suboptimality of $\mathcal{U}$ implies that $\mathcal{V}$ must satisfy the norm condition $\|S_{11}^{1/2}\mathcal{V}(-S_{22})^{-1/2}\| \le 1$ (cf. the proof of Theorem 9.1). By part (i), $\mathcal{U}$ also satisfies condition (b) in part (i).

If, on the other hand, $\mathcal{U}$ satisfies condition (b) in part (ii), i.e., if $\mathcal{X}_{22} + \mathcal{X}_{21}\mathcal{U}$ has an inverse in $TIC(W)$, then we argue essentially in the same way, but replace Figure 15 by the equivalent Figure 16. We leave the details to the reader.

(iii) We get a uniform bound on the norm of $\mathcal{U}$ by using (9.1), (9.2), and the fact that Lemma 8.2(ii) gives us a uniform bound on $\|(\mathcal{X}_{11}^{-1} - \mathcal{V}\mathcal{X}_{21})^{-1}\|$ (cf. the proof of Theorem 9.1).

(iv) We claim that the set of all suboptimal compensators $\mathcal{U}$ obtained in Theorem 9.1 is both open and closed in the set of all suboptimal compensators. To see that it is open it suffices to observe that the set of compensators satisfying the two equivalent invertibility conditions in part (i) is open in $TIC(W;U)$; hence open in the set of suboptimal compensators. To see that it is closed it suffices to observe that (9.4) give us an priori bounds on the norms of $(\mathcal{M}_{11} - \mathcal{U}\mathcal{M}_{21})^{-1}$ and $(\mathcal{X}_{22} + \mathcal{X}_{21}\mathcal{U})^{-1}$, hence if we take a sequence of suboptimal compensators $\mathcal{U}_k$ of the type described in Theorem 9.1 converging to an arbitrary suboptimal compensator $\mathcal{U}$, then this limiting compensator must also satisfy conditions (a) and (b) in part (i). Thus, we conclude that the set of all compensators $\mathcal{U}$ parameterized by Theorem 9.1 is a bounded component of the set of all suboptimal compensators (more precisely, the component that contains the central compensator $\mathcal{F}_{12}^{\curvearrowleft}$).

To complete the proof of (iv) it suffices to show that the set of all suboptimal compensators is convex, hence connected. However, this follows from the positivity of $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ which implies that, for each fixed $x_0 \in H$ and $w \in L^2(\mathbf{R}^+; W)$, the cost function $Q(0, u, w)$ is convex in u. $\square$

10 Separation of Feedback and Feed-Forward Terms

In the primary state feedback/feedforward representation of the minimax solution given in Theorem 4.5 there is no direct reference to possible feedforward terms, and it is in fact impossible to include such a reference, due to the fact that for an arbitrary well-posed system it is not possible to separate a possible feedforward term from the feedback term. To do this we need an extra *regularity* assumption on the system introduced by George Weiss. In [Weiss 1994a, Theorem 5.8] he gives eight equivalent characterizations of the needed regularity notion, one of which is the following:

Definition 10.1 *(i) A causal time-invariant operator $\mathcal{D}\colon L^2(\mathbf{R}; V) \to L^2(\mathbf{R}; Y)$ is called regular if, for every $v_0 \in V$, the strong Abel mean*

$$Dv_0 = \lim_{\lambda \to +\infty} \widehat{\mathcal{D}}(\lambda) v_0$$

exists for every $v_0 \in V$; here λ tends to infinity along the real axis and $\widehat{\mathcal{D}}$ is the transfer function (the distribution Laplace transform) of $\mathcal{D}$.

(ii) The operator $D\colon V \to Y$ defined in (i) is called the feedthrough operator of $\mathcal{D}$.

(iii) A regular map $\mathcal{D}\colon L^2(\mathbf{R}; V) \to L^2(\mathbf{R}; Y)$ is called strictly proper if its feedthrough operator vanishes.

(iv) We say that $\mathcal{D}$ is regular together with its adjoint if, in addition to (i), the strong Abel mean $\lim_{\lambda \to +\infty} \widehat{\mathcal{D}}^*(\lambda) y_0$ exists for every $y_0 \in Y$. (This limit is equal to $D^* v_0$ whenever it exists.)

(v) A well-posed linear time-invariant system Ψ is regular [together with its adjoint] if its input/output map is regular [together with its adjoint].

We borrow the following result (due to Weiss [1994a]) from [Staffans 1997, Proposition 39] (and at the same time extend it slightly to cover also the situation described in Theorem 6.1):

Proposition 10.2 *Suppose that $\mathcal{D}$ is J-coercive and that $\mathcal{D}$ a (J,S)-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$. In addition, suppose that the extended system $\Psi^{\text{ext}} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \left[\begin{smallmatrix} \mathcal{C} \\ \mathcal{K} \end{smallmatrix}\right] & \left[\begin{smallmatrix} \mathcal{D} \\ \mathcal{F} \end{smallmatrix}\right] \end{bmatrix}$ constructed in Theorem 4.5 is regular together with its adjoint, i.e., $\mathcal{D}$ and $\mathcal{F}$ are regular together with their adjoints.*

(i) Then all the input/output maps appearing in Theorem 4.5 are regular together with their adjoints. If we denote the feedthrough operators of $\mathcal{D}$, $\mathcal{N}$, $\mathcal{X}$, $\mathcal{M}$, $\mathcal{F}$, and $\mathcal{F}_{\circlearrowleft}$ by D, N, X, M, F, and $F_{\circlearrowleft}$, respectively, then

$$D = NX, \quad X = M^{-1}, \quad F = I - X, \quad F_{\circlearrowleft} = M - I.$$

In particular, X and M are invertible.

(ii) If, in addition, the factorization $\mathcal{N}\mathcal{X}$ is feasible, then all the input/output maps appearing in Theorem 6.1 and Lemma 6.2 are regular together with their adjoints. If we denote the feedthrough operators of $\mathcal{D}^{\curvearrowleft}$ and $\mathcal{F}^{\curvearrowleft}$ by $D^{\curvearrowleft}$ and $F^{\curvearrowleft}$, respectively, then

$$\begin{bmatrix} D_1^{\curvearrowleft} & D_2^{\curvearrowleft} \\ F_{11}^{\curvearrowleft} & F_{12}^{\curvearrowleft} \\ F_{21}^{\curvearrowleft} & F_{22}^{\curvearrowleft} \end{bmatrix} = \begin{bmatrix} D_1 X_{11}^{-1} & D_2 - D_1 X_{11}^{-1} X_{12} \\ X_{11}^{-1} - I & -X_{11}^{-1} X_{12} \\ -X_{21} X_{11}^{-1} & I - X_{22} + X_{21} X_{11}^{-1} X_{12} \end{bmatrix}$$
$$= \begin{bmatrix} N_1 - N_2 M_{22}^{-1} M_{21} & N_2 M_{22}^{-1} \\ M_{11} - I - M_{12} M_{22}^{-1} M_{21} & M_{12} M_{22}^{-1} \\ M_{22}^{-1} M_{21} & I - M_{22}^{-1} \end{bmatrix}.$$

In particular, X_{11} and M_{22} are invertible.

(iii) There is a unique (J,S)-inner-outer factorization $\mathcal{N}\mathcal{X}$ in Theorem 4.5 for which the feedthrough operator of $\mathcal{F}$ is zero (i.e., there is "no feed-forward term inside the feedback loop"), namely the one where $M = X = I$ and $D = N$. The formulas in part (ii) then simplify into

$$D^{\curvearrowleft} = D = N, \qquad F^{\curvearrowleft} = 0.$$

Observe that these relations between the feedthrough maps simply reflect the same relations valid between the corresponding input/output maps; cf. the formulas in Theorems 4.5 and 6.1. Part (iii) corresponds to the "standard" classical normalization described in, for example, [Green and Limebeer 1995, Chapter 6].

It is possible to parameterize the set of all central compensators using their feedforward operator as a parameter:

Lemma 10.3 *Under the assumptions of Proposition 10.2(ii) every central compensator is determined uniquely by its feedthrough operator $F_{\widehat{12}}$, and it has a (unique) representation induced by a factorization for which the feedthrough operator of the spectral factor $\mathcal{X}$ is given by*

$$X = \begin{bmatrix} I & -F_{\widehat{12}} \\ 0 & I \end{bmatrix}.$$

Proof Take an arbitrary parameterization of the central compensator. Then, by Proposition 10.2, the feedthrough operator X of the spectral factor $\mathcal{X}$ has an invertible upper left corner X_{11}. This means that we can factor X into the LU-form

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} X_{11} & 0 \\ X_{21} & X_{22} - X_{21}X_{11}^{-1}X_{12} \end{bmatrix} \begin{bmatrix} I & X_{11}^{-1}X_{12} \\ 0 & I \end{bmatrix}.$$

If we multiply $\mathcal{X}$ by $\left[\begin{smallmatrix} X_{11} & 0 \\ X_{21} & X_{22}-X_{21}X_{11}^{-1}X_{12} \end{smallmatrix}\right]^{-1}$ to the left, then we get a new equivalent spectral factor whose feedthrough operator is of the required form $\left[\begin{smallmatrix} I & -F_{\widehat{12}} \\ 0 & I \end{smallmatrix}\right]$. This spectral factor is unique (since a spectral factor is determined uniquely by its feedthrough operator), hence the corresponding central compensator is also determined uniquely by its feedthrough operator. $\square$

Theorem 10.4 *In addition to the assumptions of Proposition 10.2, suppose that $\pi_+\mathcal{D}_1^*J\mathcal{D}_1 >> 0$. Let $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ be the special $(J, \widetilde{S})$-inner-outer factorization (with zero feedthrough operator) of $\mathcal{D}$ described in Proposition 10.2(iii).*

(i) There exists at most one strictly proper central compensator. Such a compensator exists if and only if the factorization $\widetilde{\mathcal{N}}\widetilde{\mathcal{X}}$ is feasible, in which case it is induced by this factorization. It is [uniformly] suboptimal iff $\widetilde{S}_{22}$ is [uniformly] negative.

(ii) If $\widetilde{S}_{11} >> 0$, then there exists a uniformly suboptimal central compensator if and only if the factorization with feedthrough operator

$$X = \begin{bmatrix} I & \widetilde{S}_{11}^{-1}\widetilde{S}_{12} \\ 0 & I \end{bmatrix}$$

is uniformly suboptimal, i.e., if and only if this factorization is feasible and $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} << 0$. The sensitivity operator of this factorization is $\left[\begin{smallmatrix} \widetilde{S}_{11} & 0 \\ 0 & \widetilde{S}_{22}-\widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \end{smallmatrix}\right]$.

Proof (i) This follows from Corollary 7.3 and Lemma 10.3.

(ii) Suppose that a uniformly suboptimal central compensator $\mathcal{U}$ does exist (in particular, this is true if the special factorization given in (ii) is uniformly suboptimal). Since the feedthrough operator of $\widetilde{\mathcal{X}}$ is the identity, the corresponding outer factor produced by the parameterization in Lemma 10.3 is equal to

$$\mathcal{X} = \begin{bmatrix} I & -G \\ 0 & I \end{bmatrix} \widetilde{\mathcal{X}},$$

where G is the feedthrough operator of $\mathcal{U}$. Comparing this formula to the formula in Lemma 4.10 we find that the sensitivity of this factorization is given by (note that $\left[\begin{smallmatrix} I & -G \\ 0 & I \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & G \\ 0 & I \end{smallmatrix}\right]$)

$$\begin{aligned} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} &= \begin{bmatrix} I & 0 \\ G^* & I \end{bmatrix} \begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{21} & \widetilde{S}_{22} \end{bmatrix} \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{S}_{11} & \widetilde{S}_{11}G + \widetilde{S}_{12} \\ G^*\widetilde{S}_{11} + \widetilde{S}_{21} & G^*\widetilde{S}_{11}G + G^*\widetilde{S}_{12} + \widetilde{S}_{21}G + \widetilde{S}_{22} \end{bmatrix}. \end{aligned}$$

In particular, $S_{11} = \widetilde{S}_{11} >> 0$, and $G^*\widetilde{S}_{11}G + G^*\widetilde{S}_{12} + \widetilde{S}_{21}G + \widetilde{S}_{22} = S_{22} << 0$. By rearranging the terms (completing the square) we can rewrite this operator in the form

$$S_{22} = \widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} + \left(G + \widetilde{S}_{11}^{-1}\widetilde{S}_{12}\right)^* \widetilde{S}_{11} \left(G + \widetilde{S}_{11}^{-1}\widetilde{S}_{12}\right) << 0.$$

Thus $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} << 0$. By Lemma 8.2(iv), the factorization with sensitivity operator $\left[\begin{smallmatrix} \widetilde{S}_{11} & 0 \\ 0 & \widetilde{S}_{22}-\widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \end{smallmatrix}\right]$ is feasible. But this is exactly the factorization that we get by taking $G = -\widetilde{S}_{11}^{-1}\widetilde{S}_{12}$. $\square$

Theorem 10.4 stresses the importance of the particular sensitivity operator $\widetilde{S}$ corresponding to the factorization in Theorem 10.2(iii). What do we know about this sensitivity operator? Not very much in general, except that it is bounded from below by D^*JD whenever the Riccati operator Π is nonnegative on the reachable subspace:

Lemma 10.5 *In addition to the assumptions of Proposition 10.2, suppose that* $\Pi \geq 0$ *on the reachable subspace. Then the special sensitivity operator* $\widetilde{S}$ *that corresponds to the case of a zero feedthrough term in the feedback loop described in Theorem 10.4 satisfies* $\widetilde{S} \geq D^*JD$*. In particular,* $\widetilde{S}_{11} \geq D_1^*JD_1$*, hence* $\widetilde{S}_{11} >> 0$ *whenever* $D_1^*JD_1 >> 0$*.*

Proof. The inequality $\widetilde{S} \geq D^*JD$ follows from [Staffans 1998c, Theorem 6.13]. Trivially, this implies that $\widetilde{S}_{11} \geq D_1^*JD_1$. $\square$

Remark 10.6 *In particular, Lemma 10.5 applies to the full information problem* (1.4)–(1.5)*, because for the cost function* (1.5) *we have* $Q(x_0, u, 0) \geq 0$*, hence for all* $x_0 \in H$*,*

$$\langle x_0, \Pi x_0 \rangle_H = Q^{\mathrm{crit}}(x_0) = \max_{w \in L^2(\mathbf{R}^+;W)} Q^{\min}(x_0, w) \geq Q^{\min}(x_0, 0) \geq 0.$$

Thus, in this case $\widetilde{S}_{11} >> 0$ *whenever* $D_1^*D_1 >> 0$*.*

For the convenience of the reader, let us end this section by recalling from Staffans [1998bc] that the Riccati operator Π satisfies an algebraic Riccati equation. To formulate this result we need a few more facts about the general theory about well-posed linear systems. More precisely, it is known (see, e.g., Weiss [1994ab] or [Staffans 1997, Propositions 29 and 36]) that, in the case where $u \in W^{1,2}(\mathbf{R}^+; U)$, $w \in W^{1,2}(\mathbf{R}^+; W)$, and $Ax(0) + B_1u(0) + B_2w(0) \in H$ (where A is the generator of $\mathcal{A}$ and B_1 and B_2 are the two control operators; see the formula below), the input-state-output relations of the extended system appearing in Theorem 10.4(ii) can be written in the form (for all $t \in \mathbf{R}^+$)

$$\begin{aligned} x'(t) &= Ax(t) + B_1u(t) + B_2w(t), \\ y(t) &= \overline{C}x(t) + D_1u(t) + D_2w(t), \\ z_1(t) &= \overline{K}_1x(t) + F_{12}w(t), \\ z_2(t) &= \overline{K}_2x(t), \end{aligned}$$

where $F_{12} = -\widetilde{S}_{11}^{-1}\widetilde{S}_{12}$. The operators $\overline{C}$, $\overline{K}_1$, and $\overline{K}_2$ are the Weiss extensions of the observation operators C, K_1, and K_2 defined on dom(A), i.e.,

$$\overline{C} = \lim_{\lambda\to+\infty} \lambda C(\lambda I - A)^{-1}, \quad \overline{K}_i = \lim_{\lambda\to+\infty} \lambda K_i(\lambda I - A)^{-1}, \quad i = 1, 2.$$

The adjoints B_1^* and B_2^* of the operators B_1 and B_2 are defined on dom(A*), and they are extended in a similar way into $\overline{B}_i^* = \lim_{\lambda\to+\infty} \lambda B_i^*(\lambda I - A^*)^{-1}$, $i = 1, 2$. Moreover, we define

$$K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \quad \overline{B}^* = \begin{bmatrix} \overline{B}_1^* \\ \overline{B}_2^* \end{bmatrix}, \quad F = \begin{bmatrix} 0 & F_{12} \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} \widetilde{S}_{11} & 0 \\ 0 & \widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \end{bmatrix}.$$

With these notations we have the following result:

Theorem 10.7 *The Riccati operator* Π *and the feedback operator* K *satisfy the following two equations for all* $x_0 \in \mathrm{dom(A)}$ *and* $x_1 \in \mathrm{dom(A)}$*:*

$$\begin{aligned} &\langle Ax_0, \Pi x_1\rangle_H + \langle x_0, \Pi A x_1\rangle_H \\ &\qquad = -\langle Cx_0, JCx_1\rangle_Y + \langle Kx_0, SKx_1\rangle_U , \\ &Kx_0 = -S^{-1}(I - F^*)^{-1}\left(\overline{B}^*\Pi + D^*JC\right)x_0. \end{aligned} \tag{10.1}$$

Proof. This follows from [Staffans 1998b, Theorem 6.1] and [Staffans 1998c, Remark 5.2]. $\square$

By combining the two equations in (10.1) we get an algebraic Riccati equation for Π.

It is possible to write out the two components K_1 and K_2 of K explicitly in terms of the data: a substitution into (10.1) gives

$$\begin{aligned} K_1 &= -\widetilde{S}_{11}^{-1}\left(\overline{B}_1^*\Pi + D_{11}^*C_1\right), \\ K_2 &= -\left(\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12}\right)^{-1}\left(\overline{B}_2^*\Pi + D_{12}^*C_1 - \widetilde{S}_{21}K_1\right). \end{aligned}$$

Note that both of these operators appear in the algebraic Riccati equation for Π that we get from Theorem 10.7, but that only K_1 is used in the actual central control, i.e., in the feedback/feedforward formula

$$u(t) = \overline{K}_1 x(t) + F_{12}w(t)$$

for u. The role of K_2 is to reproduce the "worst possible" disturbance $w(t) = \overline{K}_2 x(t)$ in feedback form.

We observe that the Riccati equation that we get differs from the usual one in the sense that there is an extra unknown parameter $\widetilde{S}$ that does not normally occur in the continuous time case (although it is standard in the discrete time case).[1] This parameter can be computed from the Riccati operator:

Theorem 10.8 *The sensitivity operator $\widetilde{S}$ can be computed as the strong limit*

$$\widetilde{S}v_0 = D^*JDv_0 + \lim_{\lambda\to\infty} \overline{B}^*\Pi(\lambda I - A)^{-1}Bv_0$$

for each $v_0 \in U \times W$; here λ tends to $+\infty$ along the positive real axis.

Proof. This follows from [Staffans 1998b, Corollary 7.2] and [Staffans 1998c, Remark 5.2]. $\square$

11 (J, S)-Lossless Factorizations

The purpose of this final short section is to relate our central notion of a "uniformly suboptimal (J, S)-inner-outer factorization" $\mathcal{N}\mathcal{X}$ of $\mathcal{D}$ to the more commonly known notion of a (J, S)-lossless-outer factorization used in, e.g., Green [1992] and Curtain and Green [1997]. A formal definition of a (J, S)-lossless-outer factorization in the spirit of the present work is given in [Staffans 1998c, Definition 6.1]. This definition does not refer to any minimax properties of the problem. It is stated in terms of the *inner* factor $\mathcal{N}$ as opposed to the *outer* factor $\mathcal{X}$ used by Definitions 8.1, and it does not depend on how S is chosen. The notion of a lossless factorization is also easy to connect to the Riccati operator: according to [Staffans 1998c, Theorem 6.5], an (J, S)-inner-outer factorization of $\mathcal{D}$ is lossless if and only if the corresponding Riccati operator Π is nonnegative on the reachable subspace. We have implicitly used this fact in the proof Lemma 10.5; cf. [Staffans 1998c, Theorem 6.13].

Although the two notions are related, they are not identical. For example, it is not true that every uniformly suboptimal (J, T)-inner-outer factorization is (J, T)-lossless. To see this it suffices to take the dimension of W to be zero,

[1] The reader may compare the formulas for K_1 and K_2 given above to those valid in the discrete case; see, e.g., [Green and Limebeer 1995, Appendix B]. It is natural to expect a feedforward term from w to u in our case, too, since the class of discrete systems can be imbedded in the class of well-posed linear systems. See Staffans [1996].

which reduces the minimax problem to the minimization problem studied in Staffans [1998b]. For example, the Riccati operators corresponding to the bounded and positive real lemmas presented in [Staffans 1998b, Section 8] are negative definite, and this prevents the corresponding factorizations from being lossless. (To get an example where the dimension of W is nonzero we can simply combine this example with another independent example of full information type.)

On the other hand, if we replace the general problem (1.1)–(1.2) by the special full information problem (1.4)–(1.5), then every (J, S)-inner-outer factorization is lossless; this follows from [Staffans 1998c, Theorem 6.5] and Remark 10.6. (The factorization is not uniformly suboptimal unless $S_{22} << 0$; cf. Corollary 7.3.) As a matter of fact, one of the key conditions used in our proof of Theorem 6.4, namely (6.2), can be interpreted as a "losslessness" condition; cf. [Staffans 1998c, Lemma 4.11 and Definition 6.1], and our first proof of Theorem 1.3 (which was inspired by Green [1992] and Curtain and Green [1997]) was based on the fact that certain (J, S)-lossless factorizations are uniformly suboptimal. (That proof was more complicated than the present one and it produced a weaker result: it applied only to the full information problem (1.4)–(1.5), and it required W to be finite dimensional.)

Recently we have together with Kalle Mikkola studied the suboptimal Nehari problem in Mikkola and Staffans [1998]. Here, too, the notion of a uniformly suboptimal factorization seems to simpler and more useful than the notion of a lossless factorization. This has to do with the fact that, whereas the outer factor is still causal in the Nehari problem, the inner factor is neither causal nor anticausal, and the definition of a lossless factorization becomes more complicated.

References

T. Başar and P. Bernard. *H^∞-Optimal Contral and Related Minimax Design Problems, A Dynamic Game Approach.* Birkhäuser, Boston, Basel, Berlin, 1991.

T. Başar and G. J. Olsder. *Dynamic Noncooperative Game Theory.* Academic Press, London and New York, second edition, 1995.

R. F. Curtain and M. Green. Analytic system problems and J-lossless coprime factorization for infinite-dimensional linear systems. *J. Linear Algebra*, 257:121–161, 1997.

Y. Fourès and I. E. Segal. Causality and analyticity. *Transactions of the American Mathematical Society*, 78:385–405, 1955.

M. Green. H^∞ controller synthesis by J-lossless coprime factorization. *SIAM J. Control and Optimization*, 30:522–547, 1992.

M. Green and D. L. L. Limebeer. *Linear Robust Control.* Prentice Hall, Englewood Cliffs, New Jersey, 1995.

K. Mikkola. On the stable H^2 and H^∞ infinite-dimensional regulator problems and their algebraic Riccati equations. Technical Report A383, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 1997.

K. Mikkola and O. J. Staffans. A minimax formulation of the infinite-dimensional Nehari problem. In preparation, 1998.

O. J. Staffans. On the discrete and continuous time infinite-dimensional algebraic Riccati equations. *Systems and Control Letters*, 29:131–138, 1996.

O. J. Staffans. Quadratic optimal control of stable well-posed linear systems. *Transactions of American Mathematical Society*, 349:3679–3715, 1997.

O. J. Staffans. Coprime factorizations and well-posed linear systems. To appear in SIAM Journal on Control and Optimization, 1998a.

O. J. Staffans. Quadratic optimal control of well-posed linear systems. To appear in SIAM Journal on Control and Optimization, 1998b.

O. J. Staffans. Feedback representations of critical controls for well-posed linear systems. To appear in International Journal of Robust and Nonlinear Control, 1998c.

E. G. F. Thomas. Vector-valued integration with applications to the operator-valued H^∞ space. *IMA Journal on Mathematical Control and Information*, 14:109–136, 1997.

G. Weiss. Transfer functions of regular linear systems. Part I: Characterizations of regularity. *Transactions of American Mathematical Society*, 342: 827–854, 1994a.

G. Weiss. Regular linear systems with feedback. *Mathematics of Control, Signals, and Systems*, 7:23–57, 1994b.

M. Weiss. Riccati equation theory for Pritchard-Salamon systems: a Popov function approach. *IMA Journal of Mathematical Control & Information*, 14:1–37, 1997.

K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice Hall, Englewood Cliffs, New Jersey, 1996.