# Quadratic Optimal Control through Spectral and Coprime Factorization

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**Abstract**. We study the infinite horizon quadratic cost minimization problem for a linear time-invariant distributed parameter system with finitely may inputs and outputs. We work in an input/output framework, and reduce the unstable case to the stable case by the use of a right coprime factorization of the impulse response and a preliminary stabilizing feedback. The stable case is then solved through spectral factorization. The theory is illustrated with two examples involving pure time delays.

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## 1 Introduction

In the first half of this paper we solve the linear quadratic optimal control problem on an infinite horizon for linear systems whose impulse response is a locally finite matrix measure. It is an extension of our earlier work [15], where the same problem was solved in the stable case, i.e., when the measure is finite. Here the idea is to assume the existence of a right coprime factorization of the impulse response (as a convolution operator) in the ring of finite measures. By making use of this coprime factorization we first stabilize the system with a preliminary feedback, and then analyze the stabilized system by means of a spectral factorization as in [15]. Similar arguments have been presented in a more general setting in [19, 20], but in the special case considered here we are able to provide more details and to avoid some of the technical difficulties encountered in [19, 20].

The second half of the paper is devoted to the presentation of two examples illuminating the theory. Both of these have an impulse response which contains a pure time delay, and for this reason they cannot be analyzed with the classical theory based on the solution of a standard Riccati equation.

In order to provide some motivation for the approach that we use, let us begin by discussing the standard quadratic cost minimization problem for an infinite-dimensional time-invariant exponentially stabilizable and detectable system with

bounded control and observation operators. Suppose that we have such a system $\Sigma = (A, B, C, D)$ with finite-dimensional input space $U = \mathbf{R}^m$, (possibly) infinite-dimensional state space $H$, and finite-dimensional output space $Y = \mathbf{R}^n$, given by

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \qquad t \geq 0, \\ x(s) &= x_0. \end{aligned} \tag{1}$$

The spaces $U$, $H$, and $Y$ are Hilbert spaces, $A$ is the generator of a strongly continuous semigroup $S$ on $H$, and $B \in \mathcal{L}(U; H)$, $C \in \mathcal{L}(H; Y)$, and $D \in \mathcal{L}(U; Y)$ are bounded linear operators. The object is to find the optimal control $u^{\text{opt}} \in L^2(\mathbf{R}^+; U)$ that minimizes the cost function

$$J(u) = \int_0^\infty \left( |y(t)|^2 + |R^{1/2}u(t)|^2 \right) dt, \tag{2}$$

where $R$ is positive semi-definite. In particular, this means that $u$ must be chosen in such a way that $y \in L^2(\mathbf{R}^+; \mathbf{R}^n)$. To avoid the so called singular case we assume throughout that $D^*D + R$ is strictly positive definite.

We approach the problem described above in the following way. Since $\Sigma$ is stabilizable, we can choose some stabilizing state feedback operator $F$, i.e., we can choose some bounded operator $F \in \mathcal{L}(H; U)$ such that $A + BF$ generates an exponentially stable semigroup $S^F$. We define a new auxiliary variable $x$ by

$$x(t) = u(t) - Fz(t), \quad t \geq 0, \tag{3}$$

and rewrite (1) in the form

$$\begin{aligned} z'(t) &= (A + BF)z(t) + Bx(t), \quad t \geq 0, \\ y(t) &= (C + DF)z(t) + Dx(t), \quad t \geq 0, \\ u(t) &= Fz(t) + x(t), \qquad\quad t \geq 0, \\ z(0) &= z_0. \end{aligned} \tag{4}$$

See Figure 1 for a diagram of this system (the arrows labelled $f$, $g$, $\mu$ and $\nu$ refer to formula (8) below). According to the standard variation of constants formula,

$$z(t) = S^F(t)z_0 + \int_0^t S^F(t-s)Bx(s)\, ds, \quad t \geq 0, \tag{5}$$

hence

$$\begin{aligned} y(t) &= C^F S^F(t)z_0 + \int_0^t C^F S^F(t-s)Bx(s)\, ds + Dx(t), \quad t \geq 0, \\ u(t) &= FS^F(t)z_0 + \int_0^t FS^F(t-s)Bx(s)\, ds + x(t), \qquad t \geq 0, \end{aligned} \tag{6}$$

where $C^F = C + DF$. Define

$$\begin{aligned} \nu(\,ds) &= D\delta_0(\,ds) + C^F S^F(s)B\, ds, \\ \mu(\,ds) &= I\delta_0(\,ds) + FS^F(s)B\, ds, \qquad s \geq 0, \\ f(t) &= C^F S^F(t)z_0, \\ g(t) &= FS^F(t)z_0, \qquad\qquad t \geq 0, \end{aligned} \tag{7}$$

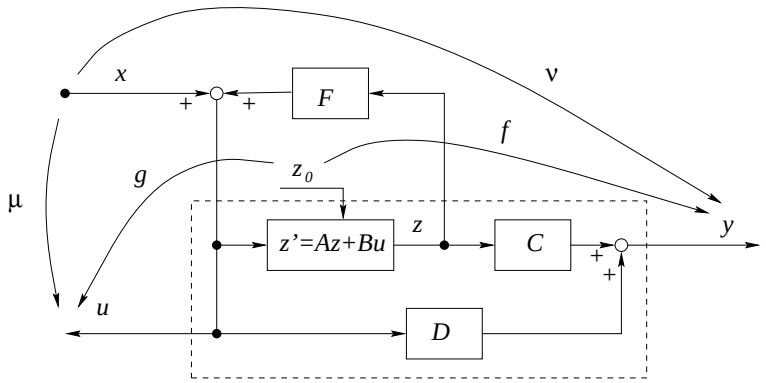


Figure 1: Preliminary stabilizing feedback

where $\delta_0$ represents a unit atom at zero and $I$ is the identity matrix. Then $\nu$ and $\mu$ are matrix-valued measures that consist of an atom at zero plus a function in $L^1(\mathbf{R}^+)$, and $f$ and $g$ are $L^2$-functions (of appropriate dimensions). Moreover, equations (6) can be written in the form (cf. Figure 1)

$$\begin{aligned} y &= \nu * x + f, \\ u &= \mu * x + g. \end{aligned} \tag{8}$$

The convolutions are defined in the usual way, i.e.,

$$\begin{aligned} (\nu * x)(t) &= \int_{[0,t]} \nu(\,ds) x(t-s), \\ (\mu * x)(t) &= \int_{[0,t]} \mu(\,ds) x(t-s), \quad t \in [0,\infty). \end{aligned}$$

In this work we study more general equations of the type (8), allowing $f$ and $g$ to be arbitrary $L^2$-functions on $(0,\infty)$, without assuming anything about the existence of an underlying system $\Sigma = (A,B,C,D)$ of the type presented in (1). In all the major proofs we work exclusively with the data given in (8), i.e., $\nu$, $\mu$, $f$, and $g$, under assumptions that will be explained in a moment. The functions $u$ and $y$ are considered to be the control and observation, respectively, and the object is to find the control $u \in L^2(\mathbf{R}^+;\mathbf{R}^m)$ that minimizes the cost function $J$ defined in (2), given the initial data $f$, and $g$. The results that we obtain can be used to derive both the standard results for the system $\Sigma = (A,B,C,D)$ in (1), and to derive some new results for systems with unbounded control operator $B$ and observation operator $C$.

Our major results are presented in Section 2. These results are applied to two examples in Sections 3 and 4. One of them is the system that we get by adding an input delay to (1), and the other is a distortion free transmission line, with the control and observation located at opposite ends.

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**Notations**.

$\mathbf{R}$, $\mathbf{R}^+$, $\overline{\mathbf{R}}^+$, $\mathbf{R}^-$, $\overline{\mathbf{R}}^-$: $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^+ = (0, \infty)$, $\overline{\mathbf{R}}^+ = [0, \infty)$, $\mathbf{R}^- = (-\infty, 0)$, and $\overline{\mathbf{R}}^- = (-\infty, 0]$.

$\mathcal{L}(U)$, $\mathcal{L}(U; V)$: The Banach space of bounded linear operators mapping $U$ into itself or $U$ into $V$, respectively, with the operator norm.

$L^p(E; \mathbf{R}^n)$: The Banach space of $\mathbf{R}^n$-valued $L^p$-functions, $1 \le p \le \infty$, on the interval $E$, with the usual norm.

$B^\infty(E; \mathbf{R}^n)$: The Banach space of $\mathbf{R}^n$-valued bounded Borel measurable functions on the interval $E$, with the sup-norm.

$B_0^\infty(E; \mathbf{R}^n)$: The subspace of $B^\infty(E; \mathbf{R}^n)$ of functions tending to zero at infinity.

$BC(E; \mathbf{R}^n)$, $BC_{\text{right}}(E; \mathbf{R}^n)$: The Banach spaces of bounded and continuous or right-continuous $\mathbf{R}^n$-valued functions on the interval $E$, with the sup-norm.

$BUC(E; \mathbf{R}^n)$: The Banach space of bounded and uniformly continuous $\mathbf{R}^n$-valued functions on the interval $E$, with the sup-norm.

$BUC_0(E; \mathbf{R}^n)$: The subspace of $BUC(E; \mathbf{R}^n)$ of functions tending to zero at infinity.

$M(E; \mathbf{R}^{n\times m})$: The set of $n \times m$-dimensional matrix-valued measures of bounded variation on the interval $E$, with the total variation norm.

$\mathcal{F}$, $\mathcal{G}$: $\mathcal{F} = L^2(\mathbf{R}^+; \mathbf{R}^n)$, and $\mathcal{G} = L^2(\mathbf{R}^+; \mathbf{R}^m)$.

$\check{\nu}$: The measure obtained from the measure $\nu$ through a reflection of the time axis, combined with the passing to the matrix adjoint, i.e., $\check{\nu}(E) = \nu(-E)^*$ for each Borel set $E$. The convolution operator $\check{\nu}*$ is the adjoint of the convolution operator $\nu*$.

$\hat{\nu}$: The Laplace (Stieltjes) transform of the measure $\nu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n\times m})$.

$\delta_0$: The unit atom at zero (the Dirac delta).

$\mu^{-1}$: The convolution inverse of the measure $\mu$, i.e., $\mu * \mu^{-1} = \mu^{-1} * \mu = I\delta_0$.

$\tau_t$: The translation operator $\tau_t f(s) = f(t+s)$ (this is a left-shift when $t > 0$).

$\chi_E$: The characteristic function of $E \subset \mathbf{R}$.

$\pi_E$: The (projection) operator that maps a function $f$ defined on $\mathbf{R}$ into $\pi_E f = \chi_E f$.

$\pi_+, \overline{\pi}_+$: $\pi_+ = \pi_{(0,\infty)}$ and $\overline{\pi}_+ = \pi_{[0,\infty)}$. These operators are the same in $L^p$, but they differ from each other in $B^\infty$.

$\pi_-, \overline{\pi}_-$: $\pi_- = \pi_{(-\infty,0)}$ and $\overline{\pi}_- = \pi_{(-\infty,0]}$. These operators are the same in $L^p$, but they differ from each other in $B^\infty$.

## 2 Summary of Results

We make the following basic assumptions on the data in (8). The functions $f$ and $g$ are supposed to belong to $\mathcal{F} = L^2(\mathbf{R}^+; \mathbf{R}^n)$, and $\mathcal{G} = L^2(\mathbf{R}^+; \mathbf{R}^m)$, respectively. The matrix-valued measures $\nu$ and $\mu$ (of dimensions $n \times m$ and $m \times m$, respectively) are required to be of bounded variation on $\overline{\mathbf{R}}^+$, and they should be right coprime in the following sense.

**Definition 1** *The measures $\nu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n\times m})$ and $\mu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$ are right coprime in $M(\overline{\mathbf{R}}^+)$ iff the Bezout identity*

$$\nu_1 * \nu + \mu_1 * \mu = I\delta_0 \tag{9}$$

*has a solution $\nu_1 \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times n})$ and $\mu_1 \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$. Here $I$ is the identity matrix, and $\delta_0$ is the Dirac delta.*

The measure $\mu$ is supposed to have an invertible feed-through operator:

**Definition 2** *[15, Definition 2.4] Write $\nu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n\times m})$ in the form $\nu(ds) = D_\nu\delta_0(ds) + \nu_+(ds)$, where $\delta_0$ is the Dirac delta, $D_\nu$ is a matrix, and $\nu_+$ has no atom at zero. Then $D_\nu\delta_0$ is called the (instantaneous) feed-through part of $\nu$, $D_\nu$ is called the (instantaneous) feed-through matrix of $\nu$, and $\nu_+$ is called the strictly causal part of $\nu$.*

If, for example, the Laplace transforms of $\mu$ and $\nu$ belong to the matrix-valued Callier-Desoer algebra $\mathcal{M}\hat{\mathcal{A}}_-(0)$ and $D_\mu$ is invertible, then the coprimeness of $\mu$ and $\nu$ can be tested by the standard rank test in the right half plane; see [6, Lemma 7.2.11].

According to [14, Theorem 1.2], in order for the observation $y$ in (8) to be well-defined and to depend continuously on $u$ in $L^2_{\mathrm{loc}}(\mathbf{R}^+)$, it is necessary and sufficient that $\mu$ has an invertible feed-through matrix. We assume that this is the case, and, without loss of generality, we take this feed-through matrix to be the identity matrix.[1] Thus $\mu$ has a convolution inverse $\mu^{-1}$, a measure supported on $\overline{\mathbf{R}}^+$, which is otherwise of the same type as $\mu$ except that its total variation may be infinite, cf. [11, Theorem 1.5, p. 114]. Eliminating the auxiliary variable $x$ from (8) we get the input/output relation

$$y = \nu * \mu^{-1} * (u - g) + f. \tag{10}$$

In particular, by taking the "transient" terms $f$, and $g$ to be zero, we find that the impulse response of the system is given by

$$\gamma = \nu * \mu^{-1}. \tag{11}$$

The formula above defines a right coprime factorization of $\gamma$ in $M(\overline{\mathbf{R}}^+)$ in the sense of [25, Definition 1, p. 331]. Thus, the class of impulse responses that we are able to handle is characterized by the fact that they have a right coprime factorization in $M(\overline{\mathbf{R}}^+)$.[2]

In order to solve the minimization problem presented in the introduction we have to assume that the pair $(\nu, \mu)$ is coercive:

**Definition 3** *The pair $(\nu, \mu) \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n\times m}) \times M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$ is coercive iff*

$$(\hat{\nu}(\mathrm{i}\omega))^*\hat{\nu}(\mathrm{i}\omega) + (\hat{\mu}(\mathrm{i}\omega))^* R\hat{\mu}(\mathrm{i}\omega) \geq \epsilon^2 I, \quad \omega \in \mathbf{R}, \tag{12}$$

*for some $\epsilon > 0$, here $\hat{\nu}$ and $\hat{\mu}$ are the Laplace (Stieltjes) transforms of $\nu$ and $\mu$.*

[1] Multiply the second equation in (8) by $D_\mu^{-1}$, and replace $D_\mu^{-1}\mu$, $D_\mu^{-1}u$, and $D_\mu^{-1}g$ by $\mu$, $u$, and $g$.

[2] In addition we need the Spectral Factorization Hypothesis 6.

Under these assumptions the minimization problem presented in the introduction has a unique solution:

**Lemma 4** *Let $\nu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n\times m})$ and $\mu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$ be right coprime, let $f \in L^2(\mathbf{R}^+; \mathbf{R}^n)$, $g \in L^2(\mathbf{R}^+; \mathbf{R}^m)$, and suppose that $(\nu, \mu)$ is coercive. Then the function $J(u)$ defined in (2), with $y$ given by (8), achieves its minimum at some $u^{\mathrm{opt}} \in L^2(\mathbf{R}; \mathbf{R}^m)$. The minimizing function $u^{\mathrm{opt}}$ is unique, and at the minimum, the optimal output $y^{\mathrm{opt}}$ and optimal control $u^{\mathrm{opt}}$ are given by*

$$\begin{aligned} y^{\mathrm{opt}} &= \nu * x^{\mathrm{opt}} + f, \\ u^{\mathrm{opt}} &= \mu * x^{\mathrm{opt}} + g, \end{aligned} \tag{13}$$

*where $x^{\mathrm{opt}}$ is the solution of the symmetric Wiener-Hopf equation*

$$((\check{\nu} * \nu + \check{\mu} * R\mu) * x^{\mathrm{opt}})(t) = -(\check{\nu} * f)(t) - (\check{\mu} * Rg)(t), \quad t \geq 0. \tag{14}$$

To prove this lemma it suffices to observe that because of the coprimeness assumption, $x$ can be regarded as a *free* parameter in $L^2(\mathbf{R}; \mathbf{R}^m)$ (see [20, Lemma 3.3]); hence we can carry out the minimization exactly in the same way as in [15, Lemma 3.2] with the replacements

$$Y \to Y \times U, \quad y \to \begin{pmatrix} y \\ R^{1/2}u \end{pmatrix}, \quad \gamma \to \begin{pmatrix} \nu \\ R^{1/2}\mu \end{pmatrix}, \quad f \to \begin{pmatrix} f \\ R^{1/2}g \end{pmatrix}, \quad \varphi \to 0. \tag{15}$$

At this stage we observe that we have arrived at exactly the same Wiener-Hopf equation as in [15] (with the replacements listed above), and we can apply the theory given in [15, Sections 2–3] to solve this equation. For completeness, we cite a number of results from there. The key step in the solution of the Wiener-Hopf equation (14) is to find an *invertible* solution $\xi \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$ of the equation

$$\check{\xi} * \xi = \check{\nu} * \nu + \check{\mu} * R * \mu. \tag{16}$$

Such a solution is called a *spectral factor*:

**Definition 5** *[15, Definition 2.1] A measure $\xi \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$ is called a (canonical) spectral factor of the measure $\check{\nu} * \nu + \check{\mu} * R * \mu$ if $\xi$ has a convolution inverse $\xi^{-1} \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$ and $\xi$ satisfies (16).*

Throughout this work we make the following hypothesis:

**Hypothesis 6** *[15, Hypothesis 2.3] The measure $\check{\nu} * \nu + \check{\mu} * R * \mu$ has a spectral factor $\xi \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$.*

In particular, this hypothesis is true if $\nu$ and $\mu$ are of the following type:

**Lemma 7** *[15, Lemma 2.1] Let $\nu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n\times m})$ and $\mu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$. Suppose that $(\nu, \mu)$ is coercive and that neither $\nu$ nor $\mu$ has a singular non-atomic part (in particular, this implies that the atomic part of $(\nu, \mu)$ is coercive). Then $\check{\nu} * \nu + \check{\mu} * R * \mu$ has a spectral factor $\xi \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$ in (at least) the two following cases:*

1. $m = 1$ *(the control is scalar);*
2. *the atoms of* $\nu$ *and* $\mu$ *are rationally dependent, in the sense that they are located at points that are integer multiples of one fixed time* $T$ *(in particular, this is true in case 4 below).*

*Moreover, in these cases,*

3. $\xi$ *and* $\xi^{-1}$ *have no singular non-atomic part.*
4. *If* $\nu$ *and* $\mu$ *are the sum of an atom at zero plus a function in* $L^1 \cap L^p(\mathbf{R}^+)$ *for some* $p \in [1, \infty]$*, then* $\xi$ *and* $\xi^{-1}$ *are of the same type.*
5. *If* $\int_{\overline{\mathbf{R}}^+} \mathrm{e}^{\epsilon_1 s}\big(|\nu|(\,ds) + |\mu|(\,ds)\big) < \infty$ *for some* $\epsilon_1 > 0$*, then* $\int_{\overline{\mathbf{R}}^+} \mathrm{e}^{\epsilon_2 s}|\xi|(\,ds) < \infty$ *and* $\int_{\overline{\mathbf{R}}^+} \mathrm{e}^{\epsilon_2 s}|\xi^{-1}|(\,ds) < \infty$ *for some* $\epsilon_2 > 0$ *(i.e, if* $\nu$ *and* $\mu$ *are of Callier-Desoer type, then so are* $\xi$ *and* $\xi^{-1}$*; cf. [2, Definition 2.3]).*

With the aid of the spectral factorization we are able to rewrite (14) as follows. We replace $\check{\nu} * \nu + \check{\mu} * R * \mu$ in (14) by $\check{\xi} * \xi$, and then convolve the second equation by $\check{\xi}^{-1}$ (which is supported on $\overline{\mathbf{R}}^-$) to get the following delay equation satisfied by $x^{\mathrm{opt}}$:

**Lemma 8** *[15, Formula (30)] Let* $\nu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n\times m})$*,* $\mu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$*, and let Hypothesis 6 hold. Then equation (14) can be rewritten in the following form:*

$$(\xi * x^{\mathrm{opt}})(t) = -(\check{\xi}^{-1} * (\check{\nu} * f + \check{\mu} * Rg))(t), \quad t \geq 0. \tag{17}$$

This is a delay equation for the unknown $x^{\mathrm{opt}}$, with a kernel $\xi$ and a forcing term $-\check{\xi}^{-1} * (\check{\nu} * f + \check{\mu} * Rg)$. This equation can be solved in a standard way (through an inversion of the kernel $\xi$), and we recall the following result:

**Theorem 9** *[15, Theorem 3.3] Let* $\nu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n\times m})$*,* $\mu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{m\times m})$*, let Hypothesis 6 hold, and let* $\mathcal{B}$ *be any one of the spaces (see the list of notations)*

$$L^p \text{ with } p \in [1, \infty],\ B^\infty,\ B_0^\infty,\ \text{or } BC_{\mathrm{right}}. \tag{18}$$

*Then for each* $f \in \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^n)$ *and* $g \in \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^m)$*, equation (17) has a unique solution* $x^{\mathrm{opt}} \in \mathcal{B}(\mathbf{R}; \mathbf{R}^m)$*, given by*

$$x^{\mathrm{opt}} = -\xi^{-1} * \overline{\pi}_+\big(\check{\xi}^{-1} * (\check{\nu} * f + \check{\mu} * Rg)\big). \tag{19}$$

*If, in addition,* $f \in \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^n)$ *and* $g \in \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^m)$*, where* $\mathcal{B}$ *is any one of the spaces*

$$BC,\ BUC,\ \text{or } BUC_0, \tag{20}$$

*and if*

$$\big(\check{\xi}^{-1} * (\check{\nu} * f + \check{\mu} * Rg)\big)(0) = 0, \tag{21}$$

*then* $x^{\mathrm{opt}} \in \mathcal{B}(\mathbf{R}; \mathbf{R}^m)$ *and* $x^{\mathrm{opt}}(0) = 0$*. If* $f \in \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^n)$ *and* $g \in \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^m)$*, where* $\mathcal{B}$ *is any one of the spaces listed in (20), and if we instead of (21) suppose that* $\nu$ *and* $\mu$ *have no singular part (apart from a feed-through part), then* $x^{\mathrm{opt}} \in \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^m)$ *(but* $x^{\mathrm{opt}}(0)$ *may be nonzero). In all cases listed above, the optimal output* $y^{\mathrm{opt}}$ *and optimal control* $u^{\mathrm{opt}}$ *defined in (13) belong to* $\mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^n)$ *and* $\mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^m)$*, respectively.*

In the case of the system $\Sigma = (A, B, C, D)$ in (1) it is well-known that the optimal control $u^{\rm opt}$ is of feedback form, i.e, $u^{\rm opt}(t) = Kx^{\rm opt}(t)$ for some bounded feedback operator $K$. In particular, taking $t = 0$ we get $u^{\rm opt}(0) = Kx_0$. This equation can be used to define the operator $K$ if we have an independent method of computing $u^{\rm opt}(0)$. Indeed, we have just developed such an independent method. Let us denote the operator that maps the initial data $(f, g)$ into $u^{\rm opt}(0)$ by $K_\Phi$, i.e.,

$$K_\Phi \begin{pmatrix} f \\ g \end{pmatrix} = u^{\rm opt}(0). \tag{22}$$

Then, by replacing $\mu$, $\nu$, $f$ and $g$ by the measures and functions defined in (7), we can compute the feedback operator $K$ from $K_\Phi$. In this sense $K_\Phi$ plays the role of a "universal feedback operator". This operator has the following properties:

**Theorem 10** *The operator $K_\Phi$ defined in (22) is a continuous linear operator from $BC_{\rm right}(\overline{\mathbf{R}}^+; \mathbf{R}^n) \times BC_{\rm right}(\overline{\mathbf{R}}^+; \mathbf{R}^m)$ into $\mathbf{R}^n$, and it is given by the following formula (cf. Definition 2; here we assume that the feed-through operator of $\mu$ is the identity):*

$$\begin{aligned} K_\Phi \begin{pmatrix} f \\ g \end{pmatrix} &= u^{\rm opt}(0) = x^{\rm opt}(0) + g(0) \\ &= g(0) - D_\xi^{-1} \check{\xi}^{-1} * \begin{pmatrix} \check{\nu} & \check{\mu} R \end{pmatrix} * \begin{pmatrix} f \\ g \end{pmatrix}(0). \end{aligned} \tag{23}$$

This theorem is a straightforward extension of [15, Proposition 4.1].

In the same way it is possible to define a "universal Riccati operator", from which the Riccati operator of the system $\Sigma = (A, B, C, D)$ can be computed. It is not difficult to show that the optimal cost $J(u^{\rm opt})$ is a quadratic function of the initial data, i.e., it is possible to find a positive self-adjoint operator $\Pi_\Phi$ such that the optimal cost is given by

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \Pi_\Phi \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle = J(u^{\rm opt}), \tag{24}$$

where the inner product is the usual inner product in $\mathcal{F} \times \mathcal{G}$. This operator has the following properties:

**Theorem 11** *Let $\nu \in M(\overline{\mathbf{R}}^+; \mathbf{R}^{n \times m})$ satisfy Hypothesis 6. Then the Riccati operator $\Pi_\Phi$ defined in (24) is given by*

$$\Pi_\Phi \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ Rg \end{pmatrix} - \begin{pmatrix} \nu \\ R\mu \end{pmatrix} * \xi^{-1} * \overline{\pi}_+ \check{\xi}^{-1} * \begin{pmatrix} \check{\nu} & \check{\mu} R \end{pmatrix} * \begin{pmatrix} f \\ g \end{pmatrix}. \tag{25}$$

*In particular, it maps $\mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^n) \times \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^m)$ continuously into itself, where $\mathcal{B}$ is any one of the spaces listed in (18). Moreover, if $\nu$ has no singular part (apart from a feed-through part), then $\Pi_\Phi$ maps $\mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^n) \times \mathcal{B}(\overline{\mathbf{R}}^+; \mathbf{R}^m)$ continuously into itself, where $\mathcal{B}$ is any one of the spaces listed in (20).*

This theorem is a straightforward extension of [15, Theorem 5.1].

By using Theorems 10 and 11 one can easily prove the following result:

**Theorem 12** *Let $\Sigma = (A, B, C, D)$ be a coercive exponentially stabilizable and detectable system with bounded control and observation operators.*

1. *There is a unique $u^{\mathrm{opt}}$ that minimizes the cost function (2), where $y$ is the output of the system (1).*
2. *The optimal $u^{\mathrm{opt}}$ can be written in feedback form $u^{\mathrm{opt}} = Kz^{\mathrm{opt}}$.*
3. *Let $\Pi$ be the "optimal cost operator" satisfying $\langle x_0, \Pi x_0\rangle = J(u^{\mathrm{opt}})$. Then $K$ can be computed from $\Pi$ through the formula*

$$K = -\left(D^*D + R\right)^{-1}\left(B^*\Pi + D^*C\right), \tag{26}$$

*and $\Pi$ can be computed from $K$ through*

$$\begin{aligned}\langle Ax_1, \Pi x_0\rangle_H + \langle x_1, \Pi A x_0\rangle_H + \langle x_1, C^*Cx_0\rangle_H,\\ = \langle x_1, K^*\left(D^*D + R\right)Kx_0\rangle_H, \quad x_0,\ x_1 \in \mathrm{dom}(A).\end{aligned} \tag{27}$$

*By combining these two equations we get the algebraic Riccati equation*

$$\begin{aligned}\langle Ax_1, \Pi x_0\rangle_H + \langle x_1, \Pi A x_0\rangle_H + \langle x_1, C^*Cx_0\rangle_H,\\ = \left\langle x_1, \left(\Pi B + C^*D\right)\left(D^*D + R\right)^{-1}\left(B^*\Pi + D^*C\right)x_0\right\rangle_H,\\ x_0,\ x_1 \in \mathrm{dom}(A).\end{aligned} \tag{28}$$

Of course, this theorem is well-known, but our proof based on Theorems 10 and 11 might be new (see [15, 17] for details).

The spectral factorization method for the solution of the quadratic cost minimization problem presented above does not depend explicitly on the boundedness of the operators $B$ and $C$. For the spectral factorization itself it suffices if the impulse response is of the right type, and even the derivation of the Riccati equation is valid for unbounded $B$ and $C$, as long as $x_0$ can be chosen in such a way that the functions $f$ and $g$ defined in (7) are continuous (in the proofs we need to evaluate $u^{\mathrm{opt}}(0)$ and $\Pi_\Phi\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)(0)$; see [15]). This makes it possible to extend Theorem 12 to a more general class of systems. We refer the reader to [15, 16, 17, 18, 20] for the exact statements and proofs. One of the most striking features of this extension is that the formulas in Theorem 12 need to be modified in a nontrivial way. More precisely, the operator $D^*D + R$ must be replaced throughout by the $D_\xi^*D_\xi$, where $D_\xi$ is the feed-through operator of the spectral factor $\xi$ of $\nu^*\nu + \mu^*R\mu$. We shall see examples in Sections 3 and 4 where these two operators differ from each other.

Finally, let us comment on how our results relate to the literature.

Some recent publications of Grabowski [9, 10] contain results that are quite close to some of ours. He uses spectral factorization to solve quadratic minimization problems, but he has not developed a general Riccati equation theory for systems with unbounded control and observation operators. We discuss Grabowski's example [10] in detail in Section 4.

Our results for systems with bounded control and observation operators extend those of Callier and Winkin [3, 4], in particular, Theorem 12 extends [4, Theorem 3] in several ways. There it is not proved that the optimal solution $u^{\mathrm{opt}}$ is of feedback type; that result is borrowed from the standard Riccati equation theory. Likewise, Callier

and Winkin borrow (26) from the standard Riccati equation theory instead of proving this equation.

In his thesis [29], Martin Weiss studies a spectral factorization problem of a more general type in the Pritchard-Salamon setting. In the Pritchard-Salamon case (that we did not include) our Theorem 12 becomes related to the sufficiency part of [29, Theorem 4.20].

The recent book [1] of Bensoussan, Da Prato, Delfour, and Mitter uses the classical Riccati equation approach to the quadratic cost minimization problem for differential systems with delays (and also for partial differential equations). The delay systems considered there are of retarded type and have a finite delay, and they do not cover the two examples that we give below.[3] The exact relationship between the theory presented here and in [15, 16, 18, 20, 21, 22, 23, 30] on one hand and in, e.g., [1, 12, 13] on the other hand, is still not clear. (These books do not mention spectral factorization.)

In [8] Flandoli, Lasiecka, and Triggiani verify that the optimal cost operator $\Pi$ satisfies a standard (as opposed to our nonstandard) algebraic Riccati equation in a particular case. The assumptions in [8] are quite different from ours: No assumption is made on the impulse response directly; instead it is assumed that the observation operator $C$ is bounded, and that the control operator $B$ is "trace regular". A recent example [28] by Weiss and Zwart shows that the same phenomenon that we encounter here can also occur under the assumptions used in [8]. The different conclusions here and in [8] are explained by the fact that in order to formulate the exact result it is necessary the extend $B^*$ to a larger domain, and our extension differs from the one in [8]. See [8, 12, 16, 18, 19, 20] for details.

# 3 Example: A system with an input delay

Below we present two examples to which our theory applies. They are particularly interesting because of the fact that the anomaly that we mentioned above (the change in one of the coefficients in the Riccati equation) occurs; yet they are simple enough so that we can find exact analytic solutions.

We begin by considering the example (1) with one additional input delay.[4] The defining equations become in this case

$$\begin{aligned} z'(t) &= Az(t) + Bu(t-T), \\ y(t) &= Cz(t) + Du(t-T), \quad t \in [0,\infty), \\ z(0) &= z_0, \\ u(t) &= \psi(t), \quad t \in [-T,0), \end{aligned} \tag{29}$$

where $T > 0$ is a constant delay, $\psi \in L^2((-T,0);\mathbf{R}^m)$, and the rest of the setting is the same as in (1). This example has been studied extensively in different connections, see, e.g., [5], [24], and the references mentioned therein, but we have not encountered the complete state space solution before.

[3] More general systems are discussed in [7], but independently of how the realization is chosen the input and output operators in our two examples are too unbounded for the Riccati equation theory presented in [1] to apply. This can be seen from the fact that the impulse response contains a pure delay term (except for our first example with $D = 0$).

[4] An output delay is even simpler to handle.

This system has a simple realization $\Psi = (A_\Psi, B_\Psi, C_\Psi, D_\Psi)$ as a regular linear system. It is the serial connection of a delay line (a left-shift on $(-T, 0)$ of $u$), with unbounded control and observation operators, connected in series with the system $\Sigma$ in (1). We refer the reader to [5, Example 4.1] for a closer description of this realization. For our purposes it suffices to know that its state space $H_\Psi$ is $H \times L^2((-T, 0); \mathbf{R}^m)$, its input space is $U = \mathbf{R}^m$, its output space is $Y = \mathbf{R}^n$, the controlled state at time $t \geq 0$ is given by $\left(\begin{smallmatrix} z(t) \\ u_t \end{smallmatrix}\right)$, where $u_t(s) = u(t+s)$ for $-T \leq s \leq 0$, and that the generating operators are

$$\begin{aligned} &\operatorname{dom}(A_\Psi) = \operatorname{dom}(A) \times \left\{ \psi \in W^{1,2}([-T, 0]; \mathbf{R}^m) \mid \psi(0) = 0 \right\}, \\ &A_\Psi \begin{pmatrix} z_0 \\ \psi \end{pmatrix} = \begin{pmatrix} Az_0 + B\psi(-T) \\ \psi' \end{pmatrix}, \quad B_\Psi u = \begin{pmatrix} 0 \\ \delta_0 u \end{pmatrix}, \\ &C_\Psi \begin{pmatrix} z_0 \\ \psi \end{pmatrix} = Cz_0 + D\psi(-T), \qquad D_\Psi = 0. \end{aligned} \tag{30}$$

It is possible to derive a system of the type (8) in the following way. We start by choosing some stabilizing state feedback operator $F$ for the system (1), i.e., we choose an operator $F \in \mathcal{L}(H; U)$ such that $A + BF$ generates an exponentially stable semigroup $S^F$. For the moment it does not really matter how this operator $F$ is chosen, but looking ahead, it will simplify the final formulae if we choose $F$ to be the optimal feedback operator $K$ for the system (1) with the cost function (2). In other words, we take $F = K$, where $K$ is given by (26)–(28). This time we define the auxiliary variable $x$ by

$$x(t) = u(t) - Kz(t+T), \quad t \geq -T. \tag{31}$$

Then (29) can be rewritten in the form

$$\begin{aligned} z'(t) &= (A + BK)z(t) + Bx(t-T), && t \in [0, \infty), \\ y(t) &= (C + DK)z(t) + Dx(t-T), && t \in [0, \infty), \\ u(t) &= Kz(t+T) + x(t), && t \in [-T, \infty), \\ z(0) &= z_0, \\ x(t) &= \varphi(t), \quad t \in [-T, 0), \end{aligned} \tag{32}$$

provided $\varphi$ is chosen as explained below. The restriction of the function $z$ to the interval $[0, T]$ depends only on $z_0$ and the initial functions $\psi$ and $\varphi$, and the further evolution of the two systems depends only on $z(T)$ and the restrictions of $x$ and $u$ to $[0, \infty)$. Solving $z$ from (29) we get

$$z(t) = S(t)z_0 + \int_{-T}^{t-T} S(t-T-s)B\psi(s)\, ds, \qquad t \in [0, T]. \tag{33}$$

If, instead, we solve $z$ from (32), then we get

$$z(t) = S^K(t)z_0 + \int_{-T}^{t-T} S^K(t-T-s)B\varphi(s)\, ds, \qquad t \in [0, T]. \tag{34}$$

By using either (33) or (34) in (31) we find that the systems (29) and (32) become identical if we choose $\psi$ and $\varphi$ to satisfy

$$\begin{aligned}\varphi(t) &= \psi(t) - K\left(S(t+T)z_0 + \int_{-T}^{t} S(t-s)B\psi(s)\,ds\right), \qquad t \in [-T,0],\\ \psi(t) &= \varphi(t) + K\left(S^K(t+T)z_0 + \int_{-T}^{t} S^K(t-s)B\psi(s)\,ds\right), \quad t \in [-T,0].\end{aligned} \tag{35}$$

In particular, taking $t = T$ in (33) and (34) we get

$$\begin{aligned}z(T) &= S(T)z_0 + \int_{-T}^{0} S(-s)B\psi(s)\,ds\\ &= S^K(T)z_0 + \int_{-T}^{0} S^K(-s)B\varphi(s)\,ds.\end{aligned} \tag{36}$$

Whereas it was fairly obvious how to rewrite (29) into the form (8), it is less obvious how to do this for (32) (the third equation is the difficult one.) However, this can be done. A computation similar to the one in the introduction shows that, by defining (with $C^K = C + DK$)

$$\begin{aligned}\nu(\,ds) &= D\delta_0(\,ds) + C^K S^K(s)B\,ds,\\ \mu(\,ds) &= I\delta_0(\,ds) + KS^K(s)B\,ds, \quad s \in \overline{\mathbf{R}}^+,\\ \nu_T &= \delta_T * \nu,\\ f(t) &= \begin{cases} C^K z(t) + D\varphi(t-t), & 0 \le t < T,\\ C^K S^K(t-T)z(T), & t \ge T,\end{cases}\\ g(t) &= KS^K(t)z(T), \qquad t \in \mathbf{R}^+,\end{aligned} \tag{37}$$

we can turn (32) into a system of the type (8), namely

$$\begin{aligned}y(t) &= (\nu_T * x)(t) + f(t), \quad t \in \overline{\mathbf{R}}^+,\\ u(t) &= (\mu * x)(t) + g(t), \quad t \in \overline{\mathbf{R}}^+.\end{aligned} \tag{38}$$

It is well-known that $\nu_T$ and $\mu$ are right coprime.[5] The objective is to minimize the cost function (2).

The crucial part in our solution was to find a spectral factor of the measure $\check{\nu}_T * \nu_T + \check{\mu} * R\mu$. The Laplace transform of this measure, restricted to the imaginary axis (with $s = \mathrm{i}\omega$) is

$$\hat{\nu}(\mathrm{i}\omega)^* \mathrm{e}^{\mathrm{i}\omega T}\mathrm{e}^{-\mathrm{i}\omega T}\hat{\nu}(\mathrm{i}\omega) + \hat{\mu}(\mathrm{i}\omega)^* R\hat{\mu}(\mathrm{i}\omega) = \hat{\nu}(\mathrm{i}\omega)^*\hat{\nu}(\mathrm{i}\omega) + \hat{\mu}(\mathrm{i}\omega)^* R\hat{\mu}(\mathrm{i}\omega).$$

But this is exactly the same function that we have to factor in the case where there is no delay, and we are in the case (1) discussed earlier. For this factorization we may use the formulae in Section 2. In particular, we conclude that

$$D_\xi^* D_\xi = D^* D + R. \tag{39}$$

[5] See, for example, [5, Example 4.1] or [24, p. 932].

It turns out that for this particular choice of $F$ (and this is the reason for why we chose $F = K$ in the first place) that the strictly causal part $\xi_+$ of $\xi$ is zero, and

$$\xi = (D^*D + R)^{1/2}\delta_0, \qquad \xi^{-1} = (D^*D + R)^{-1/2}\delta_0.$$

Comparing (39) to $D_\Psi^* D_\Psi + R = R$ we observe that $D_\xi^* D_\xi \neq D_\Psi^* D_\Psi + R$ except when $D = 0$.

The rest of the computations are straightforward, so let us only list the final results. It turns out that for the data in (37),

$$\pi_+\big(\check{\nu}_T * f + \check{\mu} * Rg\big) = 0,$$

hence

$$\begin{aligned}
x^{\mathrm{opt}}(t) &= 0, & t &\in [0,\infty),\\
u^{\mathrm{opt}} &= Kz^{\mathrm{opt}}(t+T), & t &\in [0,\infty),\\
z^{\mathrm{opt}}(t) &= S^K(t-T)z(T), & t &\in [T,\infty),\\
z^{\mathrm{opt}}(t+T) &= S^K(T)z^{\mathrm{opt}}(t), & t &\in [T,\infty),
\end{aligned}$$

where $z(T)$ is given by (36).

By rewriting the optimal feedback operator $K_\Phi$ in terms of the original data $z_0$ and $\psi$ in (29) and calling the rewritten operator $K_\Psi$ we get

$$K_\Psi \begin{pmatrix} z_0 \\ \psi \end{pmatrix} = K_\Phi \begin{pmatrix} f \\ g \end{pmatrix} = Kz(T), \tag{40}$$

or, equivalently,

$$K_\Psi \begin{pmatrix} z_0 \\ \psi \end{pmatrix} = K\bigg(S(T)z_0 + \int_0^T S(s)B\psi(-s)\,ds\bigg). \tag{41}$$

Evidently, the operator $K_\Psi$ in (41) is bounded on the state space $H_\Psi = H \times L^2((-T,0);\mathbf{R}^m)$ of $\Psi$.

By replacing the open loop control $u$ in (30) by the closed loop control

$$\begin{aligned}
u(t) &= K_\Psi \begin{pmatrix} z(t) \\ u_t \end{pmatrix} + v(t) = Kz(t+T) + v(t)\\
&= KS(T)z(t) + K\int_0^T S(s)Bu(t-s)\,ds + v(t),
\end{aligned}$$

we get a closed loop system described by the delay-integrodifferential equation

$$\begin{aligned}
z'(t) &= Az(t) + Bu(t-T),\\
y(t) &= Cz(t) + Du(t-T),\\
u(t) &= KS(T)z(t) + K\int_0^T S(s)Bu(t-s)\,ds + v(t), \quad t \in [0,\infty),\\
z(0) &= z_0,\\
u(t) &= \psi(t), \quad t \in [-T,0),
\end{aligned} \tag{42}$$

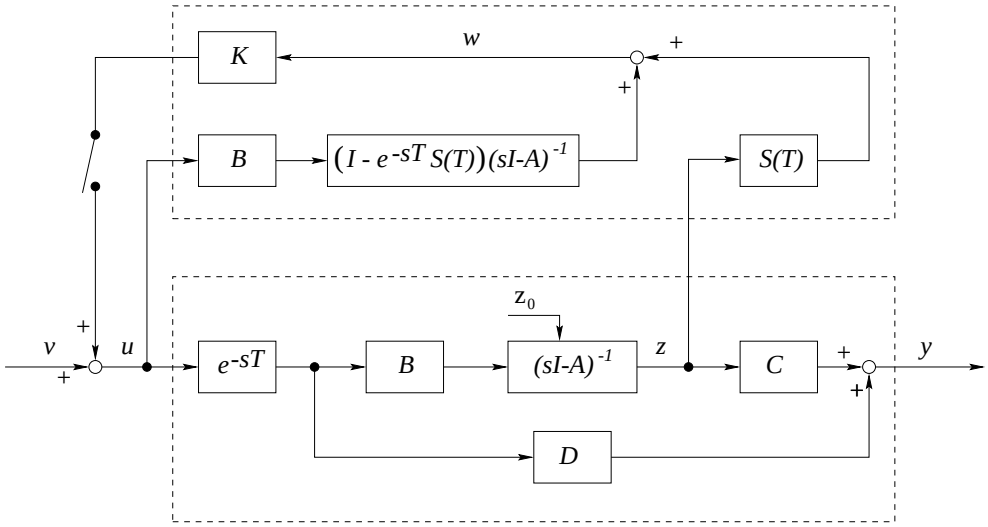


Figure 2: Optimal compensator for system $\Psi$

The action of the closed loop generator is formally the same as the action of the open loop generator, i.e.,

$$A_\Psi^K \begin{pmatrix} z_0 \\ \psi \end{pmatrix} = \begin{pmatrix} Az_0 + B\psi(-T) \\ \psi' \end{pmatrix},$$

but it has a different domain, namely

$$\mathrm{dom}(A_\Psi^K) = \left\{ \begin{pmatrix} z_0 \\ \psi \end{pmatrix} \in \mathrm{dom}(A) \times W^{1,2}([-T,0];\mathbf{R}^m) \;\middle|\; \psi(0) = K_\Psi \begin{pmatrix} z_0 \\ \psi \end{pmatrix} \right\}. \tag{43}$$

The easiest way to describe the optimal cost $\Pi_\Psi$ is to write it in the form

$$\left\langle \begin{pmatrix} \psi \\ z_0 \end{pmatrix}, \Pi_\Psi \begin{pmatrix} \psi \\ z_0 \end{pmatrix} \right\rangle = \int_0^T \|y(t)\|^2 \, dt + \langle z(T), \Pi z(T) \rangle,$$

where $y(t) = Cz(t) + D\psi(t-T)$ and $z(T)$ is given by (36), and where $\Pi$ is the Riccati operator for the non-delayed system (1) with cost function (2).

Figure 2 describes the optimal feedback system. This figure should be interpreted as follows. For $t < 0$ we choose $v(t) = \psi(t)$ and keep the feedback switch open. This initializes the delay line in the correct way, and also provides the feedback with the needed information about the initial function $\psi$ of $u$. At time $t = 0$ the system $\Sigma$ that is part of $\Psi$ is initialized to the value $z_0$, and the feedback switch is closed, activating the feedback.

The optimal feedback operator has a very interesting separation structure as indicated in Figure 2: It is a composition of two operators, namely the operator which maps the present state $\binom{z(t)}{u_t}$ into $w(t) = S(T)z(t) + \int_{t-T}^t S(t-s)Bu(s)\,ds$, and the optimal feedback $K$ for the non-delayed system (1). The signal $w$ is simply a prediction of the future state $z(t+T)$, and $K$ is applied to this predicted state. As a matter

of fact, this prediction is exact! To see this, it suffices to rewrite (42) in the form

$$\begin{aligned} w'(t) &= Aw(t) + Bu(t), \\ y(t+T) &= Cw(t) + Du(t), \\ u(t) &= Kw(t) + v(t), \quad t \in [0,\infty), \\ w(0) &= z(T), \end{aligned} \tag{44}$$

where $w(t) = z(t+T) = S(T)z(t) + \int_{t-T}^{t} S(t-s)Bu(s)\,ds$. This has the form of a closed loop system with an output delay. In particular, this means that the closed loop impulse response from $v$ to $y$ is simply a delayed version of the optimal closed loop impulse response for the non-delayed case (1). We remark that this is the same optimal system that we get if we replace the input delay in (31) by an output delay, except for the fact that in the latter case $z(T)$ is the given initial value of $w$, instead of being derived from the initial data through (36).

As the discussion above shows, the solution is "robust" with respect to the delay $T$ in the sense that the norm of the optimal impulse response from $v$ to $y$ is independent of the time delay (the extra delay in the optimal impulse response does not affect its norm). However, this is not the whole truth. If $\Sigma$ is unstable and $T$ large, then the norm of the feedback operator $K_\Psi$ can be very large, making it difficult to approximate this operator with a finite rank operator. Moverover, in this case the initialization phase in Figure 2 can be quite critical, too, since there is no control on the growth of $y$ and $z$ during initialization. This is reflected in the fact that the the "uncontrollable" part of the optimal cost caused by the delay (the $L^2$-norm of $y$ over $(0,T)$ and a possible large value of $\|z(T)\|$) can be large.

There is a simplification to Figure 2 which suggests itself: the transfer function from $Bu$ to $w - S(T)z$ is the product of $(I - \mathrm{e}^{-sT}S(T))$ and $(sI - A)^{-1}$ (in arbitrary order). However, since this is the Laplace transform of a function with compact support, it is analytic in the whole plane, and this means that the product contains a zero-pole cancelation at every spectral point of $A$. These zero-pole cancelation can be tolerated if the original system is stable, but in the unstable case a separation of the two factors will lead to a closed loop system which contains either uncontrollable or unobservable unstable modes (depending on the order of the factors). Thus, in this case the feedback must be implemented as a numerical approximation of the integral in (41).

# 4 Example: a transmission line

Our second example is the same example that Grabowski presents in [10]. It is a controlled $RLCG$ transmission line of length one without distortion, i.e., $R/L = G/C$, driven by a control voltage $u$ at one end, and loaded by a resistance $R_1$ at the opposite end.[6] If we let $i(x,t)$ represent the current and $v(x,t)$ the voltage of the line at the

[6] The following derivation of the state equations is essentially the same as in [10]. When comparing our formulae with those in [10], one should replace our notations $u$, $T$, $e(t)$, $Z$, and $\beta$ by Grabowski's $w$, $r$, $2x_2(t)$, $z$, and $1/\rho$, respectively. The physical explanation of the signal $e$ that we present is not found in [10].

point $x \in [0,1]$ at time $t \in [0,\infty)$, then these satisfy the equations

$$\begin{aligned} C\,\frac{\partial v(x,t)}{\partial t} &= -\frac{\partial i(x,t)}{\partial x} - G\,v(x,t), \\ L\,\frac{\partial i(x,t)}{\partial t} &= -\frac{\partial v(x,t)}{\partial x} - R\,i(x,t), \quad x \in [0,1]; \quad t \in [0,\infty), \\ i(1,t)R_1 &= v(1,t), \\ v(0,t) &= u(t), \\ y(t) &= v(1,t), \qquad\qquad t \in [0,\infty). \end{aligned} \tag{45}$$

The objective is to minimize the cost function $J$ in (2) with $R = 1$. The pair of d'Alembert solutions of (45) are

$$\begin{aligned} i(x,t) &= \frac{\mathrm{e}^{-\alpha t}}{2Z}\left[\phi(x - t/T) - \psi(x + t/T)\right], \\ v(x,t) &= \frac{\mathrm{e}^{-\alpha t}}{2}\left[\phi(x - t/T) + \psi(x + t/T)\right], \quad x \in [0,1]; \quad t \in [0,\infty), \end{aligned} \tag{46}$$

where $\alpha = R/L = G/C$ is the decay rate, $T = \sqrt{LC}$ is the time that it takes a wave to travel from one end of the line to the other ($1/T$ is the wave speed), and $Z = \sqrt{L/C} = \sqrt{R/G}$ is the wave impedance of the line. Observe that $\phi(s)$ is defined for $s \le 1$ and $\psi(s)$ is defined for $s \ge 0$. Substitute the boundary conditions in (45) into these equations to get

$$\begin{aligned} u(t) &= \frac{\mathrm{e}^{-\alpha t}}{2}[\phi(-t/T) + \psi(t/T)], \\ \psi(1 + t/T) &= \kappa\phi(1 - t/T), \\ y(t) &= \frac{\mathrm{e}^{-\alpha t}}{2}[\phi(1 - t/T) + \psi(1 + t/T)] \\ &= \frac{\mathrm{e}^{-\alpha t}}{2}(1 + \kappa)\phi(1 - t/T), \qquad\qquad t \ge 0 \end{aligned} \tag{47}$$

where $\kappa$ is the reflection coefficient at the output end, i.e., $\kappa = (R_1 - Z)/(R_1 + Z)$. We introduce the two new variables

$$\begin{aligned} e(t) &= \mathrm{e}^{-\alpha t}\phi(-t/T), \quad t \ge -T, \\ w(t) &= \mathrm{e}^{-\alpha t}\psi(t/T) \qquad t \ge 0. \end{aligned} \tag{48}$$

Then

$$\begin{aligned} e(t) &= v(0,t) + Zi(0,t) = u(t) + Zi(0,t), \\ w(t) &= v(0,t) - Zi(0,t) = u(t) - Zi(0,t), \qquad t \ge 0, \end{aligned} \tag{49}$$

hence

$$\begin{aligned} e(t) + w(t) &= 2u(t), \\ e(t) - w(t) &= 2Zi(0,t), \qquad t \ge 0. \end{aligned} \tag{50}$$

It follows from the first equation in (47) and from (48) that $w(t) = \kappa\beta^2 e(t - 2T)$ for $t \ge T$, where $\beta = \mathrm{e}^{-\alpha T}$ is the attenuation of the line. This identity can be

extended to the interval $0 \le t < T$ if we choose $e(t)$ in the interval $[-2T, T)$ to satisfy $\kappa e(t) = \mathrm{e}^{-\alpha t}\psi(2 + t/T)$.[7] We denote the restriction of $e$ to the interval $[-2T, 0]$ by $\epsilon$, and arrive at the delay equation

$$\begin{aligned} e(t) &= \epsilon(t), && t = [-2T, 0], \\ y(t) &= \tfrac{1}{2}(1+\kappa)\beta e(t-T), && t \in [0, \infty), \\ e(t) &= 2u(t) - w(t), && t \in [0, \infty), \\ w(t) &= \kappa\beta^2 e(t-2T), && t \in [0, \infty). \end{aligned} \tag{51}$$

This system is almost of the type (8). The initial function $\epsilon$ can be absorbed in the forcing terms $f$ and $g$, but there is an extra factor 2 in front of $u(t)$ (which means that if we write this system directly into the form (8) then the feed-through part of $\mu$ will not be one). To get rid of this factor we can, for example, define

$$x = \tfrac{1}{2}e, \qquad \varphi = \tfrac{1}{2}\epsilon, \tag{52}$$

and get the system

$$\begin{aligned} x(t) &= \varphi(t), && t = [-2T, 0), \\ y(t) &= (1+\kappa)\beta x(t-T), && t \in [0, \infty), \\ u(t) &= x(t) + \kappa\beta^2 x(t-2T), && t \in [0, \infty). \end{aligned} \tag{53}$$

This system is of the type (8) with

$$\begin{aligned} \nu &= (1+\kappa)\beta\delta_T, \\ \mu &= \delta_0 + \kappa\beta^2\delta_{2T}, \\ f &= \begin{cases} (1+\kappa)\beta\varphi(t-T), & 0 \le t < T, \\ 0, & t \ge T, \end{cases} \\ g &= \begin{cases} \kappa\beta^2\varphi(t-2T), & 0 \le t < 2T, \\ 0, & t \ge 2T, \end{cases} \end{aligned} \tag{54}$$

The system (51) is (exponentially) stable, unless $\kappa\beta^2 = \pm 1$.[8] The trivial solution for the case $\kappa = -1$ is $u = 0$, so let us exclude this case from the following discussion, and in the sequel take $\kappa > -1$. Let us immediately observe that our basic assumption about the coprimeness of $\nu$ and $\mu$ is satisfied in the stable case, due to the fact that in this case $\mu^{-1} \in M(\mathbf{R}^+; \mathbf{R})$. It is also true that $\nu$ and $\mu$ are coprime in the case $\kappa = \beta = 1$, as can be easily seen (but not in the case $\kappa = -1$ and $\beta = 1$).

The measure to be factorized is this time $\check{\nu} * \nu + \check{\mu} * \mu$. A short computation shows that

$$|\hat{\nu}(\mathrm{i}\omega)|^2 + |\hat{\mu}(\mathrm{i}\omega)|^2 = a + 2b\cos(2T\omega), \qquad \omega \in (-\infty, \infty),$$

where

$$a = \left(1 + (1+\kappa)^2\beta^2 + \kappa^2\beta^4\right), \qquad b = \kappa\beta^2.$$

[7] We get this identity for free if the original wave equation holds for all $t \ge -T$.

[8] This corresponds to the case where there is no internal damping in the line ($R = G = 0$ and $\beta = 1$) and the output end is either open ($R_1 = \infty$ and $\kappa = 1$) or shunted ($R_1 = 0$ and $\kappa = -1$).

Observe, in particular, that $a > 1 + |b|^2$, hence $a > 2|b|$. The spectral factor $\xi$ of $\check{\nu} * \nu + \check{\mu} * \mu$ can be found by direct inspection: it suffices to take

$$\xi = p\delta_0 + q\delta_{2T},$$

with $p > |q|$ (in order to guarantee the invertibility of $\xi$), and to choose the coefficients $p$ and $q$ to satisfy

$$p^2 + q^2 = a, \qquad pq = b, \qquad p > |q|.$$

This set of equations have a unique solution, namely

$$p = \tfrac{1}{2}\left(\sqrt{a+2b} + \sqrt{a-2b}\right), \qquad |q| = \tfrac{1}{2}\left(\sqrt{a+2b} - \sqrt{a-2b}\right). \tag{55}$$

We remark that both $p$ and $|q|$ are nonnegative solutions of the equation

$$p^2 + b^2/p^2 = a,$$

with $p > |q|$. For $p = 1$ the left hand side becomes $1 + b^2$, which is less that $a$. Thus,

$$p > 1, \qquad |q| < |b| = |\kappa|\beta^2 \leq 1. \tag{56}$$

There exist several realizations of the the system (51). The simplest one is the exactly controllable realization described in, e.g., [15, section 6.1] (see, in particular, the discussion of the the closed loop system (52) in that paper). The feed-through operator is independent of the realization; it is simply equal to $D_\nu$ (if we normalize $D_\mu = I$). In this case $D = D_\nu = 0$. On the other hand, $D_\xi = p$. Thus, we have found another example where $D^*D + R = 1 \neq p^2 = D_\xi^* D_\xi$.

Equation (23) rewritten in terms of the initial function $\varphi$ in (53) becomes

$$u^{\mathrm{opt}}(0+) = \left(\kappa\beta^2 - q/p\right)\varphi(-2T). \tag{57}$$

and the optimal cost is given by

$$\int_{-2}^{0} \langle \varphi(t), \Pi\varphi(t)\rangle \, dt \tag{58}$$

where

$$\left(\Pi\varphi\right)(t) = \begin{cases} (p^2-1)\varphi(t), & -T < t \leq 0, \\ q^2(p^2-1)\varphi(t), & -2T < t \leq -T. \end{cases} \tag{59}$$

We leave the verifications of (57)–(59) to the reader.

The formulae that we have developed above can be interpreted as shown in Figure 3. To get a physical interpretation of the variable $e$ it suffices to take a closer look at the first equation in (49). For $t < 0$, $e$ can be interpreted as the voltage of a signal generator with internal resistance $Z$ matching the wave impedance of the line used to transmit a signal into the line. During this stage the switch in Figure 3 is kept in its top position. Because of the matching terminating resistance at the left end of the line, there are no reflections at this end (although the signal may be reflected at the output end), and the output voltage $y$ follows the input $e$ with no distortion, but

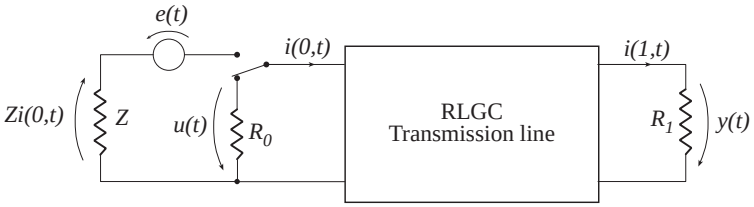


Figure 3: Optimal controller for the transmission line

with the attenuation of $\frac{1}{2}(1+\kappa)\beta$ and a time delay of $T$ time units. At time zero the signal generator is switched off, and it is desired to minimize the spill-over energy $\left(\int_0^\infty \left(\|u(t)\|^2 + \|y(t)\|^2\right)\, dt\right)^{1/2}$.

We can use (51), (52), and (57) to write the optimal control $u$ in the form

$$u(t) = \frac{p^2-1}{2p^2} w(t) = \frac{p^2-1}{2p^2}\big(u(t) - Zi(0,t)\big),$$

which gives

$$u(t) = -R_0 i(0,t), \tag{60}$$

with $R_0 = \frac{p^2-1}{p^2+1} Z$. Clearly, this is the formula for the voltage over a resistor of size $R_0$ carrying a current $i(0,t)$. Thus, the optimal controller is simply a terminating resistance of size $R_0$ at the left end, as drawn in Figure 3, where the switch is moved to its lower position at time $t = 0$. We remark that $R_0 < Z$, and that the reflection coefficient at the input end is $-1/p^2$. Note that the intuitive choice $R_0 = Z$, which leads to the extinction of the signals $u$ and $y$ in the finite time $2T$, is not the optimal one. It would have been optimal if we had not put any cost on the input voltage $u$, only on the output voltage $y$.

The optimal cost of the input signal is (cf. (59))

$$J(u^{\rm opt}) = \tfrac{1}{2}\left\{ q^2(p^2-1) \int_{-2T}^{T} \|e(t)\|^2\, dt + (p^2-1) \int_{-T}^{0} \|e(t)\|^2\, dt \right\}^{1/2}. \tag{61}$$

There is an obvious explanation for the different weights of the two time intervals $(-2T, T)$ and $(-T, 0)$: at time $t = 0$ the signal that entered the transmission line during the time interval $(-2T, T)$ has been reflected and travels to the left, whereas the signal that entered during the time interval $(-T, 0)$ is still traveling to the right. The part of the signal that entered before time $t = -2T$ is no longer present in the system, so the cost of this part is zero. If $\kappa = 0$ (hence $q = 0$), then there is no reflection at the output end of the line, and the cost of the part of $e$ that entered during the time interval $(-2T, -T)$ is zero, too.

Let us finally remark that, although the structure of the optimal solution is very simple, the formula (55) for the crucial number $p$ is nontrivial. In particular, this formula could not have been deduced from the standard continuous time Riccati equation theory.[9]

[9] However, it can be derived from the discrete time Riccati equation theory. See [16].

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