Canonical Conservative State/Signal Shift Realizations of Passive Discrete Time Behaviors

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Abstract

A passive linear discrete time invariant s/s (state/signal) system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ consists of a Hilbert (state) space $\mathcal{X}$, a Kreĭn (signal) space $\mathcal{W}$, a maximal nonnegative (generating) subspace V of the Kreĭn space $\mathfrak{K} := -\mathcal{X} [\dotplus] \mathcal{X} [\dotplus] \mathcal{W}$, and the sets of trajectories $(x(\cdot); w(\cdot))$ generated by V on the discrete time intervals $I \subset \mathbb{Z}$ that are defined by

$$(x(n+1); x(n); w(n)) \in V, \qquad n \in I.$$

This system is forward conservative, or backward conservative, or conservative if $V \subset V^{[\perp]}$, $V^{[\perp]} \subset V$, or $V^{[\perp]} = V$, respectively. The set $\mathfrak{W}_+^\Sigma$ of all signal components $w(\cdot)$ of trajectories $(x(\cdot); w(\cdot))$ of Σ on $I = \mathbb{Z}^+$ with $x(0) = 0$ and $w(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{W})$ is called the future time domain behavior of Σ. The Fourier transform $\widehat{\mathfrak{W}}_+^\Sigma$ of $\mathfrak{W}_+^\Sigma$ is called the future frequency domain behavior of Σ. This set is a maximal nonnegative right-shift invariant subspace in the Kreĭn space $K^2(\mathbb{D}; \mathcal{W})$ that as a topological vector space coincides with the usual Hardy space $H^2(\mathbb{D}; \mathcal{W})$, but has the indefinite Kreĭn space inner product inherited from $\mathcal{W}$. A subspace of $K^2(\mathbb{D}; \mathcal{W})$ with the above properties is called a *passive future frequency domain behavior on* $\mathcal{W}$. It has been shown earlier by the present authors that every passive future frequency domain behavior $\widehat{\mathfrak{W}}_+$ on $\mathcal{W}$ may be realized as the future frequency domain behavior of some passive s/s system Σ, and that it is possible to require, in addition, that Σ is (a) controllable and forward conservative, (b) observable and backward conservative, or (c) simple and conservative. These three types of realizations are determined by $\widehat{\mathfrak{W}}_+$ up to unitary similarity. Canonical functional shift realizations of the types (a) and (b) have been obtained earlier by the present authors, and their connection to the classical deBranges–Rovnyak models have been discussed. Here we present analogous results for a realization of the type (c).

Keywords

Passive, conservative, behavior, state/signal system, de Branges–Rovnyak model, input/state/output system, transfer function, scattering matrix, Krein space, fundamental decomposition.

Contents

1 Introduction

This article may be regarded as a continuation of [AS09a] and [AS09b], which in turn continued the development of a passive time-invariant linear s/s (state/signal) systems theory in discrete time that was begun in [AS05]–[AS07c]. Some further comments on the earlier history are given at the end of this introduction, and also in [AS09b].

A linear discrete time invariant s/s (state/signal) system Σ consists of a Hilbert (state) space $\mathcal{X}$, a Kreĭn (signal) space $\mathcal{W}$, and a family of trajectories $(x(\cdot), w(\cdot))$ of Σ on each discrete time interval I defined by an equation of the form

$$x(n+1) = F(x(n), w(n)), \qquad n \in I, \tag{1.1}$$

where F is a bounded linear operator from a closed domain $\mathcal{D}(F) \subset \mathcal{X}[\dotplus]\mathcal{W}$ into $\mathcal{X}$ with the extra property that for every $x \in \mathcal{X}$ there exists at least one $w \in \mathcal{W}$ such that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F)$. The three most important cases are $I = \mathbb{Z}^+ := \{0, 1, 2, \ldots\}$, $I = \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, and $I = \mathbb{Z}^- := \{-1, -2, \ldots\}$.

By a *past*, *full*, and *future* trajectory of Σ we mean a trajectory of Σ on $\mathbb{Z}^-$, $\mathbb{Z}$, and $\mathbb{Z}^+$, respectively. The extra property of F mentioned above is equivalent to the requirement that for each $x_0 \in \mathcal{X}$ and for each interval I with finite left end-point m there exists at least one trajectory $(x(\cdot), w(\cdot))$ of Σ on I satisfying $x(m) = x_0$. However, instead of working directly with equation (1.1) we shall use the graph form of (1.1), given by

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in I, \tag{1.2}$$

where V is the graph of F defined by the formula

$$V := \left\{ \begin{bmatrix} F(x,w) \\ x \\ w \end{bmatrix} \in \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F) \right\}. \tag{1.3}$$

The subspace V is called the *generating subspace* of Σ, since it generates the sets of all trajectories of Σ on the discrete intervals I by formula (1.2). We denote the system by $\Sigma = (V; \mathcal{X}, \mathcal{W})$. The properties of F listed above can be rewritten in terms of conditions on V, as was done in [AS05]. If V is a maximal nonnegative subspace of the Kreĭn *node space* $\mathfrak{K} := -\mathcal{X} [\dot{+}] \mathcal{X} [\dot{+}] \mathcal{W}$ then these conditions are satisfied, and hence every maximal nonnegative subspace V is the generating subspace of a s/s system. By a *passive* s/s system we mean a system whose generating subspace V is maximal nonnegative. In addition to passivity we shall often assume that Σ is *forward conservative*, *backward conservative*, or *conservative*, which means that $V \subset V^{[\perp]}$, $V^{[\perp]} \subset V$, or $V^{[\perp]} = V$, respectively, where $V^{[\perp]}$ is the orthogonal companion to V in $\mathfrak{K}$. Passivity implies that all trajectories $(x(\cdot), w(\cdot))$ of Σ on I satisfy

$$-\|x(n+1)\|^2_{\mathcal{X}} + \|x(n)\|^2_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}} \geq 0, \qquad n \in I, \tag{1.4}$$

and forward conservativity means that (1.4) holds in form of an equality

$$-\|x(n+1)\|^2_{\mathcal{X}} + \|x(n)\|^2_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}} = 0, \qquad n \in I. \tag{1.5}$$

See Sections 2.1–2.2 for details.

The *future behavior* $\mathfrak{W}^{\Sigma}_+$ of a passive s/s system $\Sigma = (V; \mathcal{X}; \mathcal{W})$ consists of all the signal components $w(\cdot)$ of all trajectories $(x(\cdot), w(\cdot))$ of Σ on $I = \mathbb{Z}^+$ with $x(0) = 0$ and $w(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{W})$. This set is a maximal nonnegative right-shift invariant subspace of the Kreĭn space $k^2_+(\mathcal{W})$. As a topological vector space this space coincides with the Hilbert space $\ell^2(\mathbb{Z}^+; \mathcal{W})$, but it has the indefinite Kreĭn space inner product inherited from $\mathcal{W}$. A subspace $\mathfrak{W}_+$ of $k^2_+(\mathcal{W})$ with the above properties is called a *passive future behavior on $\mathcal{W}$*.

By replacing the interval $\mathbb{Z}^+$ by either $\mathbb{Z}^-$ or $\mathbb{Z}$ we get two more behaviors induced by the passive s/s system Σ, namely the *past* behavior $\mathfrak{W}_-^\Sigma$ consisting of sequences in $k_-^2(\mathcal{W})$, and the *full* behavior $\mathfrak{W}^\Sigma$ consisting of sequences in $k^2(\mathcal{W})$, where $k_-^2(\mathcal{W})$ and $k^2(\mathcal{W})$ are topologically equal to $\ell^2(\mathbb{Z}^-;\mathcal{W})$ and $\ell^2(\mathbb{Z};\mathcal{W})$, respectively, but carry the inner products inherited from $\mathcal{W}$. In the definition of these behaviors we replace the condition $x(0) = 0$ in the definition of $\mathfrak{W}_+^\Sigma$ by the condition $x(k) \to 0$ as $k \to -\infty$. The set $\mathfrak{W}_-^\Sigma$ is a maximal nonnegative right-shift invariant subspace of $k_-^2(\mathcal{W})$. A subspace $\mathfrak{W}_-$ of $k_-^2(\mathcal{W})$ with the above properties is called a *passive past behavior on* $\mathcal{W}$. The set $\mathfrak{W}^\Sigma$ is a maximal nonnegative subspace of $k^2(\mathcal{W})$ which is bilaterally shift-invariant, and it has an extra *causality* property which will be explained in Section 2.3. A subspace $\mathfrak{W}$ of $k^2(\mathcal{W})$ with these properties is called *passive full behavior on* $\mathcal{W}$. It turns out that any two of the three behaviors $\mathfrak{W}_+^\Sigma$, $\mathfrak{W}^\Sigma$, and $\mathfrak{W}_-^\Sigma$ can be recovered from the third by formulas (2.29)–(2.31). The same formulas can be used to uniquely define any two of the above types of passive behaviors $\mathfrak{W}_-$, $\mathfrak{W}$, and $\mathfrak{W}_+$ by means of the third. See Section 2.3 for more details.

It was shown in [AS07a] that every passive future behavior $\mathfrak{W}_+$ on $\mathcal{W}$ may be realized as the future behavior of some passive s/s system Σ, and that it is possible to require, in addition, that Σ is (a) controllable and forward conservative, (b) observable and backward conservative, or (c) simple and conservative. These three types of realizations are determined by $\mathfrak{W}_+$ (or equivalently, by $\mathfrak{W}$ or $\mathfrak{W}_-$) up to unitary similarity. In [AS09b] the present authors obtained canonical functional shift realizations of the types (a) and (b), and discussed their connections to the respective two classical deBranges–Rovnyak models.

The main purpose of the present paper is to obtain analogous results for a realization of the type (c), and to further study the properties of simple conservative s/s systems.

This paper is organized as follows. In Section 2.1 we review the notion of a Kreĭn space and present some Kreĭn space results that will be needed later. Some background on passive s/s systems is presented in Section 2.2. Passive future, past, and full behaviors on a Kreĭn signal space and related results are presented in Section 2.3. Two crucial Hilbert spaces $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ and $\mathcal{H}(\mathfrak{W}_+)$ are introduced in Section 2.4, constructed with the help of the passive past behavior $\mathfrak{W}_-$ and the corresponding passive future behavior $\mathfrak{W}_+$, as well as the *past/future map* $\Gamma_\mathfrak{W}$, which is a linear contraction $\mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$ with some special properties.

The Hilbert spaces $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ and $\mathcal{H}(\mathfrak{W}_+)$ and the past/future map $\Gamma_\mathfrak{W}$ are used in Section 3 in our construction of a canonical simple conservative realization of a given passive full behavior $\mathfrak{W}$. The state space $\mathcal{D}(\mathfrak{W})$ of this realization is a certain subspace of the quotient space $k^2(\mathcal{W})/(\mathfrak{W}_-^{[\perp]} [\dotplus] \mathfrak{W}_+)$, and the the dynamics of the system is defined in terms of a left-shift applied to sequences in the equivalence class defined by the initial state.

Some new results on the dynamics of conservative s/s systems that follow from our canonical model are discussed in Section 4. The significant notion of incoming and outgoing inner channels of a simple conservative s/s system are discussed in Section 5. Alternative characterizations of the state space $\mathcal{D}(\mathfrak{W})$ of our canonical simple conservative model are developed in Section 6, and at the same time we discuss the properties of the inverse image $\mathcal{L}(\mathfrak{W})$ of $\mathcal{D}(\mathfrak{W})$ under the quotient map $k^2(\mathbb{Z};\mathcal{W}) \to k^2(\mathbb{Z};\mathcal{W})/(\mathfrak{W}_-^{[\perp]} [\dotplus] \mathfrak{W}_+)$.

The connection of the new canonical simple conservative s/s model to the controllable forward conservative s/s model and the observable backward conservative s/s model constructed in [AS09b] is explained in Sections 7 and 8. Finally, the connection between our canonical simple conservative s/s model and the simple conservative input/state/output de Branges–Rovnyak scattering model is discussed in Section 10. This model is formulated in frequency domain terms, and for this reason we explain in Section 9 how to convert the time domain results from Sections 2–8 into corresponding frequency domain results.

As we mentioned earlier, this article may be regarded as a continuation of [AS09a] and [AS09b], which in turn continued the development of a passive time-invariant linear s/s (state/signal) systems theory in discrete time that was begun in [AS05]–[AS07c]. Some preliminary steps in this direction were taken already in [BS06] by J. Ball and the second author. See, in particular, [BS06] for a discussion of the connection with the theory of passive and conservative behaviors presented in the papers [Wil72a, Wil72b, WT98, WT02a, WT02b] and the monograph [PW98]. As explained in [AS05], part of the motivation comes from classical passive time-invariant circuit theory, see, e.g., [Bel68] and [Woh69]. Continuous time passive s/s systems theory has been studied in [KS09] and [Kur10].

As we also mentioned above, as a corollary of our main result we recover the simple conservative input/state/output deBranges–Rovnyak model, that was originally presented in [dBR66a, dBR66b], and which can be found also, e.g., in [ADRdS97] and in [NV89, NV98].

In [NV89, NV98] Nikolskiĭ and Vasyunin present a “coordinate free” model of a simple conservative i/s/o scattering system whose scattering matrix coincides with a given Schur function. The philosophy behind the work of Nikolskiĭ and Vasyunin is very different from the philosophy underlying our work. The coordinate free Nikolskiĭ–Vasyunin model contains a “free” parameter Π, and by the appropriate choice of this parameter it is possible to recover all the standard simple conservative shift models whose characteristic function is equal to a given Schur function φ, including the Sz.-Nagy–Foiaş model, the deBranges–Rovnyak model, and the Pavlov model. In this sense the Nikolskiĭ–Vasyunin model is "universal". On the other hand, our canonical simple conservative s/s shift model is completely determined by a given future behavior, and in particular, it is "coordinate free" in the sense that it does not depend on some arbitrarily chosen fundamental decomposition $\mathcal{W} = -\mathcal{Y} [\dot{+}] \mathcal{U}$ of the given signal space $\mathcal{W}$. Different choices of such a decomposition give rise to different graph representations of the frequency domain version of the given future behavior as the graphs of the multiplication operators induced by different Schur functions φ (with varying input and output spaces), and the corresponding i/s/o representations of our canonical s/s model are equivalent to the i/s/o de Branges–Rovnyak realizations of φ.

On a conceptual level our construction of a simple conservative realization is vaguely reminiscent of the abstract realization theory presented in [KFA69, Part IV]. More precisely, the basic realization in [KFA69, Section 10.5] which uses the set of past input sequences factored over the kernel of the Hankel operator of the given Schur functions is analogous to our controllable forward conservative realization presented in [AS09b], whose state space is essentially the quotient of the past behavior over the kernel of the past/future map, and the realization mentioned in [KFA69, pp. 262–263] which uses the range of the kernel of the Hankel operator is analogous to our observable backward conservative realization presented in [AS09b], whose state space is essentially the range of the past/future map. However, the construction in [KFA69] is completely algebraic as opposed to our construction which also make crucial use of topological properties (in the form of various indefinite inner products derived from the energy balance equations). Moreover, whereas the construction in [KFA69] is based entirely on i/o considerations, our construction is completely i/o free.

Notations. The following standard notations are used below. $\mathbb{C}$ is the complex plane, $\mathbb{D}_+ := \{z \in \mathbb{C} \mid |z| < 1\}$, $\mathbb{D}_- := \{z \in \mathbb{C} \mid |z| > 1\} \cup \{\infty\}$, $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$, and $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$.

The space of bounded linear operators from one Kreĭn space $\mathcal{U}$ to another Kreĭn space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{U};\mathcal{Y})$. The domain, range, and kernel of a linear operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\mathcal{N}(A)$, respectively. The restriction of A to some subspace $\mathcal{Z} \subset \mathcal{D}(A)$ is denoted by $A|_{\mathcal{Z}}$. The identity operator on $\mathcal{U}$ is denoted by $1_{\mathcal{U}}$. The orthogonal projection onto a closed subspace $\mathcal{Y}$ of a Kreĭn space $\mathcal{K}$ is denoted by $P_{\mathcal{Y}}$.

The inner product in a Hilbert space $\mathcal{X}$ is denoted by $(\cdot, \cdot)_{\mathcal{X}}$, and the inner product in a Kreĭn space $\mathcal{K}$ is denoted by $[\cdot,\cdot]_{\mathcal{K}}$. The orthogonal sum of $\mathcal{U}$ and $\mathcal{Y}$ is denoted by $\mathcal{U} \oplus \mathcal{Y}$ in the case of Hilbert spaces, and by $\mathcal{U}\,[\dot{+}]\,\mathcal{Y}$ in the case of Kreĭn spaces. The anti-space $-\mathcal{K}$ of a Kreĭn space is algebraically the same space as $\mathcal{K}$, but it has a different inner product $[\cdot,\cdot]_{-\mathcal{K}} := -[\cdot,\cdot]_{\mathcal{K}}$.

We denote the orthogonal product of two Kreĭn or Hilbert spaces $\mathcal{Y}$ and $\mathcal{U}$ by $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. We sometimes identify $\left[\begin{smallmatrix}\mathcal{Y}\\ 0\end{smallmatrix}\right]$ with $\mathcal{Y}$ and $\left[\begin{smallmatrix}0\\ \mathcal{U}\end{smallmatrix}\right]$ with $\mathcal{U}$ and write $\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ instead of $\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$ in order to simplify the typesetting.

If $w(\cdot)$ is a sequence with values in a Kreĭn or Hilbert space $\mathcal{W}$ defined on $\mathbb{Z}$, then $S^{\pm 1}w$ is the sequence $w(\cdot)$ shifted one step to the right or left, respectively. For sequences $w(\cdot)$ defined on $\mathbb{Z}^+$ we define $(S_+w)(n) = w(n-1)$, $n \geq 1$, $(S_+w)(0) = 0$, and for sequences $w(\cdot)$ defined on $\mathbb{Z}^-$ we define $(S_-w)(n) = w(n-1)$, $n \in \mathbb{Z}^-$.

Some additional notations will be introduced in Sections 2 and 3.

2 Preliminary Notions and Results.

2.1 Kreĭn Spaces

Throughout this work both the signal space $\mathcal{W}$ and the node space $\mathfrak{K}$ will be a Kreĭn space. We therefore begin with a review of the most important Kreĭn space notions and results that will be needed here.

A Kreĭn space $\mathcal{W}$ is a (possibly infinite-dimensional) vector space with an inner product $[\cdot,\cdot]_{\mathcal{W}}$ that satisfies all the standard properties required by an inner product, except for the condition $[w,w]_{\mathcal{W}} \geq 0$, with strict inequality if $w \neq 0$. In addition, it is required that the space $\mathcal{W}$ can be decomposed into a direct sum $\mathcal{W} = -\mathcal{Y} \dotplus \mathcal{U}$, such that the following conditions are satisfied:

1) $\mathcal{U}$ and $-\mathcal{Y}$ are orthogonal to each other with respect to the inner product $[\cdot,\cdot]_{\mathcal{W}}$, i.e., $[y,u]_{\mathcal{W}} = 0$ for all $u \in \mathcal{U}$ and all $y \in -\mathcal{Y}$.

2) $\mathcal{U}$ is a Hilbert space with the inner product $(u,u')_{\mathcal{U}} := [u,u']_{\mathcal{W}}$, u, $u' \in \mathcal{U}$, inherited from $\mathcal{W}$.

3) $-\mathcal{Y}$ is an anti-Hilbert space with the inner product $[y,y']_{-\mathcal{Y}} := [y,y']_{\mathcal{W}}$, y, $y' \in -\mathcal{Y}$, inherited from $\mathcal{W}$.

Here and later we shall use the notation $-\mathcal{Y}$ for the *anti-space* of a vector space $\mathcal{Y}$ equipped with a (possibly indefinite) inner product. This is algebraically the same space as $\mathcal{Y}$, but the inner product $[\cdot,\cdot]_{\mathcal{Y}}$ in $\mathcal{Y}$ has been replaced by the inner product$[y,y']_{-\mathcal{Y}} := -[y,y']_{\mathcal{Y}}$, y, $y' \in -\mathcal{Y}$. The condition that $-\mathcal{Y}$ is an *anti-Hilbert space* with the inner product inherited from

$\mathcal{W}$ is equivalent to saying that $\mathcal{Y}$ is a Hilbert space with the inner product $(y, y')_{\mathcal{Y}} := -[y, y']_{\mathcal{W}},\ y,\ y' \in -\mathcal{Y}$, inherited from $-\mathcal{W}$. Since $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal to each other we shall denote the direct sum by $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$.

Any decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ with the properties listed above is called a *fundamental decomposition* of $\mathcal{W}$. If the space $\mathcal{W}$ itself is neither a Hilbert space nor an anti-Hilbert space, then it has infinite many fundamental decompositions. If $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$, then

$$[w, w]_{\mathcal{W}} = -\|y\|_{\mathcal{Y}}^2 + \|u\|_{\mathcal{U}}^2,\ w = u + y,\ u \in \mathcal{U},\ y \in \mathcal{Y}. \tag{2.1}$$

The dimensions of the positive space $\mathcal{U}$ and the negative space $-\mathcal{Y}$ do not depend on the particular decomposition. These dimensions are called the positive and negative indices of $\mathcal{W}$, respectively, and they are denoted by $\mathrm{ind}_+\mathcal{W}$ and $\mathrm{ind}_-\mathcal{W}$.

An arbitrary choice of fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ determines a Hilbert space norm on $\mathcal{W}$ by

$$\|w\|_{\mathcal{Y}\oplus\mathcal{U}}^2 = \|y\|_{\mathcal{Y}}^2 + \|u\|_{\mathcal{U}}^2,\ w = u + y,\ u \in \mathcal{U},\ y \in \mathcal{Y}. \tag{2.2}$$

While the norm $\|\cdot\|_{\mathcal{Y}\oplus\mathcal{U}}$ itself depends on the choice of fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ for $\mathcal{W}$, all these norms are equivalent and the resulting strong and weak topologies are each independent of the choice of the fundamental decomposition. Thus, we can define topological notions, such as convergence, or closedness, with respect to any one of these norms. Any norm on $\mathcal{W}$ arising in this way from some choice of fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$ for $\mathcal{W}$ we shall call an *admissible norm* on $\mathcal{W}$, and we shall refer to the corresponding positive inner product on $\mathcal{Y} \oplus \mathcal{U}$ as an *admissible Hilbert space inner product* on $\mathcal{W}$.

A subspace $\mathcal{L}$ of $\mathcal{W}$ is *positive* if every nonzero vector $w \in \mathcal{L}$ is positive ($[w, w]_{\mathcal{W}} > 0$), it is *neutral* if every vector $w \in \mathcal{L}$ is neutral ($[w, w]_{\mathcal{W}} = 0$), and *negative* if every nonzero vector $w \in \mathcal{L}$ is negative ($[w, w]_{\mathcal{W}} < 0$). *Nonnegative* and *nonpositive* subspaces are defined in the analogous way. A nonnegative subspace which is not strictly contained in any other nonnegative subspace is called *maximal nonnegative*, and the notion of a *maximal nonpositive subspace* is defined in an analogous way. Every nonnegative subspace is contained in some maximal nonnegative subspace, and every nonpositive subspace is contained in some maximal nonpositive subspace. Maximal nonnegative or nonpositive subspaces are always closed.

The *orthogonal companion* $\mathcal{L}^{[\perp]}$ of an arbitrary subset $\mathcal{L} \subset \mathcal{W}$ with respect to the Kreĭn space inner product $[\cdot, \cdot]_{\mathcal{W}}$ consists of all vectors in $\mathcal{W}$ that are orthogonal to all vectors in $\mathcal{L}$, i.e.,

$$\mathcal{L}^{[\perp]} = \{w' \in \mathcal{W} \mid [w', w]_{\mathcal{W}} = 0 \text{ for all } w \in \mathcal{L}\}.$$

This is always a closed subspace of $\mathcal{W}$, and $\mathcal{L} = (\mathcal{L}^{[\perp]})^{[\perp]}$ if and only if $\mathcal{L}$ is a closed subspace. If $\mathcal{W}$ is a Hilbert space, then we write $\mathcal{L}^{\perp}$ instead of $\mathcal{L}^{[\perp]}$. Note that, by definition, a subspace $\mathcal{L}$ is neutral if and only if $\mathcal{L} \subset \mathcal{L}^{[\perp]}$. A stronger notion than an neutral subspace is that of a Lagrangian subspace: a subspace $\mathcal{L} \subset \mathcal{W}$ is called *Lagrangian* if $\mathcal{L} = \mathcal{L}^{[\perp]}$.

A direct sum decomposition $\mathcal{W} = \mathcal{F} \dotplus \mathcal{E}$ of $\mathcal{W}$ where both $\mathcal{F}$ and $\mathcal{E}$ are neutral is called a *Lagrangian decomposition* of $\mathcal{W}$. The subspaces $\mathcal{F}$ and $\mathcal{E}$ are automatically Lagrangian in this case. Such a decomposition exists if and only if $\mathrm{ind}_+\mathcal{W} = \mathrm{ind}_-\mathcal{W}$ (this index may be finite or infinite).

If we fix a fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$, then we may view elements of $\mathcal{W}$ as consisting of column vectors

$$w = \begin{bmatrix} y \\ y \end{bmatrix} \in \begin{bmatrix} -\mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

where we view $\mathcal{Y}$ and $\mathcal{U}$ as Hilbert spaces, and the Kreĭn space inner product on $\mathcal{W}$ is given by

$$\begin{aligned} \left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} &= \left(\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\begin{bmatrix} y' \\ u' \end{bmatrix}\right)_{\mathcal{Y}\oplus\mathcal{U}} \\ &= -(y, y')_{\mathcal{Y}} + (u, u')_{\mathcal{U}}. \end{aligned} \tag{2.3}$$

In this representation, nonnegative, neutral, nonpositive, and Lagrangian subspaces are characterized as follows.

Proposition 2.1. *Let $\mathcal{W}$ be a Kreĭn space represented in the form $\mathcal{W} = \left[\begin{smallmatrix} -\mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ with Kreĭn space inner product equal to the quadratic form $[\cdot,\cdot]_J$ induced by the operator $J = \left[\begin{smallmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$ in the Hilbert space inner product of $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ as explained above, and let $\mathcal{L}$ be a subspace of $\mathcal{W}$. Then the following claims are true:*

1) *$\mathcal{L}$ is nonnegative if and only if there is a linear Hilbert space contraction $K_+ \colon \mathcal{D}_+ \mapsto \mathcal{Y}$ from some domain $\mathcal{D}_+ \subset \mathcal{U}$ into $\mathcal{Y}$ such that*

$$\mathcal{L} = \begin{bmatrix} K_+ \\ 1_{\mathcal{U}} \end{bmatrix} \mathcal{D}_+ = \left\{ \begin{bmatrix} K_+ d_+ \\ d_+ \end{bmatrix} \;\middle|\; d_+ \in \mathcal{D}_+ \right\}. \tag{2.4}$$

$\mathcal{L}$ is maximal nonnegative if and only if, in addition, $\mathcal{D}_+ = \mathcal{U}$.

2) *$\mathcal{L}$ is nonpositive if and only if there is a linear contraction $K_- \colon \mathcal{D}_- \mapsto \mathcal{U}$ from some domain $\mathcal{D}_- \subset \mathcal{Y}$ into $\mathcal{U}$ such that*

$$\mathcal{L} = \begin{bmatrix} 1_{\mathcal{Y}} \\ K_- \end{bmatrix} \mathcal{D}_- = \left\{ \begin{bmatrix} d_- \\ K_- d_- \end{bmatrix} \;\middle|\; d_- \in \mathcal{D}_- \right\}. \tag{2.5}$$

$\mathcal{L}$ is maximal nonpositive if and only if, in addition, $\mathcal{D}_- = \mathcal{Y}$.

3) *$\mathcal{L}$ is neutral if and only if there is an isometry K_+ mapping a subspace $\mathcal{D}_+$ of $\mathcal{U}$ isometrically onto a subspace $\mathcal{D}_-$ of $\mathcal{Y}$, or equivalently, an isometry K_- mapping $\mathcal{D}_- \subset \mathcal{Y}$ isometrically onto $\mathcal{D}_+ \subset \mathcal{U}$, such that*

$$\mathcal{L} = \begin{bmatrix} K_+ \\ 1_{\mathcal{U}} \end{bmatrix} \mathcal{D}_+ = \begin{bmatrix} 1_{\mathcal{Y}} \\ K_- \end{bmatrix} \mathcal{D}_-. \tag{2.6}$$

$\mathcal{L}$ is Lagrangian if and only if, in addition, $\mathcal{D}_+ = \mathcal{U}$ and $\mathcal{D}_- = \mathcal{Y}$.

4) *$\mathcal{L}$ is maximal nonnegative if and only if $\mathcal{L}$ is closed and $\mathcal{L}^{[\perp]}$ is maximal nonpositive. More precisely, if $\mathcal{L}$ has the representation* (2.4) *with $\mathcal{D}_+ = \mathcal{U}$, then $\mathcal{L}^{[\perp]}$ has the representation*

$$\mathcal{L}^{[\perp]} = \begin{bmatrix} 1_{\mathcal{Y}} \\ K_+^* \end{bmatrix} \mathcal{Y}, \tag{2.7}$$

where K_+^* is computed with respect to the Hilbert space inner product in $\mathcal{Y}$ (instead of the anti-Hilbert space inner product in $-\mathcal{Y}$ inherited from $\mathcal{W}$).

5) *$\mathcal{L}$ is maximal nonnegative if and only if $\mathcal{L}$ is closed and nonnegative and $\mathcal{L}^{[\perp]}$ is nonpositive. In particular, $\mathcal{L}$ is Lagrangian if and only if $\mathcal{L}$ is both maximal nonnegative and maximal nonpositive.*

Proof. See [AI89, Section 1.8, pp. 48–64] or the following theorems in [Bog74]: Theorem 11.7 on p. 54, Theorems 4.2 and 4.4 on pp. 105–106, and Lemma 4.5 on p. 106. □

The fundamental decompositions that we have considered above are a special case of *orthogonal decompositions* $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of $\mathcal{W}$, where $\mathcal{Y}$ and $\mathcal{U}$ are orthogonal with respect to $[\cdot,\cdot]_{\mathcal{W}}$, and both $\mathcal{Y}$ and $\mathcal{U}$ are Kreĭn spaces with the inner products inherited from $-\mathcal{W}$ and $\mathcal{W}$, respectively. Thus, if $w = y + u$ with $y \in \mathcal{Y}$ and $u \in \mathcal{U}$, then

$$[w,w]_{\mathcal{W}} = [y,y]_{\mathcal{W}} + [u,u]_{\mathcal{W}} = -[y,y]_{\mathcal{Y}} + [u,u]_{\mathcal{U}}. \tag{2.8}$$

This orthogonal decomposition is fundamental if and only if $\mathcal{Y}$ and $\mathcal{U}$ are Hilbert spaces, i.e., if they are both nonnegative.

The next lemma will be used later to find out if certain subspaces of a Kreĭn space with a special orthogonal decomposition are maximal nonnegative, or maximal nonpositive, or Lagrangian.

Lemma 2.2. *Let $\mathcal{X}$ and $\mathcal{Z}$ be two Hilbert spaces and $\mathcal{W}$ a Kreĭn space, and let $\mathfrak{K}$ be the Kreĭn space $\mathfrak{K} = \begin{bmatrix} -\mathcal{Z} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$.*

1) *A nonnegative subspace V of $\mathfrak{K}$ is maximal nonnegative if and only if conditions (a) and (b) below hold:*

   (a) For each $x \in \mathcal{X}$ there exists some $z \in \mathcal{Z}$ and $w \in \mathcal{W}$ such that $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$;

   (b) The set of all $w \in \mathcal{W}$ for which there exists some $z \in \mathcal{Z}$ such that $\begin{bmatrix} z \\ 0 \\ w \end{bmatrix} \in V$ is maximal nonnegative in $\mathcal{W}$.

2) *A nonpositive subspace V of $\mathfrak{K}$ is maximal nonpositive if and only if conditions (c) and (d) below hold:*

   (c) For each $z \in \mathcal{Z}$ there exists some $x \in \mathcal{X}$ and $w \in \mathcal{W}$ such that $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$;

   (d) The set of all $w \in \mathcal{W}$ for which there exists some $x \in \mathcal{X}$ such that $\begin{bmatrix} 0 \\ x \\ w \end{bmatrix} \in V$ is maximal nonpositive in $\mathcal{W}$.

3) *A neutral subspace V of $\mathfrak{K}$ is Lagrangian if and only if conditions (a)–(d) above hold.*

Proof of 1). Assume first that (a) and (b) hold. Let $\mathcal{W} = -\mathcal{Y} \,[\dot{+}]\, \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. Then $-\begin{bmatrix} \mathcal{Z} \\ 0 \\ \mathcal{Y} \end{bmatrix} [\dot{+}] \begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ is a fundamental decomposition of $\mathfrak{K}$. By assertion 1) in Proposition 2.1 V has a representation

$$V = \left\{ \begin{bmatrix} K_1 \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \\ x_0 \\ K_2 \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] + u_0 \end{bmatrix} \,\middle|\, \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in D_+ \right\}, \tag{2.9}$$

where $\left[\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right]$ is a contraction defined on some subspace D_+ of $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ with values in $\left[\begin{smallmatrix} \mathcal{Z} \\ \mathcal{Y} \end{smallmatrix}\right]$. By Proposition 2.1, in order to show that V is maximal nonnegative it suffices to show that $D_+ = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$.

Let x be an arbitrary vector in $\mathcal{X}$. Then by (a), there exist $z_1 \in \mathcal{X}$ and $w_1 \in \mathcal{W}$ such that $\begin{bmatrix} z_1 \\ x \\ w_1 \end{bmatrix} \in V$. Since the set in (b) is maximal nonnegative it follows that for any $u \in \mathcal{U}$ there exist $z_2 \in \mathcal{Z}$ and $w_2 \in \mathcal{W}$ such that $P_{\mathcal{U}} w_2 = u - P_{\mathcal{U}} w_1$ and $\begin{bmatrix} z_2 \\ 0 \\ w_2 \end{bmatrix} \in V$. Since V is a subspace, also

$$\begin{bmatrix} z_1 \\ x \\ w_1 \end{bmatrix} + \begin{bmatrix} z_2 \\ 0 \\ w_2 \end{bmatrix} = \begin{bmatrix} z_1 + z_2 \\ x \\ w_1 + w_2 \end{bmatrix} \in V,$$

with $P_{\mathcal{U}}(w_1 + w_2) = u$. Thus $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in D_+$, $z_1 + z_2 = K_1 \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, and $P_{\mathcal{Y}}(w_1 + w_2) = K_2 \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$. Since $x \in \mathcal{X}$ and $u \in \mathcal{U}$ are arbitrary we find that $D_+ = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$. This proves that V is maximal nonnegative.

Conversely, suppose that V maximal nonnegative. By Proposition 2.1, V has a representation of the form (2.9) for some contraction $\left[\begin{smallmatrix} K_1 \\ K_1 \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$. Clearly this implies that (a) holds. Moreover, the set in (b) is given by $\left\{u_0 + K_2 \left[\begin{smallmatrix} 0 \\ u_0 \end{smallmatrix}\right] \,\middle|\, u_0 \in \mathcal{U}\right\}$, and by Proposition 2.1 it is maximal nonnegative. Thus also (b) holds.

Proof of 2). The proof of 2) is analogous to the proof of 1).

Proof of 3). This follows from 1) and 2) together with assertion 5) in Proposition 2.1. □

The Hilbert Space $\mathcal{H}(\mathcal{Z})$. In [AS09a] was constructed a Hilbert space $\mathcal{H}(\mathcal{Z})$, where $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space. Below we give a short review of this construction.

Let $\mathcal{Z}$ be a maximal nonnegative subspace of the Kreĭn space $\mathcal{K}$, and let $\mathcal{K}/\mathcal{Z}$ be the quotient of $\mathcal{K}$ modulo $\mathcal{Z}$. We define $\mathcal{H}(\mathcal{Z})$ by

$$\mathcal{H}(\mathcal{Z}) = \{h \in \mathcal{K}/\mathcal{Z} \mid \sup\{-[x,x]_{\mathcal{K}} \mid x \in h\} < \infty\}. \tag{2.10}$$

It turns out that $\sup\{-[x,x]_{\mathcal{K}} \mid x \in h\} \geq 0$ for all $h \in \mathcal{H}(\mathcal{Z})$, that $\mathcal{H}(\mathcal{Z})$ is a subspace of $\mathcal{K}/\mathcal{Z}$, that $\mathcal{H}(\mathcal{Z})$ is a Hilbert space with the norm

$$\|h\|_{\mathcal{H}(\mathcal{Z})} = \big(\sup\{-[x,x]_{\mathcal{K}} \mid x \in h\}\big)^{1/2}, \qquad h \in \mathcal{H}(\mathcal{Z}), \tag{2.11}$$

and that $\mathcal{H}(\mathcal{Z})$ is continuously contained in $\mathcal{X}/\mathcal{Z}$. We denote the equivalence class $h \in \mathcal{K}/\mathcal{Z}$ that contains a particular vector $x \in \mathcal{K}$ by $h = x + \mathcal{Z}$. Thus, with this notation, (2.10) and (2.11) can be rewritten in the form

$$\mathcal{H}(\mathcal{Z}) = \{x + \mathcal{Z} \in \mathcal{K}/\mathcal{Z} \mid \|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} < \infty\}, \tag{2.12}$$

$$\|x + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = \big(\sup\{-[x+z, x+z]_{\mathcal{K}} \mid z \in \mathcal{Z}\}\big), \qquad x \in \mathcal{H}(\mathcal{Z}). \tag{2.13}$$

A very important (and easily proved fact) is that if we define

$$\mathcal{H}^0(\mathcal{Z}) := \big\{z^\dagger + \mathcal{Z} \,\big|\, z^\dagger \in \mathcal{Z}^{[\perp]}\big\}, \tag{2.14}$$

then $\mathcal{H}^0(\mathcal{Z})$ is a subspace of $\mathcal{H}(\mathcal{Z})$. However, even more is true: $\mathcal{H}^0(\mathcal{Z})$ is a *dense subspace* of $\mathcal{H}(\mathcal{Z})$, and for every $z^\dagger \in \mathcal{Z}^{[\perp]}$ it is true that

$$\|z^\dagger + \mathcal{Z}\|^2_{\mathcal{H}(\mathcal{Z})} = -[z^\dagger, z^\dagger]_{\mathcal{K}}, \qquad z^\dagger \in \mathcal{Z}^{[\perp]}. \tag{2.15}$$

Furthermore, if we denote

$$\mathcal{K}(\mathcal{Z}) = \{x \in \mathcal{K} \mid x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})\}. \tag{2.16}$$

then $\mathcal{Z} + \mathcal{Z}^{[\perp]} \subset \mathcal{K}(\mathcal{Z}) \subset \overline{\mathcal{Z} + \mathcal{Z}^{[\perp]}}$ and

$$(z^\dagger + \mathcal{Z}, x + \mathcal{Z})_{\mathcal{H}(\mathcal{Z})} = -[z^\dagger, x]_{\mathcal{K}}, \quad z^\dagger \in \mathcal{Z}^{[\perp]}, \quad x \in \mathcal{K}(\mathcal{Z}). \tag{2.17}$$

See [AS09a] for more details.

Connection between $\mathcal{H}(\mathcal{Z})$ and de Branges Complementary Space. Let $A \in \mathcal{B}(\mathcal{U};\mathcal{Y})$ be a contractive operator between the Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. The de Branges complementary space $\mathcal{H}(A)$ is defined by the formulas

$$\mathcal{H}(A) = \{y \in \mathcal{Y} \mid \|y\|_{\mathcal{H}(A)} < \infty\}, \tag{2.18}$$

where

$$\|y\|_{\mathcal{H}(A)} = \sup\{\|y - Au\|^2_{\mathcal{Y}} - \|u\|^2_{\mathcal{Y}} \mid u \in \mathcal{U}\}. \tag{2.19}$$

This is a Hilbert space continuously contained in $\mathcal{Y}$. It was introduced and used in [dBR66a, dBR66b], with A replaced by the operator $\widehat{\mathfrak{D}}_+$ defined in formula (10.2), as the state space in the canonical de Branges–Rovnyak model of a scattering i/s/o observable backward passive system with a given Schur class scattering matrix Φ. We shall derive this model from our s/s model in the Section 10.

Later it was observed that $\mathcal{H}(A)$ has another alternative characterization:

$$\begin{aligned} \mathcal{H}(A) &= \mathcal{R}\big((1 - AA^*)^{1/2}\big), \\ \|y\|_{\mathcal{H}(A)} &= \|[(1 - AA^*)^{1/2}]^{[-1]}y\|_{\mathcal{Y}}, \quad y \in \mathcal{H}(A), \end{aligned} \tag{2.20}$$

where the upper index ${}^{[-1]}$ represents a pseudo-inverse, i.e., $B^{[-1]}\colon \mathcal{R}(B) \to (\mathcal{N}(B))^\perp$ is the inverse of the injective operator $B|_{(\mathcal{N}(B))^\perp} \to \mathcal{R}(B)$. The operator $(1 - AA^*)^{1/2}$ is usually called the *defect operator* of the contraction A^*. See [ADRdS97] and [Sar94] for more details.

In [AS09a] it was explained how the space $\mathcal{H}(\mathcal{Z})$ defined earlier in this section is related to the space $\mathcal{H}(A)$, where A is the contraction appearing in the graph representation

$$\mathcal{Z} = \left\{ \left[\begin{smallmatrix} Au \\ u \end{smallmatrix}\right] \,\middle|\, u \in \mathcal{U} \right\}$$

of the maximal nonnegative subspace $\mathcal{Z}$ of $\mathcal{K}$ with respect to some fundamental decomposition $\mathcal{K} = -\mathcal{Y}[\dotplus]\mathcal{U}$. The connection is the following. There exists a unitary map $T\colon \mathcal{H}(\mathcal{Z}) \to \mathcal{H}(A)$ with the property that the image of $x + \mathcal{Z} \in \mathcal{H}(\mathcal{Z})$ under T is the unique vector y in this equivalence class whose projection onto $\mathcal{U}$ is zero. Explicitly this means that

$$\begin{aligned} T\big(\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] + \mathcal{Z}\big) &= y - Au, \qquad \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{K}(\mathcal{Z}), \\ T^{-1}y &= \left[\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right] + \mathcal{Z}, \qquad y \in \mathcal{H}(A). \end{aligned} \tag{2.21}$$

The operator T maps $\mathcal{H}^0(\mathcal{Z})$ one-to-one onto the dense subspace $\mathcal{R}(1 - AA^*)$ of $\mathcal{H}(A)$. In the sequel we denote $\mathcal{H}^0(A) := \mathcal{R}(1 - AA^*)$.

2.2 Passive and Conservative State/Signal Systems

A s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called *forward passive* if the inequality (1.4) holds for every interval I and every trajectory $\begin{bmatrix} x(\cdot) \\ w(\cdot) \end{bmatrix}$ of Σ on I. This is equivalent to the requirement that V is a nonnegative subspace of the node space $\mathfrak{K}$. We call Σ *forward conservative* if the equality (1.5) holds. This stronger notion is equivalent to the condition that V is neutral, i.e., $V \subset V^{[\perp]}$.

The notions of a passive and a conservative s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ depend on the notion of the adjoint s/s system $\Sigma_* = (V_*; \mathcal{X}, -\mathcal{W})$. This system has same state space $\mathcal{X}$ as Σ, its signal space is $-\mathcal{W}$, its node space is $\mathfrak{K}_* = -\mathcal{X} \,[\dot{+}]\, \mathcal{X} \,[\dot{+}]\, -\mathcal{W}$, and its generating subspace V_* is defined by

$$V_* := \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \mathcal{I} \end{bmatrix} V^{[\perp]}, \tag{2.22}$$

where $\mathcal{I}$ is the identity operator acting from $\mathcal{W}$ to $-\mathcal{W}$.

Proposition 2.3 ([AS07a, Proposition 4.6]). *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s system with the adjoint $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$.*

1) *A sequence $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$ if and only if*

$$-(x(n+1), x_*(0))_{\mathcal{X}} + (x(0), x_*(n+1))_{\mathcal{X}} + \sum_{k=0}^{n} \langle w(k), \mathcal{I}^{-1} w_*(n-k) \rangle_{\mathcal{W}} = 0 \tag{2.23}$$

for all trajectories of Σ_* on $\mathbb{Z}^+$ and all $n \in \mathbb{Z}^+$.

2) *A sequence $(x_*(\cdot), w_*(\cdot))$ is a trajectory of Σ_* on $\mathbb{Z}^+$ if and only if* (2.23) *holds for all trajectories of Σ on $\mathbb{Z}^+$ and all $n \in \mathbb{Z}^+$.*

From this proposition follows that $(\Sigma_*)_* = \Sigma$.

By a *backward conservative* s/s system Σ we mean a system whose dual system Σ_* is forward conservative. A s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is called *passive* if both Σ and the dual system Σ_* are forward passive, and it is called *conservative* if both Σ and the dual system Σ_* are forward conservative, or in other words, Σ is both forward and backward conservative. The dual node space $\mathfrak{K}_*$ and the dual generating subspace V_* have been defined in such a way that V_* is nonnegative in $\mathfrak{K}_*$ if and only if $V^{[\perp]}$ is nonpositive in $\mathfrak{K}$, and therefore, by Proposition 2.1, Σ is passive if and only if V is a maximal nonnegative subspace of $\mathfrak{K}$. Likewise, V_* is a neutral subspace of $\mathfrak{K}_*$, i.e., $V_*^{[\perp]} \subset V_*$, if and only if $V^{[\perp]} \subset V$, and hence Σ is conservative if and only if $V = V^{[\perp]}$, i.e., V is a Lagrangian subspace of $\mathfrak{K}$.

A trajectory $(x(\cdot), w(\cdot))$ defined on some interval I with a finite left end-point m is called *externally generated* on I if $x(m) = 0$. In the case where the left end-point of I is $-\infty$ we replace this condition by $\lim_{k\to-\infty} x(k) = 0$.

By (1.4), if Σ is passive, and if $\left[\begin{smallmatrix} x(\cdot) \\ w(\cdot) \end{smallmatrix}\right]$ is an externally generated trajectory of Σ on some interval I with left end-point $m \geq -\infty$, then

$$\|x(n+1)\|_{\mathcal{X}}^2 \leq \sum_{k=m}^{n} [w(k), w(k)]_{\mathcal{W}}, \qquad n \in I. \tag{2.24}$$

If $w(\cdot) \in \ell^2(I; \mathcal{W})$ with respect to some admissible norm in $\mathcal{W}$, then the sum $\sum_{k=m}^{n} [w(k), w(k)]_{\mathcal{W}}$ has an upper bound independent of n, and it follows from (2.24) that the sequence $x(\cdot)$ is bounded, i.e., $x(\cdot) \in \ell^\infty(I; \mathcal{X})$. We call a trajectory $(x(\cdot), w(\cdot))$ of Σ on some interval I *stable* if $w(\cdot) \in \ell^2(I; \mathcal{W})$ and $x(\cdot) \in \ell^\infty(I; \mathcal{X})$. Thus, externally generated trajectories of a passive s/s system are stable on I whenever the signal part belongs to $\ell^2(I; \mathcal{W})$.

Every passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is well-posed in the forward time direction in the following sense:

1) For every $x_0 \in \mathcal{X}$ there exists $x_1 \in \mathcal{X}$ and $w_0 \in \mathcal{W}$ such that $\left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$;

2) For every $\left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ there exists a stable future trajectory $(x(\cdot), w(\cdot))$ of Σ satisfying $x(0) = x_0$, $x(1) = x_1$, and $w(0) = w_0$;

see, e.g., [AS07a, Proposition 5.12] and [AS09b, Lemma 2.3, assertion 7)]. If Σ is conservative then it is also well-posed in the backward time direction in the sense that the following two conditions hold:

3) For every $x_0 \in \mathcal{X}$ there exists $x_{-1} \in \mathcal{X}$ and $w_{-1} \in \mathcal{W}$ such that $\left[\begin{smallmatrix} x_0 \\ x_{-1} \\ w_{-1} \end{smallmatrix}\right] \in V$;

4) For every $\left[\begin{smallmatrix} x_0 \\ x_{-1} \\ w_{-1} \end{smallmatrix}\right] \in V$ there exists a stable past trajectory $(x(\cdot), w(\cdot))$ of Σ satisfying $x(0) = x_0$, $x(-1) = x_{-1}$, and $w(-1) = w_{-1}$;

this follows from [AS09b, Lemma 3.1] and the fact that the adjoint system Σ^* is well-posed in the forward time direction.

The subspace of $\mathcal{X}$ that we get by taking the closure in $\mathcal{X}$ of all states $x(n)$ that appear in externally generated trajectories $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+$ is called the (approximately) *reachable subspace*, and we denote it by $\mathfrak{R}_\Sigma$. If $\mathfrak{R}_\Sigma = \mathcal{X}$, then Σ is called *controllable*. The subspace of all $x_0 \in \mathcal{X}$ with the property that there exists some trajectory $(x(\cdot), w(\cdot))$ of Σ on $\mathbb{Z}^+$ with $x(0) =$

x_0 for which $w(\cdot)$ vanishes identically is called the *unobservable subspace*, and it is denoted by $\mathfrak{U}_\Sigma$. If $\mathfrak{U}_\Sigma = \{0\}$, then Σ is called (approximately) *observable*. A s/s system Σ is called *simple* if $\mathcal{X} = \overline{\mathfrak{R}_\Sigma + \mathfrak{U}_\Sigma^\perp}$, or equivalently, if $\mathfrak{U}_\Sigma \cap \mathfrak{R}_\Sigma^\perp = \{0\}$, and it is *minimal* if it is both controllable and observable.

Throughout the rest of this paper all s/s systems that we shall consider will be assumed to be passive. The main object of study in this paper is the subclass of simple conservative s/s systems.

2.3 Future, Past, and Full Behaviors

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system. By the (stable) *behavior of* Σ on the discrete time interval $I \subset \mathbb{Z}$ we mean the set of all the signal parts $w(\cdot)$ of all stable externally generated trajectories of Σ on I. We denote this set by $\mathfrak{W}^\Sigma(I)$, and introduce the abbreviations

$$\mathfrak{W}_-^\Sigma := \mathfrak{W}^\Sigma(\mathbb{Z}^-),\ \mathfrak{W}^\Sigma := \mathfrak{W}^\Sigma(\mathbb{Z}),\ \mathfrak{W}_+^\Sigma := \mathfrak{W}^\Sigma(\mathbb{Z}^+). \tag{2.25}$$

These three behaviors $\mathfrak{W}_-^\Sigma$, $\mathfrak{W}^\Sigma$, and $\mathfrak{W}_+^\Sigma$ are called the *past behavior*, the *full behavior*, and the *future behavior* of Σ, respectively. These are the signal parts of all stable externally generated past, full, and future trajectories of Σ, respectively.

By $k^2(I; \mathcal{W})$ we denote the Kreĭn space that coincides with $\ell^2(I, \mathcal{W})$ as a topological vector space, and is equipped with the indefinite inner product

$$[w_1(\cdot), w_2(\cdot)]_{k^2(I;\mathcal{W})} = \sum_{k \in I} [w_1(k), w_2(k)]_\mathcal{W}, \tag{2.26}$$

and introduce the abbreviations

$$k_-^2(\mathcal{W}) := k^2(\mathbb{Z}^-; \mathcal{W}),\ k^2(\mathcal{W}) := k^2(\mathbb{Z}; \mathcal{W}),\ k_+^2(\mathcal{W}) := k^2(\mathbb{Z}^+; \mathcal{W}). \tag{2.27}$$

If $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$, then $k^2(I, \mathcal{W}) = -\ell^2(I, \mathcal{Y}) \,[\dotplus]\, \ell^2(I, \mathcal{U})$ is a fundamental decomposition of $k^2(I, \mathcal{W})$.

It follows from (2.24) that $\mathfrak{W}^\Sigma(I)$ is a nonnegative subspace of $k^2(I; \mathcal{W})$ for all intervals I. Actually, the *maximal* nonnegativity of V in $\mathfrak{K}$ implies that these subspaces are even *maximal* nonnegative. This was proved in the cases of $\mathfrak{W}_\pm^\Sigma$ and $\mathfrak{W}^\Sigma$ in [AS09b, Theorem 2.8], and the proof for a general interval I is similar.

Apart from being maximal nonnegative the three behaviors $\mathfrak{W}_\pm^\Sigma$ and $\mathfrak{W}^\Sigma$ are *shift-invariant* in the following sense. We denote the right-shift operator on $k_\pm^2(\mathcal{W})$ by $S_\pm$ and the right-shift operator on $k^2(\mathcal{W})$ by S. It is easy to see that $\mathfrak{W}_\pm^\Sigma$ are $S_\pm$-invariant, and that $\mathfrak{W}^\Sigma$ is S-reducing ($S\mathfrak{W}^\Sigma = \mathfrak{W}^\Sigma$). In addition $\mathfrak{W}^\Sigma_\pm$ can be recovered from $\mathfrak{W}^\Sigma$ in the following way. We let $\pi_\pm$ be the orthogonal projection of $k^2(\mathcal{W})$ onto $k^2_\pm(\mathcal{W})$. Then

$$\mathfrak{W}^\Sigma_- = \pi_- \mathfrak{W}^\Sigma, \qquad \mathfrak{W}^\Sigma_+ = \mathfrak{W}^\Sigma \cap k^2_+(\mathcal{W}). \tag{2.28}$$

It is also possible to recover $\mathfrak{W}^\Sigma$ from $\mathfrak{W}^\Sigma_-$ and from $\mathfrak{W}^\Sigma_+$ as described in Proposition 2.5 below.

The above facts motivate the following definition.

Definition 2.4. Let $\mathcal{W}$ be a Kreĭn space.

1) A maximal nonnegative S_--invariant subspace of $k^2_-(\mathcal{W})$ is called a *passive past behavior* on the (signal) space $\mathcal{W}$.

2) A maximal nonnegative S_+-invariant subspace of $k^2_+(\mathcal{W})$ is called a *passive future behavior* on the Kreĭn (signal) space $\mathcal{W}$.

3) A maximal nonnegative S-reducing subspace $\mathfrak{W}$ of $k^2(\mathcal{W})$ is *causal* if $\mathfrak{W}_- := \pi_-\mathfrak{W}$ and $\mathfrak{W}_+ := \mathfrak{W} \cap k^2_+(\mathcal{W})$ are maximal nonnegative subspaces of $k^2_-(\mathcal{W})$ and $k^2_+(\mathcal{W}$, respectively.

4) A maximal nonnegative S-reducing causal subspace of $k^2(\mathcal{W})$ is called a *passive full behavior* on the (signal) space $\mathcal{W}$.

As the following proposition shows, the two additional conditions required of $\mathfrak{W}$ in the above definition of causality are equivalent.

Proposition 2.5 ([AS09b, Theorem 2.11])**.** *Let $\mathcal{W}$ be a Kreĭn space.*

1) *If $\mathfrak{W}$ is a maximal nonnegative S-reducing subspace of $k^2(\mathcal{W})$, and if we define $\mathfrak{W}_-$ and $\mathfrak{W}_+$ by*

$$\mathfrak{W}_- := \pi_-\mathfrak{W}, \qquad \mathfrak{W}_+ := \mathfrak{W} \cap k^2_+(\mathcal{W}), \tag{2.29}$$

then $\mathfrak{W}_-$ is a passive past behavior if and only if $\mathfrak{W}_+$ is a passive future behavior (and in this case $\mathfrak{W}$ is a passive full behavior). Moreover, $\mathfrak{W}$ can be recovered from $\mathfrak{W}_+$ and from $\mathfrak{W}_-$ by the formulas

$$\mathfrak{W} = \bigcap_{n\in\mathbb{Z}^+} \{w(\cdot) \in k^2(\mathcal{W}) \mid \pi_- S^{-n} w \in \mathfrak{W}_-\}, \tag{2.30}$$

$$\mathfrak{W} = \bigvee_{n\in\mathbb{Z}^+} S^{-n}\mathfrak{W}_+. \tag{2.31}$$

2) *If $\mathfrak{W}_-$ is a passive past behavior on $\mathcal{W}$, and if we define $\mathfrak{W}$ by (2.30), then $\mathfrak{W}$ is a passive full behavior on $\mathcal{W}$ and $\mathfrak{W}_- = \pi_-\mathfrak{W}$.*

3) *If* $\mathfrak{W}_+$ *is a passive future behavior on* $\mathcal{W}$*, and if we define* $\mathfrak{W}$ *by* (2.31), *then* $\mathfrak{W}$ *is a passive full behavior on* $\mathcal{W}$ *and* $\mathfrak{W}_+ = \mathfrak{W} \cap k^2_+(\mathcal{W})$.

This proposition combined with our earlier results on the behaviors induced by a passive s/s systems imply the following result.

Proposition 2.6 ([AS09b, Theorem 2.8])**.** *Let* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *be a passive s/s system. Then the past, full, and future behaviors of* Σ *are passive past, full, and future behaviors, respectively, in the sense of Definition 2.4. Each one of these behaviors determine the two others uniquely through formulas* (2.29)*–*(2.31)*.*

This proposition has the following converse.

Proposition 2.7. *Let* $\mathcal{W}$ *be a Kreĭn space, and let* $\mathfrak{W}_-$*,* $\mathfrak{W}$*, and* $\mathfrak{W}_+$ *be past, full, and future behaviors on* $\mathcal{W}$ *connected to each other by equations* (2.29)*–*(2.31)*. Then there exists a passive s/s system* $\Sigma = (V; \mathcal{X}, \mathcal{W})$ *whose past, full, and future behaviors are equal to* $\mathfrak{W}_-$*,* $\mathfrak{W}$*, and* $\mathfrak{W}_+$*, respectively. Moreover, it is possible to require, in addition, that* Σ *is (a) controllable and forward conservative, (b) observable and backward conservative, or (c) simple and conservative. These three types of realizations are defined uniquely by the given behaviors up to unitary similarity.*

Proof. This follows from [AS09b, Theorem 1.1] and Propositions 2.5 and 2.6. □

Two canonical shift models of the type (a) and (b) were originally found in [AS09b], and they will be presented in Section 2.4.

Graph Representations of Passive Behaviors. Let $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$. Then $k^2(\mathcal{W}) = -\ell^2(\mathcal{Y}) [\dotplus] \ell^2(\mathcal{U})$ and $k^2_\pm(\mathcal{W}) = -\ell^2_\pm(\mathcal{Y}) [\dotplus] \ell^2_\pm(\mathcal{U})$ are fundamental decompositions of of the Kreĭn spaces $k^2(\mathcal{W})$ and $k^2_\pm(\mathcal{W})$, respectively. By assertion 1) and 4) of Proposition 2.1, every passive past, full, and future behavior $\mathfrak{W}_-$, $\mathfrak{W}$, and $\mathfrak{W}_+$ on $\mathcal{W}$ and their orthogonal companions have a graph representation with respect to the above fundamental decompositions of the type

$$\mathfrak{W}_\pm = \left\{ \left[\begin{smallmatrix} \mathfrak{D}_\pm u \\ u \end{smallmatrix}\right] \middle|\, u \in \ell^2_\pm(\mathcal{U}) \right\}, \qquad \mathfrak{W} = \left\{ \left[\begin{smallmatrix} \mathfrak{D} u \\ u \end{smallmatrix}\right] \middle|\, u \in \ell^2 \right\},$$
$$\mathfrak{W}_\pm^{[\perp]} = \left\{ \left[\begin{smallmatrix} y \\ \mathfrak{D}_\pm^* y \end{smallmatrix}\right] \middle|\, y \in \ell^2_\pm(\mathcal{Y}) \right\}, \qquad \mathfrak{W}^{[\perp]} = \left\{ \left[\begin{smallmatrix} y \\ \mathfrak{D}^* y \end{smallmatrix}\right] \middle|\, y \in \ell^2(\mathcal{Y}) \right\},$$

where $\mathfrak{D}_\pm$ and $\mathfrak{D}$ are linear contractions between the respective ℓ^2-spaces. Since $S_\pm \mathfrak{W}_\pm$ and $S\mathfrak{W} = \mathfrak{W}$ we have

$$S_\pm \mathfrak{D}_\pm \subset \mathfrak{D}_\pm S_\pm \text{ and } S\mathfrak{D} = \mathfrak{D} S. \tag{2.32}$$

Furthermore, if $\mathfrak{W}_\pm$ and $\mathfrak{W}$ are related to each other by the relations (2.29)–(2.31), then

$$\mathfrak{D}_+ = \mathfrak{D}|_{\ell^2_+(\mathcal{U})},\ \mathfrak{D}_- = \pi_-\mathfrak{D}|_{\ell^2_-(\mathcal{U})},\ \mathfrak{D}_+^* = \pi_+\mathfrak{D}^*|_{\ell^2_+(\mathcal{U})},\ \mathfrak{D}_-^* = \mathfrak{D}^*|_{\ell^2_-(\mathcal{U})}. \quad (2.33)$$

From (2.32) and (2.33) follow that $\mathfrak{D}_\pm$ and $\mathfrak{D}$ are convolution operators of the type

$$\begin{aligned}
(\mathfrak{D}_+u_+)(n) &= \sum_{k=0}^{n} D(n-k)u(k), && u_+ \in \ell^2_+(\mathcal{U}),\ \ n \in \mathbb{Z}^+,\\
(\mathfrak{D}_-u_-)(n) &= \sum_{k=-\infty}^{n} D(n-k)u_-(k), && u_- \in \ell^2_-(\mathcal{U}),\ \ n \in \mathbb{Z}^-,\\
(\mathfrak{D}u)(n) &= \sum_{k=-\infty}^{n} D(n-k)u(k), && u \in \ell^2(\mathcal{U}),\ \ n \in \mathbb{Z},
\end{aligned} \quad (2.34)$$

with the same sequence $\{D(k)\}_{k=0}^{\infty}$ of operators in $\mathcal{B}(\mathcal{U};\mathcal{Y})$ in the three formulas above. The contractivity of $\mathfrak{D}_+$ implies, in particular, that $D(0)$ is a contraction. The adjoint of these causal convolution operators are the anti-causal convolutions operators

$$\begin{aligned}
(\mathfrak{D}_+^*y_+)(n) &= \sum_{k=n}^{\infty} D^*(n-k)y(k), && u_+ \in \ell^2_+(\mathcal{Y}),\ \ n \in \mathbb{Z}^+,\\
(\mathfrak{D}_-^*y_-)(n) &= \sum_{k=n}^{-1} D^*(n-k)y_-(k), && u_- \in \ell^2_-(\mathcal{Y}),\ \ n \in \mathbb{Z}^-,\\
(\mathfrak{D}^*y)(n) &= \sum_{k=n}^{\infty} D^*(n-k)y(k), && u \in \ell^2(\mathcal{Y}),\ \ n \in \mathbb{Z}.
\end{aligned} \quad (2.35)$$

Lemma 2.8. *Let $\mathcal{W}$ be a Kreĭn space.*

1) *The zero section*

$$\mathfrak{W}_+(0) := \{w(0) \mid w \in \mathfrak{W}_+\}$$

of every passive future behavior $\mathfrak{W}_+$ on $\mathcal{W}$ is a maximal nonnegative subspace of $\mathcal{W}$. Conversely, every maximal nonnegative subspace $\mathcal{W}_0$ of $\mathcal{W}$ is the zero section of some passive future behavior on $\mathcal{W}$.

2) *The (-1)-section*

$$\mathfrak{W}_-^{[\perp]}(-1) := \{w(-1) \mid w \in \mathfrak{W}_-^{[\perp]}\}$$

of the orthogonal companion of every passive past behavior $\mathfrak{W}_-$ on $\mathcal{W}$ is a maximal nonpositive subspace of $\mathcal{W}$. Conversely, every maximal nonpositive subspace $\mathcal{W}_{-1}$ of $\mathcal{W}$ is the -1-section of the orthogonal companion of some passive past behavior $\mathfrak{W}_-$.

Proof. We only prove 1) below, and leave the analogous proof of 2) to the reader.

Let $\mathcal{W} = -\mathcal{Y} \, [\dot{+}] \, \mathcal{U}$ be a fundamental decomposition of $\mathfrak{W}$. By (2.34), $\mathfrak{W}_+(0)$ has the representation $\mathfrak{W}_+(0) = \left[\begin{smallmatrix} D(0)u_0 \\ u_0 \end{smallmatrix}\right]$, where $D(0)$ is a contraction $\mathcal{U} \to \mathcal{Y}$. By Proposition 2.1, this implies that $\mathfrak{W}_+(0)$ is maximal nonnegative.

Conversely, if $\mathcal{W}_0$ is maximal nonnegative in $\mathcal{W}$, then by Proposition 2.1, $\mathcal{W}_0$ has a graph representation $\mathcal{W}_0 = \{\left[\begin{smallmatrix} D_0 u_0 \\ u_0 \end{smallmatrix}\right] \mid u_0 \in \mathcal{U}\}$ with respect to the fundamental decomposition $\mathcal{W} = -\mathcal{Y} \, [\dot{+}] \, \mathcal{U}$ of $\mathcal{W}$, where $D_0 \in \mathcal{B}(\mathcal{U}; \mathcal{Y})$ is a contraction. It is easy to see that $\mathfrak{W}_+ := \left\{ \left[\begin{smallmatrix} D_0 u(\cdot) \\ u(\cdot) \end{smallmatrix}\right] \middle| \, u(\cdot) \in \ell^2_+(\mathcal{U}) \right\}$ is a S_+-invariant subspace of $k^2_+(\mathcal{W})$, and it follows from Proposition 2.1 that $\mathfrak{W}_+$ is maximal nonnegative since it is the graph of a contraction operator $\ell^2_+(\mathcal{U}) \to \ell^2_+(\mathcal{Y})$. Thus, $\mathcal{W}_0$ is the zero section of the passive future behavior $\mathfrak{W}_+$. □

2.4 Forward and Backward Conservative Canonical Models

In this section we shall present two special Hilbert spaces that play a central role throughout the rest of this article. Among others, they were used in [AS09b] as the state spaces of two of our canonical realizations of a passive behavior. These two spaces are special cases of the Hilbert space $\mathcal{H}(\mathcal{Z})$ described in the preceding section.

The Hilbert Space $\mathcal{H}(\mathfrak{W}_+)$ and the Backward Conservative Canonical Model. Let $\mathfrak{W}_+$ be a given passive future behavior on a Kreĭn signal space $\mathcal{W}$, i.e., $\mathfrak{W}_+$ is a maximal nonnegative S_+-invariant subspace of $k^2_+(\mathcal{W})$. We take $\mathcal{K} = k^2_+(\mathcal{W})$ and $\mathcal{Z} = \mathfrak{W}_+$ in the discussion in Section 2.1, and adapting our earlier formulas for $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}^0(\mathcal{Z})$ to this case we get the following result.

Theorem 2.9 ([AS09b, Theorem 4.1]). *Let $\mathfrak{W}_+$ be a passive future behavior on the Kreĭn space $k^2_+(\mathcal{W})$. Define*

$$\mathcal{H}(\mathfrak{W}_+) = \{h_+ \in k^2_+(\mathcal{W})/\mathfrak{W}_+ \mid \sup\{-[w_+, w_+]_{k^2_+(\mathcal{W})} \mid w_+ \in h_+\} < \infty\}, \tag{2.36}$$

and define $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_+)}$ *by*

$$\|h_+\|_{\mathcal{H}(\mathfrak{W}_+)} = \big(\sup\{-[w_+, w_+]_{k^2_+(\mathcal{W})} \mid w_+ \in h_+\}\big)^{1/2}, \qquad h_+ \in \mathcal{H}(\mathfrak{W}_+). \tag{2.37}$$

Then $\mathcal{H}(\mathfrak{W}_+)$ *is a Hilbert space with the norm* $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_+)}$ *that is continuously contained in* $k^2_+(\mathcal{W})/\mathfrak{W}_+$*. The set*

$$\mathcal{H}^0(\mathfrak{W}_+) := \big\{w^\dagger_+ + \mathfrak{W}_+ \;\big|\; w^\dagger_+ \in \mathfrak{W}^{[\perp]}_+\big\} \tag{2.38}$$

is a dense subspace of $\mathcal{H}(\mathfrak{W}_+)$*, and*

$$\|w^\dagger_+ + \mathfrak{W}_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} = -[w^\dagger_+(\cdot), w^\dagger_+(\cdot)]_{k^2_+(\mathcal{W})}, \qquad w^\dagger_+ \in \mathfrak{W}^{[\perp]}_+. \tag{2.39}$$

The set

$$\mathcal{K}(\mathfrak{W}_+) = \{w_+(\cdot) \in k^2_+(\mathcal{W}) \mid w_+(\cdot) + \mathfrak{W}_+ \in \mathcal{H}(\mathfrak{W}_+)\} \tag{2.40}$$

is a subspace of $k^2_+(\mathcal{W})$*, and*

$$\begin{gathered}(w^\dagger_+(\cdot) + \mathfrak{W}_+, w_+(\cdot) + \mathfrak{W}_+)_{\mathcal{H}(\mathfrak{W}_+)} = -[w^\dagger_+(\cdot), w_+(\cdot)]_{k^2_+(\mathcal{W})},\\ \text{if } w^\dagger_+(\cdot) \in \mathfrak{W}^{[\perp]}_+ \text{ and } w_+(\cdot) \in \mathcal{K}(\mathfrak{W}_+).\end{gathered} \tag{2.41}$$

Lemma 2.10 ([AS09b, Lemma 4.3])**.** *If* $w_+(\cdot) \in \mathcal{K}(\mathfrak{W}_+)$*, where* $\mathfrak{W}_+$ *is a passive future behavior on the Kreĭn space* $\mathcal{W}$*, then* $S^*_+ w_+ \in \mathcal{K}(\mathfrak{W}_+)$ *and*

$$\|S^*_+ w_+ + \mathfrak{W}_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} \le \|w_+ + \mathfrak{W}_+\|^2_{\mathcal{H}(\mathfrak{W}_+)} + [w_+(0), w_+(0)]_{\mathcal{W}}. \tag{2.42}$$

If $w_+(\cdot) \in \mathfrak{W}^{[\perp]}_+$*, then* $w_+(\cdot) \in \mathcal{K}(\mathfrak{W}_+)$ *and* (2.42) *holds as an equality.*

Theorem 2.11 ([AS09b, Theorem 7.1])**.** *Let* $\mathfrak{W}_+$ *be a passive future behavior on the Kreĭn space* $\mathcal{W}$*, and let*

$$V^{\mathfrak{W}_+}_{\mathrm{obc}} = \left\{ \begin{bmatrix} S^*_+ w + \mathfrak{W}_+ \\ w + \mathfrak{W}_+ \\ w(0) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathfrak{W}_+) \\ \mathcal{H}(\mathfrak{W}_+) \\ \mathcal{W} \end{bmatrix} \;\middle|\; w \in \mathcal{K}(\mathfrak{W}_+) \right\}, \tag{2.43}$$

where $\mathcal{K}(\mathfrak{W}_+)$ *is the space defined in* (2.40)*. Then* $\Sigma^{\mathfrak{W}_+}_{\mathrm{obc}} = (V^{\mathfrak{W}_+}_{\mathrm{obc}}; \mathcal{H}(\mathfrak{W}_+), \mathcal{W})$ *is a passive observable backward conservative s/s system whose future behavior is equal to* $\mathfrak{W}_+$*. Moreover,* $(x(\cdot), w(\cdot))$ *is a stable future trajectory of* $\Sigma^{\mathfrak{W}_+}_{\mathrm{obc}}$ *if and only if*

$$w \in \mathcal{K}(\mathfrak{W}_+) \text{ and } x(n) = (S^*_+)^n w + \mathfrak{W}_+,\ n \in \mathbb{Z}^+. \tag{2.44}$$

The Hilbert Space $\mathcal{H}(\mathfrak{W}_-^{\perp})$. Let $\mathfrak{W}_-$ be a given passive past behavior on a Kreĭn signal space $\mathcal{W}$, i.e., $\mathfrak{W}_-$ is a maximal nonnegative S_--invariant subspace of $k_-^2(\mathcal{W})$. Then $\mathfrak{W}_-^{[\perp]}$ is a maximal nonpositive S_-^*-invariant subspace of $k_-^2(\mathcal{W})$, and hence it can be interpreted as a maximal nonnegative S_-^*-invariant subspace of the anti-space $-k_-^2(\mathcal{W})$. This time we take $\mathcal{K} = -k_-^2(\mathcal{W})$ and $\mathcal{Z} = \mathfrak{W}_-^{[\perp]}$ in the definition of $\mathcal{H}(\mathcal{Z})$. Adapting our earlier formulas to this case we get the following result.

Theorem 2.12 ([AS09b, Theorem 4.4])**.** *Let $\mathfrak{W}_-$ be a passive past behavior on the Kreĭn space $k_-^2(\mathcal{W})$, and interpret $\mathfrak{W}_-^{[\perp]}$ as a maximal nonnegative S_-^*-invariant subspace of the anti-space $-k_-^2(\mathcal{W})$. Define*

$$\mathcal{H}(\mathfrak{W}_-^{[\perp]}) = \{h_- \in -k_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]} \mid \sup\big\{[w_-(\cdot), w_-(\cdot)]_{k_-^2(\mathcal{W})} \mid w_-(\cdot) \in h_-\big\} < \infty\}, \tag{2.45}$$

and define $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$ by

$$\|h_-\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = \sup\big\{[w_-(\cdot), w_-(\cdot)]_{k_-^2(\mathcal{W})} \mid w_-(\cdot) \in h_-\big\}. \tag{2.46}$$

Then $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ is a Hilbert space with the norm $\|\cdot\|_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})}$ that is continuously contained in $-k_-^2(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$. The set

$$\mathcal{H}^0(\mathfrak{W}_-^{[\perp]}) = \{w_-(\cdot) + \mathfrak{W}_-^{[\perp]} \mid w_-(\cdot) \in \mathfrak{W}_-\} \tag{2.47}$$

is a dense subspace of $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$, and

$$\|w_- + \mathfrak{W}_-^{[\perp]}\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} = [w_-(\cdot), w_-(\cdot)]_{k_-^2(\mathcal{W})}, \qquad w_-(\cdot) \in \mathfrak{W}_-. \tag{2.48}$$

The set

$$\mathcal{K}(\mathfrak{W}_-^{[\perp]}) = \{w_-(\cdot) \in k_-^2(\mathcal{W}) \mid w_-(\cdot) + \mathfrak{W}_-^{[\perp]} \in \mathcal{H}(\mathfrak{W}_-^{[\perp]})\} \tag{2.49}$$

is a subspace of $k_-^2(\mathcal{W})$, and

$$\begin{aligned}(w_-(\cdot) + \mathfrak{W}_-^{[\perp]}, v_-(\cdot) + \mathfrak{W}_-^{[\perp]})_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} &= [w_-(\cdot), v_-(\cdot)]_{k_-^2(\mathcal{W})}, \\ \text{if } w_-(\cdot) \in \mathfrak{W}_- \text{ and } v_-(\cdot) &\in \mathcal{K}(\mathfrak{W}_-^{[\perp]}).\end{aligned} \tag{2.50}$$

Lemma 2.13 ([AS09b, Lemma 4.6])**.** *If $w_-(\cdot) \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$, then $S_-w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ and*

$$\|S_-w_- + \mathfrak{W}_-^{[\perp]}\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \le \|w_- + \mathfrak{W}_-^{[\perp]}\|^2_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} - [w_-(-1), w_-(-1)]_{\mathcal{W}}. \tag{2.51}$$

If $w_-(\cdot) \in \mathfrak{W}_-$, then $w_-(\cdot) \in \mathcal{K}(\mathfrak{W}_-^{[\perp]})$ and (2.51) holds as an equality.

The Past/Future Map and the Forward Conservative Canonical Model. The passive controllable forward conservative canonical model developed in [AS09b, Section 8] used the past/future map of a passive full behavior, which will be defined below.

Lemma 2.14 ([AS09b, Lemma 6.1])**.** *Let $\mathfrak{W}$ be a passive full behavior on $\mathcal{W}$ with the corresponding passive past behavior $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and passive future behavior $\mathfrak{W}_+ = \mathfrak{W} \cap k^2_+(\mathcal{W})$. Then $\pi_+ w + \mathfrak{W}_+ \in \mathcal{H}(\mathfrak{W}_+)$ whenever $w \in \mathfrak{W}$, and there exists a unique contraction $\Gamma_{\mathfrak{W}} \colon \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$ satisfying*

$$\Gamma_{\mathfrak{W}}(\pi_- w + \mathfrak{W}_-^{[\perp]}) = \pi_+ w + \mathfrak{W}_+, \qquad w \in \mathfrak{W}. \tag{2.52}$$

Definition 2.15. The contraction $\Gamma_{\mathfrak{W}} \colon \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{H}(\mathfrak{W}_+)$ in Lemma 2.14 is called the *past/future map* of the full behavior $\mathfrak{W}$. If $\mathfrak{W}$ is the full behavior of a passive s/s system Σ, then we also call $\Gamma_{\mathfrak{W}}$ the past/future map of Σ and denote it by Γ_Σ.

Theorem 2.16 ([AS09b, Theorem 8.1 and 8.6])**.** *Let $\mathfrak{W}$ be a passive full behavior on the Kreĭn space $\mathcal{W}$, and let $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and $\mathfrak{W}_+ = \mathfrak{W} \cap k^2_+(\mathcal{W})$ be the corresponding passive past and future behaviors. Let*

$$\mathring{V}^{\mathfrak{W}_-}_{\rm cfc} = \left\{ \begin{bmatrix} w_- + \mathfrak{W}_-^{[\perp]} \\ S_- w_- + \mathfrak{W}_-^{[\perp]} \\ w_-(-1) \end{bmatrix} \in \begin{bmatrix} \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \\ \mathcal{W} \end{bmatrix} \middle| \, w_- \in \mathfrak{W}_- \right\}. \tag{2.53}$$

and let $V^{\mathfrak{W}_-}_{\rm cfc}$ be the closure of $\mathring{V}^{\mathfrak{W}_-}_{\rm cfc}$ in the Kreĭn space $\mathfrak{K}_- := -\mathcal{H}(\mathfrak{W}_-^{[\perp]}) \, [\dot{+}] \, \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \, [\dot{+}] \, \mathcal{W}$. Then

$$V^{\mathfrak{W}_-}_{\rm cfc} = \left\{ \begin{bmatrix} \pi_- S^{-1} w + \mathfrak{W}_-^{[\perp]} \\ \pi_- w + \mathfrak{W}_-^{[\perp]} \\ w(0) \end{bmatrix} \middle| \begin{array}{l} w = w_- + w_+, \ w_- \in \mathcal{K}(\mathfrak{W}_-^{[\perp]}), \ w_+ \in \mathcal{K}(\mathfrak{W}_+), \\ \text{and } w_+ + \mathfrak{W}_+ = \Gamma_{\mathfrak{W}}(w_- + \mathfrak{W}_-^{[\perp]}), \end{array} \right\} \tag{2.54}$$

and $\Sigma^{\mathfrak{W}_-}_{\rm cfc} = (V^{\mathfrak{W}_-}_{\rm cfc}; \mathcal{H}(\mathfrak{W}_-^{[\perp]}), \mathcal{W})$ is a passive controllable forward conservative s/s system with past behavior $\mathfrak{W}_-$ and full behavior $\mathfrak{W}$.

The Input and Output Maps of a Passive State/Signal System. In [AS09b, Section 5] the input and output maps of a passive s/s system were defined in the following way.

Lemma 2.17 ([AS09b, Lemma 5.10])**.** *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behavior $\mathfrak{W}_-$. Then there exists a unique linear contraction $\mathfrak{B}_\Sigma \colon \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \to \mathcal{X}$, called the* input map *of Σ, whose restriction to $\mathcal{H}^0(\mathfrak{W}_-^{[\perp]})$ is given by*

$$\mathfrak{B}_\Sigma(w_- + \mathfrak{W}_-^{[\perp]}) = x(0), \quad w_-(\cdot) \in \mathfrak{W}_-, \tag{2.55}$$

where $(x(\cdot), w_-(\cdot))$ is the unique stable externally generated past trajectory of Σ whose signal part is $w_-(\cdot)$.

Lemma 2.18 ([AS09b, Lemma 5.2]). *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with future behavior $\mathfrak{W}_+$. Then the formula*

$$\mathfrak{C}_\Sigma x_0 = \left\{ w_+ + \mathfrak{W}_+ \;\middle|\; \begin{array}{l} w_+(\cdot) \text{ is the signal part of some stable future} \\ \text{trajectory } (x(\cdot), w_+(\cdot)) \text{ of } \Sigma \text{ with } x(0) = x_0 \end{array} \right\} \tag{2.56}$$

defines a linear contraction $\mathfrak{C}_\Sigma \colon \mathcal{X} \to \mathcal{H}(\mathfrak{W}_+)$, called the output map of Σ.

Lemma 2.19 ([AS09b, Lemma 5.12]). *Let $\Sigma = (V; \mathcal{X}; \mathcal{W})$ be a passive s/s system with past behavior $\mathfrak{W}_-$, full behavior $\mathfrak{W}$, future behavior $\mathfrak{W}_+$, input map $\mathfrak{B}_\Sigma$, and output map $\mathfrak{C}_\Sigma$. Then $(x(\cdot), w(\cdot))$ is an externally generated stable past trajectory of Σ if and only if*

$$w \in \mathfrak{W}_- \text{ and } x(n) = \mathfrak{B}_\Sigma(S_-^{-n} w + \mathfrak{W}_-^{[\perp]}),\ n \leq 0, \tag{2.57}$$

and $(x(\cdot), w(\cdot))$ is an externally generated stable full trajectory of Σ if and only if

$$w \in \mathfrak{W} \text{ and } x(n) = \mathfrak{B}_\Sigma(\pi_- S^{-n} w + \mathfrak{W}_-^{[\perp]}),\ n \in \mathbb{Z}. \tag{2.58}$$

In the latter case we have, in addition,

$$\mathfrak{C}_\Sigma x(n) = \pi_+ S^{-n} w + \mathfrak{W}_+,\ n \in \mathbb{Z}. \tag{2.59}$$

Lemma 2.20 ([AS09b, Lemma 6.3]). *The past/future map Γ_Σ of a passive s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ factors into the product*

$$\Gamma_\Sigma = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma \tag{2.60}$$

of the input map $\mathfrak{B}_\Sigma$ and the output map $\mathfrak{C}_\Sigma$ of Σ.

Lemma 2.21 ([AS09b, Lemma 5.15]). *If Σ is a passive forward conservative s/s system, then the input map $\mathfrak{B}_\Sigma$ of Σ is an isometry with $\mathcal{R}(\mathfrak{B}_\Sigma) = \mathfrak{R}_\Sigma$.*

Lemma 2.22 ([AS09b, Lemma 5.20]). *If Σ is a passive backward conservative s/s system, then the output map $\mathfrak{C}_\Sigma$ of Σ is a co-isometry with $\mathcal{N}(\mathfrak{C}_\Sigma) = \mathfrak{U}_\Sigma$.*

The Null Controllable and Unconstructable Subspaces. As we mentioned earlier, every conservative s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is well-posed both in the forward and the backward time direction in the sense that to each $x_0 \in \mathcal{X}$ there exists a stable full trajectory of Σ with $x(0) = x_0$. Much of what we have said earlier remains true if we interchange the roles played by $\mathbb{Z}^+$ and $\mathbb{Z}^-$, provided we at the same time replace the notion of an externally generated trajectory by the notion of a *backward externally generated trajectory.* This notion is defined in the natural way: A trajectory

$(x(\cdot), w(\cdot))$ of Σ defined on an interval I with a finite right end-point n is backward externally generated if $x(n) = 0$, and if the right end-point of I is $+\infty$ then we replace the condition $x(n) = 0$ by $\lim_{k\to\infty} x(k) = 0$.

The *anti-causal* future, full, and past behaviors of Σ are the signal parts of all backward externally generated future, full, and past stable trajectories of Σ, respectively. Here the "past", "full", and "future" still refer to the same time intervals as before, i.e., "past" refers to $\mathbb{Z}^-$, "full" to $\mathbb{Z}$, and "future" to $\mathbb{Z}^+$. By [AS09b, Theorem 3.4], these behaviors are equal to $\mathfrak{W}_+^{[\perp]}$, $\mathfrak{W}^{[\perp]}$, and $\mathfrak{W}_-^{[\perp]}$, respectively, where $\mathfrak{W}_+$, $\mathfrak{W}$, and $\mathfrak{W}_-$ are the future, full, and past behaviors of Σ.

The *causal* versions of the input, output, and past/future maps $\mathfrak{B}_\Sigma$, $\mathfrak{C}_\Sigma$, and Γ_Σ defined above also have *anti-causal* counterparts $\mathfrak{B}_\Sigma^\dagger$, $\mathfrak{C}_\Sigma^\dagger$, and $\Gamma_\Sigma^\dagger$, which we obtain by the same constructions as before, but interchange the roles of $\mathbb{Z}^+$ and $\mathbb{Z}^-$, and also interchange the roles of $\mathcal{H}(\mathfrak{W}_+)$ and $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$. Thus, if $(x(\cdot), w(\cdot))$ is a backward externally generated stable future trajectory of Σ, then $x(0) = \mathfrak{B}_\Sigma^\dagger(w + \mathfrak{W}_+)$, $\mathfrak{C}_\Sigma^\dagger x(0)$ is the equivalence class of all the signal parts of stable past trajectories $(x(\cdot), w(\cdot))$ of Σ with $x(0) = x_0$, and

$$\Gamma_{\mathfrak{W}}^\dagger(\pi_+ w^\dagger + \mathfrak{W}_+) = \pi_- w^\dagger + \mathfrak{W}_-^{[\perp]}, \qquad w^\dagger \in \mathfrak{W}^{[\perp]}.$$

As shown in [AS09b, Lemma 5.19], $\mathfrak{B}_\Sigma^\dagger = \mathfrak{C}_\Sigma^*$, and $\mathfrak{C}_\Sigma^\dagger = \mathfrak{B}_\Sigma^*$, and by [AS09b, Lemma 6.8], $\Gamma_\Sigma^\dagger = \Gamma_\Sigma^*$.

Our earlier definition of the reachable and unobservable subspaces $\mathfrak{R}_\Sigma$ and $\mathfrak{U}_\Sigma$ also have a built-in direction of time. These two subspaces do not, in general, remain invariant under time reversal, and the subspaces $\mathfrak{R}_\Sigma$ and $\mathfrak{U}_\Sigma$ that we defined earlier are the *causal* versions of these subspaces. We denote the anti-causal counterparts of $\mathfrak{R}_\Sigma$ and $\mathfrak{U}_\Sigma$ by $\mathfrak{R}_\Sigma^\dagger$ and $\mathfrak{U}_\Sigma^\dagger$, respectively. Thus, $\mathfrak{R}^\dagger$ is the closure in $\mathcal{X}$ of all states $x(n)$ that appear in backward externally generated past trajectories $(x(\cdot), w(\cdot))$ of Σ, and $\mathfrak{U}^\dagger$ consists of all $x_0 \in \mathcal{X}$ with the property that there exists some past trajectory $(x(\cdot), w(\cdot))$ of Σ which with $x(0) = x_0$ for which $w(\cdot)$ vanishes identically. We shall follow the control theory tradition and call $\mathfrak{R}_\Sigma^\dagger$ the (approximately) *null controllable subspace*. The space $\mathfrak{U}_\Sigma^\dagger$ does not have an established name in control theory, and here we shall use the name *backward unobservable subspace*. By an *backward unobservable trajectory* we mean a past trajectory $(x(\cdot), w(\cdot))$ of Σ for which $w(\cdot)$ vanishes identically. A full trajectory $(x(\cdot), w(\cdot))$ whose signal part $w(\cdot)$ vanishes identically will be called a *bilaterally unobservable trajectory*. The restriction of such a trajectory to $\mathbb{Z}^+$ is unobservable, and the restriction to $\mathbb{Z}^-$ is backward unobservable. By [AS07a, Proposition 4.7], $\mathfrak{R}_\Sigma^\dagger = \mathcal{R}(\mathfrak{B}_\Sigma^*) = \mathcal{N}(\mathfrak{C}_\Sigma)^\perp = \mathfrak{U}_\Sigma^\perp$ and $\mathfrak{U}_\Sigma^\dagger = \mathcal{N}(\mathfrak{C}_\Sigma^*) = \mathcal{R}(\mathfrak{B}_\Sigma)^\perp = (\mathfrak{R}_\Sigma)^\perp$.

3 The Canonical Conservative Model.

Let $\mathcal{W}$ be a Kreĭn space, and let $\mathfrak{W}$ be a passive full behavior on $\mathcal{W}$, with corresponding past and future behaviors $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and $\mathfrak{W}_+ = \mathfrak{W} \cap k^2_+(\mathcal{W})$. To shorten the notations we define

$$\begin{aligned} \mathcal{H}_- &:= \mathcal{H}(\mathfrak{W}_-^{[\perp]}), \quad \mathcal{H}_-^0 := \mathcal{H}^0(\mathfrak{W}_-^{[\perp]}), \\ \mathcal{H}_+ &:= \mathcal{H}(\mathfrak{W}_+). \quad \mathcal{H}_+^0 := \mathcal{H}^0(\mathfrak{W}_+), \end{aligned} \tag{3.1}$$

Moreover, we denote

$$\begin{aligned} Q_-w &:= \pi_-w + \mathfrak{W}_-^{[\perp]}, \quad Q_+w := \pi_+w + \mathfrak{W}_+, \\ Qw &:= w + (\mathfrak{W}_-^{[\perp]} + \mathfrak{W}_+), \quad w \in k^2(\mathcal{W}). \end{aligned} \tag{3.2}$$

Below we shall encounter the quotient space $k^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$. Each vector in this space is an equivalence class of the type $x := w + (\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ for some $w \in k^2(\mathcal{W})$. Above we denoted the corresponding quotient map by Q, i.e., $x = Qw$. Since $k^2(\mathcal{W}) = k^2_+(\mathcal{W}) \, [\dotplus] \, k^2_-(\mathcal{W})$, and since $\mathfrak{W}_+$ is a closed subspace of $k^2_+(\mathcal{W})$ and $\mathfrak{W}_-^{[\perp]}$ is a closed subspace of $k^2_-(\mathcal{W})$, it follows that we can identify $k^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ with the product space $\left[\begin{smallmatrix} k^2_+(\mathcal{W})/\mathfrak{W}_+ \\ k^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]} \end{smallmatrix}\right]$. We denote the projections of $k^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ onto $k^2_+(\mathcal{W})/\mathfrak{W}_+$ and $k^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ by P_+ and P_-, respectively. Thus, $P_\pm$ is the operator which for each $w \in k^2(\mathcal{W})$ maps $x = Qw$ into $Q_\pm w$. Since $\mathcal{H}_+$ is continuously contained in $k^2_+(\mathcal{W})/\mathfrak{W}_+$ and $\mathcal{H}_-$ is continuously contained in $k^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$, this means that $\left[\begin{smallmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{smallmatrix}\right]$ can be interpreted as a continuously contained subspace of $k^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$.

Let

$$A_{\mathfrak{W}} := \begin{bmatrix} 1_{\mathcal{H}_+} & \Gamma_{\mathfrak{W}} \\ \Gamma_{\mathfrak{W}}^* & 1_{\mathcal{H}_-} \end{bmatrix}. \tag{3.3}$$

This is a bounded linear operator on $\mathcal{H}_+ \oplus \mathcal{H}_-$. It is nonnegative since $\Gamma_{\mathfrak{W}}$ is a contraction $\mathcal{H}_- \to \mathcal{H}_+$, and by the Schwarz inequality, for all $\left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right] \in \mathcal{H}_+ \oplus \mathcal{H}_-$,

$$\begin{aligned} \left(\begin{bmatrix} x_+ \\ x_- \end{bmatrix}, A_{\mathfrak{W}} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \right)_{\mathcal{H}_+ \oplus \mathcal{H}_-} &= \|x_+\|^2_{\mathcal{H}_+} + 2\Re(x_+, \Gamma_{\mathfrak{W}} x_-)_{\mathcal{H}_+} + \|x_-\|^2_{\mathcal{H}_-} \\ &\geq \|x_+\|^2_{\mathcal{H}_+} - 2\|x_+\|_{\mathcal{H}_+}\|x_-\|_{\mathcal{H}_-} + \|x_-\|^2_{\mathcal{H}_-} \geq 0. \end{aligned}$$

We define $\mathcal{D}(\mathfrak{W})$ to be the range of $A_{\mathfrak{W}}^{1/2}$, with the range norm, i.e.,

$$\left\| \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \right\|_{\mathcal{D}(\mathfrak{W})} = \left\| (A_{\mathfrak{W}}^{1/2})^{[-1]} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} \right\|_{\mathcal{H}_+ \oplus \mathcal{H}_-},$$

where $(A_{\mathfrak{W}}^{1/2})^{[-1]}$ is the pseudo-inverse of $A_{\mathfrak{W}}^{1/2}$, i.e., $\begin{bmatrix} x'_+ \\ x'_- \end{bmatrix} := (A_{\mathfrak{W}}^{1/2})^{[-1]} \left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right]$ is the unique vector in $\overline{\mathcal{R}\,(A_{\mathfrak{W}})}$ which satisfies $\left[\begin{smallmatrix} x_+ \\ x_- \end{smallmatrix}\right] = A_{\mathfrak{W}}^{1/2} \begin{bmatrix} x'_+ \\ x'_- \end{bmatrix}$. With respect to this inner product in the range space the operator $A_{\mathfrak{W}}^{1/2}|_{\overline{\mathcal{R}(A_{\mathfrak{W}})}}$ is a unitary operator mapping $\overline{\mathcal{R}\,(A_{\mathfrak{W}})}$ onto $\mathcal{D}(\mathfrak{W})$. In particular, $\mathcal{D}(\mathfrak{W})$ is a Hilbert space.

Lemma 3.1. *Define $A_{\mathfrak{W}}$ by* (3.3)*.*

1) *$\mathcal{R}\,(A_{\mathfrak{W}})$ is a dense subset of the Hilbert space $\mathcal{D}(\mathfrak{W})$, $\mathcal{D}(\mathfrak{W})$ is a dense subspace of $\overline{\mathcal{R}\,(A_{\mathfrak{W}})}$, and $\mathcal{D}(\mathfrak{W})$ is continuously contained in $\mathcal{H}_+ \oplus \mathcal{H}_-$.*

2) *$A_{\mathfrak{W}}$ is bounded as an operator $\mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$.*

3) *If $x \in \mathcal{D}(\mathfrak{W})$ and $y = A_{\mathfrak{W}} y'$, then $y \in \mathcal{D}(\mathfrak{W})$, and $(x, y)_{\mathcal{D}(\mathfrak{W})} = (x, y')_{\mathcal{H}_- \oplus \mathcal{H}_+}$.*

4) *$A_{\mathfrak{W}}|_{\mathcal{H}_-} = \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix}$ is an isometry $\mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$.*

5) *$A_{\mathfrak{W}}|_{\mathcal{H}_+} = \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix}$ is an isometry $\mathcal{H}_+ \to \mathcal{D}(\mathfrak{W})$.*

Proof of 1). Clearly $\mathcal{R}\,(A_{\mathfrak{W}}) \subset \mathcal{R}\left(A_{\mathfrak{W}}^{1/2}\right) = \mathcal{D}(\mathfrak{W})$. As is well-known, $\overline{\mathcal{R}\left(A_{\mathfrak{W}}^{1/2}\right)} = \overline{\mathcal{R}\,(A_{\mathfrak{W}})}$, and thus $\mathcal{D}(\mathfrak{W})$ is a dense subspace of $\overline{\mathcal{R}\,(A_{\mathfrak{W}})}$. Let $\mathcal{U}$ be the unitary map $\mathcal{U} := A_{\mathfrak{W}}^{1/2}|_{\overline{\mathcal{R}(A_{\mathfrak{W}})}} \colon \overline{\mathcal{R}\,(A_{\mathfrak{W}})} \to \mathcal{D}(\mathfrak{W})$. Since $\mathcal{D}(\mathfrak{W})$ is a dense subspace of $\overline{\mathcal{R}\,(A_{\mathfrak{W}})}$, the image of $\mathcal{D}(\mathfrak{W})$ under $\mathcal{U}$ is a dense subspace of $\mathcal{D}(\mathfrak{W})$. But this image is equal to $\mathcal{R}\,(A_{\mathfrak{W}})$. Thus, $\mathcal{R}\,(A_{\mathfrak{W}})$ is dense in $\mathcal{D}(\mathfrak{W})$.

To show that $\mathcal{D}(\mathfrak{W})$ is continuously contained in $\mathcal{H}_+ \oplus \mathcal{H}_-$ we take some $x \in \mathcal{D}(\mathfrak{W})$. Then $x = A_{\mathfrak{W}}^{1/2} y$ for some $y \in \overline{\mathcal{R}\,(A_{\mathfrak{W}})}$, and $\|x\|_{\mathcal{D}(\mathfrak{W})} = \|y\|_{\mathcal{H}_+ \oplus \mathcal{H}_-}$. Therefore

$$\begin{aligned} \|x\|^2_{\mathcal{H}_+ \oplus \mathcal{H}_-} &= \|A_{\mathfrak{W}}^{1/2} y\|^2_{\mathcal{H}_+ \oplus \mathcal{H}_-} = (y, A_{\mathfrak{W}} y)^2_{\mathcal{H}_+ \oplus \mathcal{H}_-} \le \|A_{\mathfrak{W}}\|^2 \|y\|^2_{\mathcal{H}_+ \oplus \mathcal{H}_-} \\ &= \|A_{\mathfrak{W}}\|^2 \|x\|^2_{\mathcal{D}(\mathfrak{W})}. \end{aligned}$$

This shows that $\mathcal{D}(\mathfrak{W})$ is continuously embedded in $\overline{\mathcal{R}\,(A_{\mathfrak{W}})}$, and hence continuously contained in $\mathcal{H}_+ \oplus \mathcal{H}_-$.

Proof of 2). With the same notation as in the proof of 1), $A_{\mathfrak{W}}$ factors into $A_{\mathfrak{W}} = \mathcal{U} A_{\mathfrak{W}}^{1/2}$, where $A_{\mathfrak{W}}^{1/2}$ is a bounded linear operator in $\mathcal{H}_+ \oplus \mathcal{H}_-$, and $\mathcal{U}$ is a unitary operator $\overline{\mathcal{R}\,(A_{\mathfrak{W}})} \to \mathcal{D}(\mathfrak{W})$. Thus $A_{\mathfrak{W}}$ is bounded as an operator $\mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$.

Proof of 3). If, in addition, $x = A_{\mathfrak{W}} x'$ for some $x' \in \mathcal{H}_+ \oplus \mathcal{H}_-$, then

$$(x, y)_{\mathcal{D}(\mathfrak{W})} = (A_{\mathfrak{W}} x', A_{\mathfrak{W}} y')_{\mathcal{D}(\mathfrak{W})} = (A_{\mathfrak{W}}^{1/2} x', A_{\mathfrak{W}}^{1/2} y')_{\mathcal{H}_+ \oplus \mathcal{H}_-}$$
$$= (A_{\mathfrak{W}} x', y')_{\mathcal{H}_+ \oplus \mathcal{H}_-} = (x, y')_{\mathcal{H}_+ \oplus \mathcal{H}_-}.$$

If x is an arbitrary vector in $\mathcal{D}(\mathfrak{W})$, then there exists a sequence $x_n \in \mathcal{R}(A_{\mathfrak{W}})$ such that $x_n \to x$ in $\mathcal{D}(\mathfrak{W})$ as $n \to \infty$. Since $\mathcal{D}(\mathfrak{W})$ is continuously contained in $\mathcal{H}_+ \oplus \mathcal{H}_-$, it is also true that $x_n \to x$ in $\mathcal{H}_+ \oplus \mathcal{H}_-$. Consequently

$$(x, y)_{\mathcal{D}(\mathfrak{W})} = \lim_{n \to \infty} (x_n, y)_{\mathcal{D}(\mathfrak{W})} = \lim_{n \to \infty} (x_n, y')_{\mathcal{H}_+ \oplus \mathcal{H}_-} = (x, y')_{\mathcal{H}_+ \oplus \mathcal{H}_-}.$$

Proof of 4). This follows from 3) since we have for all $x_- \in \mathcal{H}_-$,

$$\left\| \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} x_- \right\|^2_{\mathcal{D}(\mathfrak{W})} = \left(\begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} x_-, \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} x_- \right)^2_{\mathcal{D}(\mathfrak{W})} = (A_{\mathfrak{W}} x_-, A_{\mathfrak{W}} x_-)_{\mathcal{D}(\mathfrak{W})}$$
$$= (x_-, A_{\mathfrak{W}} x_-)_{\mathcal{H}_+ \oplus \mathcal{H}_-} = \left(x_-, \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} x_- \right)_{\mathcal{H}_+ \oplus \mathcal{H}_-}$$
$$= \|x_-\|^2_{\mathcal{H}_-}.$$

The proof of 5) is analogous. □

In the sequel we shall throughout *interpret* $A_{\mathfrak{W}}$ *as a bounded linear operator* $\mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$, instead of interpreting $A_{\mathfrak{W}}$ as a self-adjoint operator in $\mathcal{H}_+ \oplus \mathcal{H}_-$. In particular, in this setting *the operator* $A_{\mathfrak{W}}$ *is not self-adjoint unless* $\mathcal{D}(\mathfrak{W}) = \mathcal{H}_+ \oplus \mathcal{H}_-$*, i.e., unless* $\Gamma_{\mathfrak{W}} = 0$. When the duality in the range space is taken with respect to the inner product in $\mathcal{D}(\mathfrak{W})$ instead of the inner product in $\mathcal{H}_+ \oplus \mathcal{H}_-$ *the operator* $A^*_{\mathfrak{W}}$ *becomes a bounded linear operator* $\mathcal{D}(\mathfrak{W}) \to \mathcal{H}_+ \oplus \mathcal{H}_-$.

Recall that we denoted the projections of $k^2(\mathcal{W})/(\mathfrak{W}_+ \dotplus \mathfrak{W}_-^{[\perp]})$ onto $k^2_+(\mathcal{W})/\mathfrak{W}_+$ and $k^2_-(\mathcal{W})/\mathfrak{W}_-^{[\perp]}$ by P_+ and P_-, respectively. We denote the restrictions of $P_\pm$ to $\begin{bmatrix} \mathcal{H}_- \\ \mathcal{H}_+ \end{bmatrix}$ by $\Pi_\pm$, so that $\Pi_\pm \begin{bmatrix} x_+ \\ x_- \end{bmatrix} = x_\pm$ for all $\begin{bmatrix} x_+ \\ x_- \end{bmatrix} \in \begin{bmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{bmatrix}$.

Lemma 3.2. *Let* $A_{\mathfrak{W}}$ *be the operator defined in* (3.3)*, interpreted as bounded linear operator* $\mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$*, whose adjoint* $A^*_{\mathfrak{W}}$ *is a bounded linear operator* $\mathcal{D}(\mathfrak{W}) \to \mathcal{H}_+ \oplus \mathcal{H}_-$*.*

1) $A^*_{\mathfrak{W}}$ *is equal to the embedding operator* $\mathcal{D}(\mathfrak{W}) \hookrightarrow \begin{bmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{bmatrix}$.

2) $(A_{\mathfrak{W}}|_{\mathcal{H}_+})^* = \Pi_-|_{\mathcal{D}(\mathfrak{W})}$ *and* $(A_{\mathfrak{W}}|_{\mathcal{H}_-})^* = \Pi_+|_{\mathcal{D}(\mathfrak{W})}$. *(In the computation of these adjoints we interpret* $A_{\mathfrak{W}}|_{\mathcal{H}_\pm}$ *as operators* $\mathcal{H}_\pm \to \mathcal{D}(\mathfrak{W})$*.)*

Proof. By Part 3) of Lemma 3.1, for all $x \in \mathcal{D}(\mathfrak{W})$ and all $y' \in \mathcal{H}_+ \oplus \mathcal{H}_-$,

$$(x, A_{\mathfrak{W}} y')_{\mathcal{D}(\mathfrak{W})} = (x, y')_{\mathcal{H}_- \oplus \mathcal{H}_+}.$$

This proves Claim 1). If we in the same computation replace $y' \in \mathcal{H}_+ \oplus \mathcal{H}_-$ by either $y' \in \mathcal{H}_+$ or $y' \in \mathcal{H}_-$ we get Claim 2). $\square$

As the following lemma shows, the subspace $\mathcal{D}^0(\mathfrak{W})$ defined by

$$\mathcal{D}^0(\mathfrak{W}) := \left\{ Q(z + z^\dagger) \;\middle|\; z \in \mathfrak{W},\ z^\dagger \in \mathfrak{W}^{[\perp]} \right\} \tag{3.4}$$

is dense in $\mathcal{D}(\mathfrak{W})$.

We define

$$\begin{aligned} \mathcal{L}(\mathfrak{W}) &= \{ w \in k^2(\mathcal{W}) \mid Qw \in \mathcal{D}(\mathfrak{W}) \}, \\ \mathcal{L}^0(\mathfrak{W}) &= \{ z + z^\dagger \mid z \in \mathfrak{W},\ z^\dagger \in \mathfrak{W}^{[\perp]} \}, \end{aligned} \tag{3.5}$$

and

$$(w_1, w_2)_{\mathcal{L}(\mathfrak{W})} = (Qw_1, Qw_2)_{\mathcal{D}(\mathfrak{W})}, \qquad w_1,\ w_2 \in \mathcal{L}(\mathfrak{W}), \tag{3.6}$$

$$\|w\|_{\mathcal{L}(\mathfrak{W})} = \|Qw\|_{\mathcal{D}(\mathfrak{W})}, \qquad w \in \mathcal{L}(\mathfrak{W}). \tag{3.7}$$

Then $(\cdot,\cdot)_{\mathcal{L}(\mathfrak{W})}$ is a semi-inner product in $\mathcal{L}(\mathfrak{W})$ and $\|\cdot\|_{\mathcal{L}(\mathfrak{W})}$ is a semi-norm in $\mathcal{L}(\mathfrak{W})$.

Lemma 3.3. 1) *If $z \in \mathfrak{W}$ and $z^\dagger \in \mathfrak{W}^{[\perp]}$, then $Q(z + z^\dagger) = A_{\mathfrak{W}} \begin{bmatrix} Q_+ z^\dagger \\ Q_- z \end{bmatrix}$. In particular, $\mathcal{D}^0(\mathfrak{W}) \subset \mathcal{R}(A_{\mathfrak{W}})$ and $\mathcal{L}^0(\mathfrak{W}) \subset \mathcal{L}(\mathfrak{W})$.*

2) *$\mathcal{D}^0(\mathfrak{W})$ is a dense subspace of $\mathcal{D}(\mathfrak{W})$.*

3) *If $w \in \mathcal{L}(\mathfrak{W})$, $z \in \mathfrak{W}$, and $z^\dagger \in \mathfrak{W}^{[\perp]}$, then*

$$(w, z)_{\mathcal{L}(\mathfrak{W})} = (Q_- w, Q_- z)_{\mathcal{H}_-} = [\pi_- w, \pi_- z]_{k^2_-(\mathcal{W})} \tag{3.8}$$

$$(w, z^\dagger)_{\mathcal{L}(\mathfrak{W})} = (Q_+ w, Q_+ z^\dagger)_{\mathcal{H}_+} = -[\pi_+ w, \pi_+ z^\dagger]_{k^2_+(\mathcal{W})}. \tag{3.9}$$

In particular,

$$\|z\|^2_{\mathcal{L}(\mathfrak{W})} = \|Q_- z\|^2_{\mathcal{H}_-} = [\pi_- z, \pi_- z]_{k^2_-(\mathcal{W})}, \qquad z \in \mathfrak{W}, \tag{3.10}$$

$$\|z^\dagger\|^2_{\mathcal{L}(\mathfrak{W})} = \|Q_+ z^\dagger\|^2_{\mathcal{H}_+} = -[\pi_+ z^\dagger, \pi_+ z^\dagger]_{k^2_+(\mathcal{W})}, \qquad z^\dagger \in \mathfrak{W}^{[\perp]}. \tag{3.11}$$

Step 1: Proof of 1). Let $z \in \mathfrak{W}$. Then $\pi_+ z + \mathfrak{W}_+ = \Gamma_{\mathfrak{W}}(\pi_- z + \mathfrak{W}_-^{[\perp]})$, and consequently

$$Qz = \begin{bmatrix} Q_+ z \\ Q_- z \end{bmatrix} = \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} Q_- z = A_{\mathfrak{W}} Q_- z.$$

An analogous computation shows that $Qz^\dagger = A_{\mathfrak{W}} Q_+ z^\dagger$ for all $z^\dagger \in \mathfrak{W}^{[\perp]}$. Thus, $Q(z+z^\dagger) = A_{\mathfrak{W}} \left[\begin{smallmatrix} Q_+ z^\dagger \\ Q_- z \end{smallmatrix}\right]$.

Step 2: $\mathcal{D}^0$ is a dense subspace of $\mathcal{D}(\mathfrak{W})$. Since $\mathcal{H}^0_\pm$ is dense in $\mathcal{H}_\pm$, and since $\mathcal{R}\left(A_{\mathfrak{W}}\right)$ is dense in $\mathcal{D}(\mathfrak{W})$, the image of $\mathcal{H}^0_+ \oplus \mathcal{H}^0_-$ under $A_{\mathfrak{W}}$ is dense in $\mathcal{D}(\mathfrak{W})$. However, by Claim 1, this image is equal to $\mathcal{D}^0(\mathfrak{W})$.

Step 3: Proof of (3.8)–(3.11). By Part 3) of Lemma 3.1 and Theorem 2.12

$$\begin{aligned}(w,z)_{\mathcal{L}(\mathfrak{W})} &= (Qw, Qz)_{\mathcal{D}(\mathfrak{W})} = (Qw, A_{\mathfrak{W}} Q_- z)_{\mathcal{D}(\mathfrak{W})} \\ &= (Qw, Q_- z)_{\mathcal{H}_+\oplus\mathcal{H}_-} = (Q_- w, Q_- z)_{\mathcal{H}_-} \\ &= [\pi_- w, \pi_- z]_{k^2_-(\mathcal{W})}.\end{aligned}$$

This proves (3.8), and an analogous computation together with Theorem 2.9 can be used to prove (3.9). The equalities (3.10) and (3.11) follow directly from (3.8) and (3.9). □

Lemma 3.4. 1) *If $w \in \mathcal{L}(\mathfrak{W})$, then $S^{-1}w \in \mathcal{L}(\mathfrak{W})$, and*

$$\|S^{-1}w\|^2_{\mathcal{L}(\mathfrak{W})} = [w(0), w(0)]_{\mathcal{W}} + \|w\|^2_{\mathcal{L}(\mathfrak{W})}. \tag{3.12}$$

2) *If $w \in \mathcal{L}(\mathfrak{W})$, then $Sw \in \mathcal{L}(\mathfrak{W})$, and*

$$\|Sw\|^2_{\mathcal{L}(\mathfrak{W})} = -[w(-1), w(-1)]_{\mathcal{W}} + \|w\|^2_{\mathcal{L}(\mathfrak{W})}. \tag{3.13}$$

3) *If w_1, $w_2 \in \mathcal{L}(\mathfrak{W})$, then*

$$(w_1, S^{-1}w_2)_{\mathcal{L}(\mathfrak{W})} = [w_1(-1), w_2(0)]_{\mathcal{W}} + (Sw_1, w_2)_{\mathcal{L}(\mathfrak{W})}. \tag{3.14}$$

Step 1: Proof of 1). It follows from Lemma 3.3 that if $w \in \mathcal{L}^0(\mathfrak{W})$, then $S^{-1}w \in \mathcal{L}^0(\mathfrak{W})$ and (3.12) holds. Now let $w \in \mathcal{L}(\mathfrak{W})$, and choose $x_m \in \mathcal{D}^0(\mathfrak{W})$ such that $x_m \to Qw$ in $\mathcal{D}(\mathfrak{W})$ as $m \to \infty$. Let R be a bounded left-inverse of the quotient map Q, and define $w_m := w + R(x_m - Qw)$. Then $Qw_m = x_m \to Qw$ in $\mathcal{D}(\mathfrak{W})$, $w_m \in \mathcal{L}^0(\mathfrak{W})$, and $w_m \to w$ in $k^2(\mathcal{W})$ as $m \to \infty$. It then follows from (3.12) applied to $w_m \in \mathcal{L}^0(\mathfrak{W})$ that $QS^{-1}w_m$ is a Cauchy sequence in $\mathcal{D}(\mathfrak{W})$, and hence it tends to a limit y in $\mathcal{D}(\mathfrak{W})$ satisfying $\|y\|^2_{\mathcal{D}(\mathfrak{W})} = [w(0), w(0)]_{\mathcal{W}} + \|Qw\|^2_{\mathcal{D}(\mathfrak{W})}$. By the continuity of Q and S^{-1},

$$RQS^{-1}w = RQS^{-1} \lim_{m\to\infty} w_m = R \lim_{m\to\infty} QS^{-1}w_m = Ry,$$

and hence $y = QRy = QS^{-1}w$. This proves Claim 1).

Step 2: Proof of 2) This proof is analogous to the proof of 1).

Step 3: Proof of (3.14). By polarizing (3.12) we get

$$(S^{-1}w_1, S^{-1}w_2)_{\mathcal{L}(\mathfrak{W})} = [w_1(0), w_2(0)]_{\mathcal{W}} + [w_1, w_2]^2_{\mathcal{L}(\mathfrak{W})}.$$

for all w_1, $w_2 \in \mathcal{L}(\mathfrak{W})$. If we here replace w_1 by Sw_1, then we get (3.14). □

Theorem 3.5. *Let $\mathfrak{W}$ be a passive full behavior on the Kreĭn space $\mathcal{W}$, and let $\mathfrak{W}_- = \pi_-\mathfrak{W}$ and $\mathfrak{W}_+ = \mathfrak{W} \cap k^2_+(\mathcal{W})$ be the corresponding passive past and future behaviors. Let $\mathcal{D}(\mathfrak{W})$ be the range space of the operator $A^{1/2}_{\mathfrak{W}}$, where $A_{\mathfrak{W}}$ is the nonnegative self-adjoint operator on $\mathcal{H}_+ \oplus \mathcal{H}_-$ defined by* (3.3), *and define $\mathcal{L}(\mathfrak{W})$ by* (3.5). *The the subspace $V_{\rm sc}$ defined by*

$$V_{\rm sc} := \left\{ \begin{bmatrix} QS^{-1}w \\ Qw \\ w(0) \end{bmatrix} \,\middle|\, w \in \mathcal{L}(\mathfrak{W}) \right\}. \tag{3.15}$$

is the generating subspace of a simple conservative s/s system $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ whose full behavior is $\mathfrak{W}$. The input map of $\Sigma_{\rm sc}$ is $\mathfrak{B}_{\Sigma_{\rm sc}} = \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix}$ with $B^*_\Sigma = \Pi_-|_{\mathcal{D}(\mathfrak{W})}$, and the output map of $\Sigma_{\rm sc}$ is $\mathfrak{C}_{\Sigma_{\rm sc}} = \Pi_+|_{\mathcal{D}(\mathfrak{W})}$ with $\mathfrak{C}^*_{\Sigma_{\rm sc}} = \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma^*_{\mathfrak{W}} \end{bmatrix}$. Moreover, $(x(\cdot), w(\cdot))$ is a stable externally generated full trajectory of $\Sigma_{\rm sc}$ if and only if

$$w \in \mathfrak{W} \text{ and } x(n) = QS^{-n}w, \; n \in \mathbb{Z}.$$

Proof. Step 1: $V_{\rm sc}$ is a neutral subspace of $\mathfrak{K}$. This follows from equality (3.12).

Step 2: $V_{\rm sc} = V_{\rm sc}^{[\perp]}$, and hence $V_{\rm sc}$ generates a conservative s/s system $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$. Our proof of Step 2 is based on Lemma 2.2 with $\mathcal{Z} = \mathcal{X} = \mathcal{D}(\mathfrak{W})$. Clearly condition (a) in that lemma holds because of the definition of $\mathcal{L}(\mathfrak{W})$, and (c) holds because of Lemma 3.4. The set described in condition (b) is equal to the zero section $\mathfrak{W}_+(0) = \{w(0) \in \mathcal{W} \mid w \in \mathfrak{W}_+\}$, which according to Lemma 2.8 is maximal nonnegative, and and the set described in condition (d) is is equal to the -1-section $\mathfrak{W}_-^{[\perp]}(-1) = \{w(-1) \in \mathcal{W} \mid w \in \mathfrak{W}_-^{[\perp]}\}$, which according to Lemma 2.8 is maximal nonpositive. Thus, by Lemma 2.2, $V_{\rm sc}$ is Lagrangian, and hence it generates a conservative s/s system $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$.

Step 3: The behavior of $\Sigma_{\rm sc}$ is equal to $\mathfrak{W}$. If $w \in \mathfrak{W}_+$, then $Qw \in \mathcal{D}(\mathfrak{W})$, and it follows from (3.15) that $(x(\cdot), w(\cdot))$, where $x(n) = QS^{-n}w$, $n \in \mathbb{Z}^+$, is an externally generated stable future trajectory of $\Sigma_{\rm sc}$. This implies that $\mathfrak{W}_+ \subset \mathfrak{W}_+^{\Sigma_{\rm sc}}$. Since $\mathfrak{W}_+$ is maximal nonnegative and $\mathfrak{W}_+^{\Sigma_{\rm sc}}$ is nonnegative, this implies that $\mathfrak{W}_+ = \mathfrak{W}_+^{\Sigma_{\rm sc}}$. From this follows that also $\mathfrak{W}_- = \mathfrak{W}_-^{\Sigma_{\rm sc}}$ and $\mathfrak{W} = \mathfrak{W}^{\Sigma_{\rm sc}}$.

Step 4: $(x(\cdot), w(\cdot))$ is an externally generated full trajectory of $\Sigma_{\rm sc}$ if and only if $w \in \mathfrak{W}$ and $x(n) = QS^{-n}w$, $n \in \mathbb{Z}$. By the definition of $\mathfrak{W}$, if $(x(\cdot), w(\cdot))$ is an externally generated full trajectory of $\Sigma_{\rm sc}$, then $w \in \mathfrak{W}$. Conversely, let $w \in \mathfrak{W}$. Then $w \in \mathcal{L}(\mathfrak{W})$, and it follows from (3.15) that $(x(\cdot), w(\cdot))$ is a full trajectory of $\Sigma_{\rm sc}$, where $x(n) = QS^{-n}w$, $n \in \mathbb{Z}$. This trajectory is externally generated, since, according to Lemma 3.3

$$\|x(n)\|^2_{\mathcal{D}(\mathfrak{W})} = \|QS^{-n}w\|_{\mathcal{L}(\mathfrak{W})} = [\pi_-S^{-n}w, \pi_-S^{-n}w]_{k^2_-(\mathcal{W})} \to 0 \text{ as } n \to -\infty.$$

As an externally generated full trajectory $(x(\cdot), w(\cdot))$ is determined uniquely by it signal part w (see Lemma 2.19), it follows that every externally generated trajectory $(x(\cdot), w(\cdot))$ of $\Sigma_{\rm sc}$ satisfies $x(n) = QS^{-n}w$, $n \in \mathbb{Z}$.

Step 5: The input map of Σ *is* $\left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]$. According to Lemma 2.17, the operator $\mathfrak{B}_{\Sigma_{\rm sc}}$ is the unique operator $\mathcal{H}_- \to \mathcal{D}(\mathfrak{W})$ which satisfies $\mathfrak{B}_{\Sigma_{\rm sc}} Q_- w = x(0)$ for every $w \in \mathfrak{W}$, where $x(\cdot)$ is the state component of the unique externally generated trajectory $(x(\cdot), w(\cdot))$ whose signal part is w. Let $w \in \mathfrak{W}$. By Step 4,

$$x(0) = Qw = \begin{bmatrix} Q_+ w \\ Q_- w \end{bmatrix} = \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} Q_- w.$$

Thus, $\mathfrak{B}_{\Sigma_{\rm sc}} = \left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]$.

Step 6: The output map of Σ *is* $\Pi_+|_{\mathcal{D}(\mathfrak{W})}$. According to Lemma 2.18, $\mathfrak{C}_{\Sigma_{\rm sc}}$ is the operator which maps $x_0 \in \mathcal{D}(\mathfrak{W})$ into the equivalence class consisting of all the signal parts $w(\cdot)$ of all stable future trajectories $(x(\cdot), w(\cdot))$ of $\Sigma_{\rm sc}$ satisfying $x(0) = x_0$. Let $x_0 \in \mathcal{D}(\mathfrak{W})$, and choose some $w_0 \in \mathcal{L}(\mathfrak{W})$ such that $Qw_0 = x_0$. It follows from (3.15) that $(x(\cdot), w_0(\cdot))$, where $x(n) = QS^{-n}w_o$, $n \in \mathbb{Z}^+$, is a stable future trajectory of $\Sigma_{\rm sc}$ satisfying $x(0) = x_0$. If $(x_1(\cdot), w_1(\cdot))$ is another stable future trajectory of $\Sigma_{\rm sc}$ satisfying $x_1(0) = x(0) = x_0$, then $(x - x_1, w_0 - w_-)$ is an externally generated stable future trajectory of $\Sigma_{\rm sc}$, and hence $w_1 - w_0 \in \mathfrak{W}_+$. Thus, the equivalence class of all the signal parts $w(\cdot)$ of all stable future trajectories $(x(\cdot), w(\cdot))$ of $\Sigma_{\rm sc}$ satisfying $x(0) = x_0$ is equal to $Q_+ w_0$. Consequently, $\mathfrak{C}_{\Sigma_{\rm sc}} = \Pi_+|_{\mathcal{D}(\mathfrak{W})}$.

Step 7: $\Sigma_{\rm sc}$ *is simple.* According to Lemma 3.1, the linear span of the ranges of $\mathfrak{B}_{\Sigma_{\rm sc}} = \left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{W}_-} \end{smallmatrix}\right]$ and $\mathfrak{C}^*_{\Sigma_{\rm sc}} = \left[\begin{smallmatrix} 1_{\mathcal{W}_+} \\ \Gamma^*_{\mathfrak{W}} \end{smallmatrix}\right]$ is dense in the state space $\mathcal{D}(\mathfrak{W})$, and hence $\Sigma_{\rm sc}$ is simple. $\square$

Let $\mathfrak{R}$ be the reachable subspace, $\mathfrak{U}$ the unobservable subspace, $\mathfrak{R}^\dagger$ the null controllable subspace, and $\mathfrak{U}^\dagger$ the backward unobservable subspace of $\Sigma_{\rm sc}$. As we noticed earlier, $\mathfrak{R}^\dagger = \mathfrak{U}^\perp$ and $\mathfrak{U}^\dagger = \mathfrak{R}^\perp$. By Lemma 3.2 and Theorem 3.5,

$$\begin{aligned} \mathfrak{R} &= \mathcal{R}\left(\mathfrak{B}_{\Sigma_{\rm sc}}\right) = \mathcal{R}\left(\left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right]\right) = \left\{\left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} x_- \\ x_- \end{smallmatrix}\right] \mid x_- \in \mathcal{H}_-\right\}, \\ \mathfrak{U}^\dagger &= \mathcal{N}\left(\mathfrak{B}^*_{\Sigma_{\rm sc}}\right) = \mathcal{N}\left(\Pi_-|_{\mathcal{D}(\mathfrak{W})}\right) = \{Qw \mid w \in \mathcal{L}(\mathfrak{W}) \cap k^2_+(\mathcal{W})\}, \\ \mathfrak{R}^\dagger &= \mathcal{R}\left(\mathfrak{C}^*_{\Sigma_{\rm sc}}\right) = \mathcal{R}\left(\left[\begin{smallmatrix} 1_{\mathcal{H}_+} \\ \Gamma^*_{\mathfrak{W}} \end{smallmatrix}\right]\right) = \left\{\left[\begin{smallmatrix} x_+ \\ \Gamma^*_{\mathfrak{W}} x_+ \end{smallmatrix}\right] \mid x_+ \in \mathcal{H}_+\right\}, \\ \mathfrak{U} &= \mathcal{N}\left(\mathfrak{C}_{\Sigma_{\rm sc}}\right) = \mathcal{N}\left(\Pi_+|_{\mathcal{D}(\mathfrak{W})}\right) = \{Qw \mid w \in \mathcal{L}(\mathfrak{W}) \cap k^2_-(\mathcal{W})\}. \end{aligned} \tag{3.16}$$

The orthogonal projections onto these subspaces are given by

$$\begin{aligned} P_{\mathfrak{R}} &= \mathfrak{B}_{\Sigma_{\mathrm{sc}}}\mathfrak{B}_{\Sigma_{\mathrm{sc}}}^{*} = \begin{bmatrix} \Gamma_{\mathfrak{W}}\Pi_{-}|_{\mathcal{D}(\mathfrak{W})} \\ \Pi_{-}|_{\mathcal{D}(\mathfrak{W})} \end{bmatrix}, \\ P_{\mathfrak{U}^{\dagger}} &= 1_{\mathcal{D}(\mathfrak{W})} - P_{\mathfrak{R}} = \Pi_{+}|_{\mathcal{D}(\mathfrak{W})} - \Gamma_{\mathfrak{W}}\Pi_{-}|_{\mathcal{D}(\mathfrak{W})}, \\ P_{\mathfrak{R}^{\dagger}} &= \mathfrak{C}_{\Sigma_{\mathrm{sc}}}^{*}\mathfrak{C}_{\Sigma_{\mathrm{sc}}} = \begin{bmatrix} \Pi_{+}|_{\mathcal{D}(\mathfrak{W})} \\ \Gamma_{\mathfrak{W}}^{*}\Pi_{+}|_{\mathcal{D}(\mathfrak{W})} \end{bmatrix}, \\ P_{\mathfrak{U}} &= 1_{\mathcal{D}(\mathfrak{W})} - P_{\mathfrak{R}^{\dagger}} = \Pi_{-}|_{\mathcal{D}(\mathfrak{W})} - \Gamma_{\mathfrak{W}}^{*}\Pi_{+}|_{\mathcal{D}(\mathfrak{W})}, \end{aligned} \tag{3.17}$$

4 The Full Stable Trajectories of a Conservative State/Signal System.

The input and output maps $\mathfrak{B}_{\Sigma}$ and $\mathfrak{C}_{\Sigma}$ together with their anti-causal counterparts $\mathfrak{B}_{\Sigma}^{\dagger} = \mathfrak{C}_{\Sigma}^{*}$ and $\mathfrak{C}_{\Sigma}^{\dagger} = \mathfrak{B}_{\Sigma}^{*}$ of a conservative s/s system Σ can be used to describe the relationship between the state component $x(\cdot)$ and the signal component $w(\cdot)$ of an arbitrary *stable full trajectory* $(x(\cdot), w(\cdot))$ of Σ.

Theorem 4.1. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s system with behavior $\mathfrak{W}$, input map $\mathfrak{B}_{\Sigma}$, output map $\mathfrak{C}_{\Sigma}$, reachable subspace $\mathfrak{R} = \mathcal{R}(\mathfrak{B}_{\Sigma})$, unobservable subspace $\mathfrak{U}_{\Sigma} = \mathcal{N}(\mathfrak{C}_{\Sigma})$, null controllable subspace $\mathfrak{R}_{\Sigma}^{\dagger} = \mathcal{R}(\mathfrak{C}_{\Sigma}^{*})$ and backward unobservable subspace $\mathfrak{U}_{\Sigma}^{\dagger} = \mathcal{N}(\mathfrak{B}_{\Sigma}^{*})$.*

1) *The operator*
$$\mathfrak{C}_{\Sigma}^{\mathrm{full}} := \begin{bmatrix} \mathfrak{C}_{\Sigma} \\ \mathfrak{B}_{\Sigma}^{*} \end{bmatrix} \tag{4.1}$$
is a co-isometry from $\mathcal{X}$ onto $\mathcal{D}(\mathfrak{W})$, with kernel $\mathcal{X}_0 := \mathcal{N}\left(\mathfrak{C}_{\Sigma}^{\mathrm{full}}\right) = \mathfrak{U} \cap \mathfrak{U}^{\dagger}$. Thus, Σ is simple if and only if $\mathfrak{C}_{\Sigma}^{\mathrm{full}}$ is injective.

2) *Denote the adjoint of $\mathfrak{C}_{\Sigma}^{\mathrm{full}}$ by $\mathfrak{B}_{\Sigma}^{\mathrm{full}} := (\mathfrak{C}_{\Sigma}^{\mathrm{full}})^{*}$. Then $\mathfrak{B}_{\Sigma}^{\mathrm{full}}$ is an isometry $\mathcal{D}(\mathfrak{W}) \to \mathcal{X}$ with range $\mathcal{X}_0^{\perp} = \overline{\mathfrak{R} + \mathfrak{R}^{\dagger}}$, which is uniquely determined by the fact that*
$$\mathfrak{B}_{\Sigma} = \mathfrak{B}_{\Sigma}^{\mathrm{full}} \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_{-}} \end{bmatrix}, \qquad \mathfrak{C}_{\Sigma}^{*} = \mathfrak{B}_{\Sigma}^{\mathrm{full}} \begin{bmatrix} 1_{\mathcal{H}_{+}} \\ \Gamma_{\mathfrak{W}}^{*} \end{bmatrix}. \tag{4.2}$$
In articular, $\mathfrak{B}_{\Sigma}^{\mathrm{full}}$ is surjective if and only if Σ is simple.

3) *A full trajectory $(x(\cdot), w(\cdot))$ of Σ is stable if and only if $w \in k^2(\mathcal{W})$.*

4) *If $(x(\cdot), w(\cdot))$ is a stable full trajectory of Σ, then $w \in \mathcal{L}(\mathfrak{W})$, $QS^{-n}w = \mathfrak{C}_{\Sigma}^{\mathrm{full}}x(n)$, and $P_{\mathcal{X}_0^{\perp}}x(n) = \mathfrak{B}_{\Sigma}^{\mathrm{full}}QS^{-n}w$ for all $n \in \mathbb{Z}$.*

5) *Conversely, let $w \in \mathcal{L}(\mathfrak{W})$, and define $x(n) = \mathfrak{B}_{\Sigma}^{\mathrm{full}}QS^{-n}w$, $n \in \mathbb{Z}$. Then $(x(\cdot), w(\cdot))$ is a stable full trajectory of Σ.*

6) *The state component* $x(\cdot)$ *of a stable full trajectory* $(x(\cdot), w(\cdot))$ *of* Σ *is determined uniquely by the signal component* $w(\cdot)$ *if and only if* Σ *is simple.*

7) *If* Σ *is simple, then* Σ *is unitarily similar to the canonical simple conservative model* $\Sigma_{\rm sc}^{\mathfrak{W}}$ *with unitary similarity operator* $\mathfrak{C}_\Sigma^{\rm full}$.

Proof of 1). The claim about the kernel of $\mathfrak{C}_\Sigma^{\rm full}$ is trivial. Therefore, to prove 1) it suffices to show that the restriction of $\mathfrak{C}_\Sigma^{\rm full}$ to a dense subspace of $\mathcal{X}_0^\perp$ is isometric, and that the range of this restriction is a dense subspace of $\mathcal{D}(\mathfrak{W})$. We choose this dense subspace of $\mathcal{X}_0^\perp$ to be the set of all vectors $x_0 \in \mathcal{X}$ of the form

$$x_0 = \mathfrak{B}_\Sigma Q_- z + \mathfrak{C}_\Sigma^* Q_+ z^\dagger,$$

where $z \in \mathfrak{W}$ and $z^\dagger \in \mathfrak{W}^{[\perp]}$.

By Lemmas 2.21 and 2.22, both $\mathfrak{B}_\Sigma$ and $\mathfrak{C}_\Sigma^*$ are isometric, and since $\Gamma_{\mathfrak{W}} = \mathfrak{C}_\Sigma \mathfrak{B}_\Sigma$, $Q_+ z = \Gamma_{\mathfrak{W}} Q_- z$ and $Q_+ z^\dagger = \Gamma_{\mathfrak{W}}^* Q_+ z^\dagger$, we have

$$\begin{aligned}
\|x_0\|_{\mathcal{X}}^2 &= \|\mathfrak{B}_\Sigma Q_- z + \mathfrak{C}_\Sigma^* Q_+ z^\dagger\|_{\mathcal{X}}^2 \\
&= \|\mathfrak{B}_\Sigma Q_- z\|_{\mathcal{X}}^2 + \|\mathfrak{C}_\Sigma^* Q_+ z^\dagger\|_{\mathcal{X}}^2 + 2\Re(\mathfrak{B}_\Sigma Q_- z, \mathfrak{C}_\Sigma^* Q_+ z^\dagger)_{\mathcal{X}} \\
&= \|Q_- z\|_{\mathcal{H}_-}^2 + \|Q_+ z^\dagger\|_{\mathcal{H}_+}^2 + 2\Re(\Gamma_{\mathfrak{W}} Q_- z, Q_+ z^\dagger)_{\mathcal{H}_+} \\
&= \left(\begin{bmatrix} Q_+ z^\dagger \\ Q_- z \end{bmatrix}, \begin{bmatrix} Q_+ (z+z^\dagger) \\ Q_- (z+z^\dagger) \end{bmatrix} \right)_{\mathcal{H}_+ \oplus \mathcal{H}_-}.
\end{aligned}$$

On the other hand,

$$\mathfrak{C}_\Sigma^{\rm full} x_0 = \begin{bmatrix} \mathfrak{C}_\Sigma \\ \mathfrak{B}_\Sigma^* \end{bmatrix} (\mathfrak{B}_\Sigma Q_- z + \mathfrak{C}_\Sigma^* Q_+ z^\dagger) = \begin{bmatrix} \Gamma_{\mathfrak{W}} Q_- z + Q_+ z^\dagger \\ Q_- z + \Gamma^* Q_+ z^\dagger \end{bmatrix} = A_{\mathfrak{W}} \begin{bmatrix} Q_+ z^\dagger \\ Q_- z \end{bmatrix}.$$

Thus, by Lemma 3.1, $\mathfrak{C}_\Sigma^{\rm full} x_0 \in \mathcal{D}(\mathfrak{W})$, and

$$\begin{aligned}
\|\mathfrak{C}_\Sigma^{\rm full} x_0\|_{\mathcal{D}(\mathfrak{W})}^2 &= \left(\begin{bmatrix} Q_+ z^\dagger \\ Q_- z \end{bmatrix}, A_{\mathfrak{W}} \begin{bmatrix} Q_+ z^\dagger \\ Q_- z \end{bmatrix} \right)_{\mathcal{H}_+ \oplus \mathcal{H}_-} \\
&= \left(\begin{bmatrix} Q_+ z^\dagger \\ Q_- z \end{bmatrix}, \begin{bmatrix} Q_+ (z+z^\dagger) \\ Q_- (z+z^\dagger) \end{bmatrix} \right)_{\mathcal{H}_+ \oplus \mathcal{H}_-} = \|x_0\|_{\mathcal{X}}^2.
\end{aligned}$$

This proves that the restriction of $\mathfrak{C}_\Sigma^{\rm full}$ to a dense subspace of $\mathcal{X}_0^\perp$ is a isometric map of this subspace into $\mathcal{D}(\mathfrak{W})$. The image of the same subspace is equal to $\mathcal{D}^0(\mathfrak{W})$, which is dense in $\mathcal{D}(\mathfrak{W})$. Thus, $\mathfrak{C}_\Sigma^{\rm full}$ is a unitary map from $\mathcal{X}_0^\perp$ onto $\mathcal{D}(\mathfrak{W})$, and hence a co-isometric map from $\mathcal{X}$ onto $\mathcal{D}(\mathfrak{W})$.

Proof of 2). We begin by observing that (4.2) defines $\mathfrak{B}_\Sigma^{\rm full}$ uniquely, since it defines $\mathfrak{B}_\Sigma^{\rm full}$ on $\mathcal{R}(A_{\mathfrak{W}})$, which is dense in $\mathcal{D}(\mathfrak{W})$. Thus, it suffices to prove that (4.2) holds.

Let $z \in \mathfrak{W}$, and let $x_0 = \mathfrak{B}_\Sigma Q_- z$. Then by the proof of Step 1, $\mathfrak{C}_\Sigma^{\rm full} x_0 = Qz$. On the other hand, since $\mathfrak{C}_\Sigma^{\rm full}\mathfrak{B}_\Sigma^{\rm full} = 1_{\mathcal{D}(\mathfrak{W})}$ we also have

$$\mathfrak{C}_\Sigma^{\rm full}\mathfrak{B}_\Sigma^{\rm full} Qz = Qz = \mathfrak{C}_\Sigma^{\rm full}\mathfrak{B}_\Sigma Q_- z.$$

We also know that both $x_0 = \mathfrak{B}_\Sigma Q_- z$ and $\mathfrak{B}_\Sigma^{\rm full} Qz$ lie in $\mathcal{X}_0^\perp$, and that $\mathfrak{C}_\Sigma^{\rm full}$ is injective on $\mathcal{X}_0^\perp$. Thus,

$$\mathfrak{B}_\Sigma Q_- z = \mathfrak{B}_\Sigma^{\rm full} Qz = \mathfrak{B}_\Sigma^{\rm full} \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} Q_- z$$

for all $z \in \mathfrak{W}$. Thus, $\mathfrak{B}_\Sigma x_- = \mathfrak{B}_\Sigma^{\rm full} \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} x_-$ for all $x_- \in \mathcal{H}_-^0$. The subspace $\mathcal{H}_-^0$ is dense in $\mathcal{H}_-$, and thus $\mathfrak{B}_\Sigma = \mathfrak{B}_\Sigma^{\rm full} \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix}$. An analogous argument shows that $\mathfrak{C}_\Sigma^* = \mathfrak{B}_\Sigma^{\rm full} \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix}$.

Proof of 3). By definition of a stable trajectory, if $(x(\cdot), w(\cdot))$ is a stable full trajectory, then $w \in k^2(\mathcal{W})$. Conversely, if $w \in k^2(\mathcal{W})$, then it follows from the balance equation

$$-\|x(n+1)\|_{\mathcal{X}}^2 + \|x(m)\|_{\mathcal{X}}^2 + \sum_{k=m}^{n} [w(n), w(n)]_{\mathcal{W}}$$

that $\|x(n)\|_{\mathcal{X}}^2$ has a finite limit at $\pm\infty$, and so $x \in \ell^\infty(\mathfrak{W})$.

Proof of 4). If $(x(\cdot), w(\cdot))$ is a stable full trajectory of Σ, and if we shift this trajectory to the left or right, then the shifted trajectory is still a stable full trajectory of Σ. Thus, it suffices to prove Claim 4) with $n = 0$.

If $(x(\cdot), w(\cdot))$ is a stable full trajectory of Σ, then the restriction of this trajectory to $\mathbb{Z}^+$ is a stable future trajectory of Σ, and by the definition of $\mathfrak{C}_\Sigma$, this implies that $Q_+ w = \mathfrak{C}_\Sigma x(0)$. The same argument applied to the anti-causal adjoint system implies that $Q_- w = \mathfrak{B}_\Sigma^* x(0)$, and consequently, $Qw = \mathfrak{C}_\Sigma^{\rm full} x(0)$. By applying $\mathfrak{B}_\Sigma^{\rm full}$ to this identity we get $\mathfrak{B}_\Sigma^{\rm full} Qw = \mathfrak{B}_\Sigma^{\rm full}\mathfrak{C}_\Sigma^{\rm full} x(0) = P_{\mathcal{X}_0^\perp} x(0)$.

Proof of 5). We first claim that if $w = z + z^\dagger$, where $z \in \mathfrak{W}$ and $z^\dagger \in \mathfrak{W}^{[\perp]}$, and if we $x(n) = \mathfrak{B}_\Sigma^{\rm full} Q S^{-n} w$, $n \in \mathbb{Z}$, then $(x(\cdot), w(\cdot))$ is a stable full trajectory of Σ.

We first consider the case where $w = z \in \mathfrak{W}$. Then, by Claim 2),

$$x(n) = \mathfrak{B}_\Sigma^{\rm full} Q S^{-n} z = \mathfrak{B}_\Sigma^{\rm full} \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} Q_- S^{-n} z = \mathfrak{B}_\Sigma Q_- S^{-n} z,$$

and it follows from Lemma 2.19 that $(x(\cdot), w(\cdot))$ is a stable full trajectory of Σ. An analogous argument can be used in the case where $w = z^\dagger \in \mathfrak{W}^{[\perp]}$.

Let now w be an arbitrary vector in $\mathcal{L}(\mathfrak{W})$. Choose some sequence $y_m \in \mathcal{D}^0(\mathfrak{W})$ such that $y_m \to Qw$ in $\mathcal{D}(\mathfrak{W})$ as $m \to \infty$. Let R be a bounded left-inverse of the quotient map Q, and define $w_m := w + R(y_m - Qw)$. Then $Qw_m = y_m \to Qw$ in $\mathcal{D}(\mathfrak{W})$, $w_m \in \mathcal{L}^0(\mathfrak{W})$, and $w_m \to w$ in $k^2(\mathcal{W})$ as $m \to \infty$. Define $x(n) = \mathfrak{B}_\Sigma^{\rm full} QS^{-n}w$ and $x_m(n) = \mathfrak{B}_\Sigma^{\rm full} QS^{-n}w_m$, $n \in \mathbb{Z}$. Then $(x_m(\cdot), w_m(\cdot))$ is a stable full trajectory of Σ for all m, and $x_m(n) \to x(n)$ for all $n \in \mathbb{Z}$ as $m \to \infty$. Since V is closed, also $(x(\cdot), w(\cdot))$ is a full trajectory of Σ, and it is stable since $w \in k^2(\mathcal{W})$.

Proof of 6). If Σ is simple, then $\mathcal{X}_0^\perp = \mathcal{X}$, and it follows from 4) that the state component $x(\cdot)$ of a stable full trajectory $(x(\cdot), w(\cdot))$ of Σ is determined uniquely by $w(\cdot)$. Conversely, suppose that Σ has a stable full trajectory $(x(\cdot), 0)$, where $x(\cdot)$ is not identically zero. Then it follows from 4) that $\mathfrak{C}_\Sigma^{\rm full} x(n) = 0$ for all $n \in \mathbb{Z}$, and hence Σ is not simple.

Proof of 7). Let $\left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$, and choose some arbitrary stable full trajectory $(x(\cdot), w(\cdot))$ such that $x(0) = x_0$, $x(1) = x_1$, and $w(0) = w_0$; this is possible since Σ is well-posed both in the forward and in the backward time direction. By Parts 5) and 6), the unique full trajectory $(x_{\rm sc}(\cdot), w(\cdot))$ of $\Sigma_{\rm sc}$ whose signal part is $w(\cdot)$ satisfies $x_{\rm sc}(n) = QS^{-n}w$, $n \in \mathbb{Z}$, and by the same argument, $x(n) = \mathfrak{B}_\Sigma^{\rm full} QS^{-n}w$, $n \in \mathbb{Z}$. Thus, in particular, $x_0 = x(0) = \mathfrak{B}_\Sigma^{\rm full} x_{\rm sc}(0)$ and $x_1 = x(1) = \mathfrak{B}_\Sigma^{\rm full} x_{\rm sc}(1)$, where $\left[\begin{smallmatrix} x_{\rm sc}(1) \\ x_{\rm sc}(0) \\ w_0 \end{smallmatrix}\right] \in V_{\rm sc}$. This gives

$$V \subset \begin{bmatrix} \mathfrak{B}_\Sigma^{\rm full} & 0 & 0 \\ 0 & \mathfrak{B}_\Sigma^{\rm full} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_{\rm sc}.$$

By interchanging the roles of Σ and $\Sigma_{\rm sc}$ we get the opposite inclusion. Thus, Σ is unitarily similar $\Sigma_{\rm sc}^{\mathfrak{W}}$ with unitary similarity operator $(\mathfrak{B}_\Sigma^{\rm full})^{-1} = \mathfrak{C}_\Sigma^{\rm full}$. □

Alternative proof of Theorem 3.5. Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be an arbitrary simple conservative s/s realization of $\mathfrak{W}$; that such a s/s system exists follows from [AS07a, Theorem 8.6]. It follows from Theorem 4.1 that V has the image representation

$$V = \left\{ \begin{bmatrix} \mathfrak{B}_\Sigma^{\rm full} QS^{-1}w \\ \mathfrak{B}_\Sigma^{\rm full} Qw \\ w(0) \end{bmatrix} \;\middle|\; w \in \mathcal{L}(\mathfrak{W}) \right\}.$$

If we to this system apply a unitary similarity transform with similarity operator $\mathfrak{C}_\Sigma^{\rm full} = (\mathfrak{B}_\Sigma^{\rm full})^*$, then we get another simple conservative s/s realization of $\mathfrak{W}$. The generating subspace that we get in this way is the same one which is given in (3.15). □

The above proof of Theorem 3.5 is very short, but it is not fully self-contained in the sense that it is based on the knowledge that every passive full behavior has a simple conservative realization. The original proof given Section 3 is complete in the sense that it does not rely on any a priori knowledge of the existence of a simple conservative realization of $\mathfrak{W}$.

Corollary 4.2. *Let $\mathfrak{W}$ be a full behavior on the Kreĭn space $\mathcal{W}$. Then the sequence $(x(\cdot), w(\cdot))$ is a stable full trajectory of $\Sigma_{\mathrm{sc}}^{\mathfrak{W}}$ if and only if*

$$w \in \mathcal{L}(\mathfrak{W}) \text{ and } x(n) = QS^{-n}w, \ n \in \mathbb{Z}.$$

Proof. This follows from Theorem 4.1. □

Definition 4.3. We call the operators $\mathfrak{C}_{\Sigma}^{\mathrm{full}}$ and $\mathfrak{B}_{\Sigma}^{\mathrm{full}}$ defined in Theorem 4.1 the *bilateral input and output maps*, respectively, of the conservative s/s system Σ.

5 Incoming and Outgoing Inner Channels

In this section we throughout let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s system with full behavior $\mathfrak{W}$, input map $\mathfrak{B}_{\Sigma}$, and output map $\mathfrak{C}_{\Sigma}$. Let $\mathfrak{R}_{\Sigma} = \mathcal{R}(\mathfrak{B}_{\Sigma})$ be the reachable subspace, let $\mathfrak{U}_{\Sigma} = \mathcal{N}(\mathfrak{C}_{\Sigma})$ be the unobservable subspace, let $\mathfrak{R}_{\Sigma}^{\dagger} = \mathcal{R}(\mathfrak{C}_{\Sigma}^{*})$ be the null controllable subspace, and let $\mathfrak{U}_{\Sigma}^{\dagger} = \mathcal{N}(\mathfrak{B}_{\Sigma}^{*})$ be the backward unobservable subspace of Σ.

Lemma 5.1. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s system.*

1) *There exists a unique isometry A_- on $\mathfrak{U}_{\Sigma}$ such that $(x(\cdot), 0)$ is a unobservable future trajectory of Σ if and only if $x(n) = A_-^n x(0)$. Every unobservable future trajectory $(x(\cdot), 0)$ is uniquely determined by the value of $x(n)$ for any fixed $n \in \mathbb{Z}^+$.*

2) *There exists a unique isometry A_+ on $\mathfrak{U}_{\Sigma}^{\dagger}$ such that $(x(\cdot), 0)$ is a backward unobservable past trajectory of Σ if and only if $x(n) = A_+^{|n|} x(0)$, $n \in \mathbb{Z}^-$. Every backward unobservable past trajectory $(x(\cdot), 0)$ is uniquely determined by the value of $x(n)$ for any fixed $n \in \mathbb{Z}^-$.*

Proof. By the definition of $\mathfrak{U}_{\Sigma}$, for each $x_0 \in \mathfrak{U}_{\Sigma}$ there exists a unique unobservable future trajectory $(x(\cdot), 0)$ of Σ with $x(0) = x_0$. Let A_- be the mapping from x_0 to $x(1)$. That A_- is an isometry follows from the conservativity of Σ which implies that $\|x(1)\|_{\mathcal{X}}^2 = \|x(0)\|_{\mathcal{X}}^2$. If we left-shift an unobservable trajectory by n steps, then the shifted trajectory is still an unobservable trajectory, and hence $x(n+1) = A_- x(n)$ for all $n \in \mathbb{Z}^+$. Since A_-

is isometric, the condition $x(n+1) = A_- x(n)$ implies $x(n) = A_-^* x(n+1)$, and therefore

$$x(n+1) = A_- x(n), \qquad x(n) = A_-^* x(n+1), \qquad n \in \mathbb{Z}^+. \tag{5.1}$$

Clearly, if we know $x(n)$ for any fixed $n \in \mathbb{Z}^+$, then (5.1) determines the full future trajectory uniquely.

The proof of Claim 2) is analogous. This time (5.1) is replaced by

$$x(n) = A_+ x(n+1), \qquad x(n+1) = A_+^* x(n), \qquad n \in \mathbb{Z}^-. \tag{5.2}$$

□

Definition 5.2. We call $(A_+^*, \mathfrak{U}^\dagger)$ the *incoming inner channel* and $(A_-; \mathfrak{U})$ the *outgoing inner channel* of the conservative s/s system Σ, where A_+ and A_- are the operators defined in Lemma 5.1.

Theorem 5.3. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s system with bilateral input and output maps $\mathfrak{B}_\Sigma^{\text{full}}$ and $\mathfrak{C}_\Sigma^{\text{full}}$, respectively, and incoming and outgoing inner channels $(A_+^*; \mathfrak{U}_\Sigma^\dagger)$ and $(A_-; \mathfrak{U}_\Sigma)$, respectively. Then*

$$\mathfrak{U}_\Sigma \cap \mathfrak{U}_\Sigma^\dagger = \mathcal{N}\left(\mathfrak{C}_\Sigma^{\text{full}}\right) = \mathcal{R}\left(\mathfrak{B}_\Sigma^{\text{full}}\right)^\perp = \cap_{n\in\mathbb{Z}^+} A_-^n \mathfrak{U}_\Sigma = \cap_{n\in\mathbb{Z}^+} A_+^n \mathfrak{U}_\Sigma^\dagger. \tag{5.3}$$

Consequently, the following four conditions are equivalently:

1) Σ *is simple;*

2) *The operator A_- is completely non-unitary.*

3) *The operator A_+ is completely non-unitary.*

4) Σ *has no nontrivial bilaterally unobservable trajectory.*

Proof. Step 1: $\mathfrak{U}_\Sigma \cap \mathfrak{U}_\Sigma^\dagger = \mathcal{N}\left(\mathfrak{C}_\Sigma^{\text{full}}\right) = \mathcal{R}\left(\mathfrak{B}_\Sigma^{\text{full}}\right)^\perp$. This follows from Theorem 4.1.

Step 2: $\mathfrak{U} \cap \mathfrak{U}^\dagger \subset \left(\cap_{n\in\mathbb{Z}^+} A_-^n \mathfrak{U}\right) \cap \left(\cap_{n\in\mathbb{Z}^+} A_+^n \mathfrak{U}_\Sigma^\dagger\right)$. Let $x_0 \in \mathfrak{U}_\Sigma \cap \mathfrak{U}_\Sigma^\dagger$. Then by the definitions of $\mathfrak{U}_\Sigma$ and $\mathfrak{U}_\Sigma^\dagger$, Σ has a bilaterally unobservable full trajectory $(x(\cdot), 0)$ with $x(0) = x_0$. The set of all bilaterally unobservable full trajectories of Σ is invariant under both right and left shift, and together with (5.1) and (5.2) this implies that for all $n \in \mathbb{Z}$ we have $x(0) = A_-^n x(-n) = A_+^n x(n)$. Consequently $x_0 \in \left(\cap_{n\in\mathbb{Z}^+} A_-^n \mathfrak{U}\right) \cap \left(\cap_{n\in\mathbb{Z}^+} A_+^n \mathfrak{U}_\Sigma^\dagger\right)$.

Step 3: $\cap_{n\in\mathbb{Z}^+} A_-^n \mathfrak{U} \subset \mathfrak{U}\cap\mathfrak{U}^\dagger$ *and* $\cap_{n\in\mathbb{Z}^+} A_+^n \mathfrak{U}_\Sigma^\dagger \subset \mathfrak{U}\cap\mathfrak{U}^\dagger$. Denote $\mathcal{X}_0 = \cap_{n\in\mathbb{Z}^+} A_-^n \mathfrak{U}$. Clearly

$$A_-\mathcal{X}_0 \subset \mathcal{X}_0 = \cap_{n\geq 0} A_-^n \mathfrak{U} \subset \cap_{n\geq 1} A_-^n \mathfrak{U} = A_-\mathcal{X}_0.$$

Thus, $A_-\mathcal{X}_0 = \mathcal{X}_0$, and hence $A_0 := A_-|_{\mathcal{X}_0}$ maps $\mathcal{X}_0$ unitarily onto itself. Let $x_0 \in \mathcal{X}_0$, and define $x(n) = A_0^n x_0$, $n \in \mathbb{Z}$. Then $x(n+1) = A_-x(n)$, $n \in \mathbb{Z}$. By the definition of A_-, $\begin{bmatrix} A_-x(n) \\ x(n) \\ 0 \end{bmatrix} \in V$, $n \in \mathbb{Z}$, and hence $(x(\cdot), 0)$ is a bilaterally unobservable trajectory of Σ. By the definitions of $\mathfrak{U}_\Sigma$ and $\mathfrak{U}_\Sigma^\dagger$, this implies that $x(0) = x_0 \in \mathfrak{U}\cap\mathfrak{U}^\dagger$. Thus $\cap_{n\in\mathbb{Z}^+} A_-^n \mathfrak{U} \subset \mathfrak{U}\cap\mathfrak{U}^\dagger$. An analogous argument shows that also $\cap_{n\in\mathbb{Z}^+} A_+^n \mathfrak{U}_\Sigma^\dagger \subset \mathfrak{U}\cap\mathfrak{U}^\dagger$. □

Lemma 5.4. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s system with incoming and outgoing inner channels $(A_+^*; \mathfrak{U}_\Sigma^\dagger)$ and $(A_-; \mathfrak{U}_\Sigma)$, respectively. Then*

$$P_{\mathfrak{U}_\Sigma^\dagger} x_1 = A_+^* P_{\mathfrak{U}_\Sigma^\dagger} x_0 \text{ and } P_{\mathfrak{U}_\Sigma} x_0 = A_-^* P_{\mathfrak{U}_\Sigma} x_1 \text{ whenever } \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V. \tag{5.4}$$

Proof. Let $z_0 \in \mathfrak{U}_\Sigma^\dagger$ and $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V$. By Lemma 5.1, $\begin{bmatrix} z_0 \\ A_+z_0 \\ 0 \end{bmatrix} \in V$ and since $V = V^{[\perp]}$ we get

$$\begin{aligned} 0 &= \left[\left[\begin{bmatrix} z_0 \\ A_+z_0 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix}\right]\right]_{\mathfrak{K}} = -(z_0, x_1)_{\mathcal{X}} + (A_+z_0, x_0)_{\mathcal{X}} \\ &= -(z_0, P_{\mathfrak{U}_\Sigma^\dagger} x_1)_{\mathcal{U}_\Sigma^*} + (A_+z_0, P_{\mathfrak{U}_\Sigma^\dagger} x_0)_{\mathcal{U}_{\Sigma^*}} \\ &= (z_0, -P_{\mathfrak{U}_\Sigma^\dagger} x_1 + A_+^* P_{\mathfrak{U}_\Sigma^\dagger} x_0)_{\mathcal{X}}. \end{aligned}$$

This being true for all $z_0 \in \mathfrak{U}_\Sigma^\dagger$ we have $P_{\mathfrak{U}_\Sigma^\dagger} x_1 = A_+^* P_{\mathfrak{U}_\Sigma^\dagger} x_0$. An analogous computation shows that $P_{\mathfrak{U}_\Sigma} x_0 = A_-^* P_{\mathfrak{U}_\Sigma} x_1$. □

Theorem 5.5. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a conservative s/s system.*

1) *If $(x(\cdot), w(\cdot))$ is a future trajectory of Σ, then*

$$P_{\mathfrak{U}_\Sigma^\dagger} x(n) = (A_+^*)^n P_{\mathfrak{U}_\Sigma^\dagger} x(0), \qquad n \geq 0. \tag{5.5}$$

In particular, $\|P_{\mathfrak{U}_\Sigma^\dagger} x(n)\|_{\mathcal{X}}$ is a nonincreasing function of n.

2) *If $(x(\cdot), w(\cdot))$ is a past trajectory of Σ, then*

$$P_{\mathfrak{U}_\Sigma} x(n) = (A_-^*)^{|n|} P_{\mathfrak{U}_\Sigma} x(0), \qquad n \leq 0. \tag{5.6}$$

In particular, $\|P_{\mathfrak{U}_\Sigma} x(n)\|_{\mathcal{X}}$ is a nonincreasing function of $|n|$.

3) *The following three assertions are equivalent:*

   (a) Σ *is simple;*

   (b) $P_{\mathfrak{U}_\Sigma^\dagger} x(n) \to 0$ *in* $\mathcal{X}$ *as* $n \to \infty$ *for every future trajectory* $(x(\cdot), w(\cdot))$ *of* Σ*;*

   (c) $P_{\mathfrak{U}_\Sigma} x(n) \to 0$ *in* $\mathcal{X}$ *as* $n \to -\infty$ *for every past trajectory* $(x(\cdot), w(\cdot))$ *of* Σ*;*

Proof. That (5.5) and (5.6) hold follows from Lemma 5.4, and the monotonicity of the norm follows from the fact that A_+^* and A_-^* are contractions.

By the Wold decomposition (see, e.g., [SF70, Theorem 1.1, p. 3]), $A_\pm$ is completely non-unitary if and only if $A_\pm^*$ is strongly stable, and hence 3) follows from 1) and 2) combined with Theorem 5.3. □

Suppose that Σ is simple, and denote $\mathfrak{N}_- = \mathcal{N}\left(A_-^*\right)$. Since A_- is completely non-unitary it follows from the Wold decomposition [SF70, Theorem 1.1, p. 3] that $\mathfrak{U}_\Sigma = \oplus_{n=0}^\infty A_-^n \mathfrak{N}_-$. This makes it possible to define a unitary map $U_- \colon \mathfrak{U}_\Sigma \to \ell_-^2(\mathfrak{N}_-)$ by

$$U_- x = \{P_{\mathfrak{N}_-} (A_-^*)^{-(k+1)}\}_{k=-1}^{-\infty}, \qquad x \in \mathfrak{U}_\Sigma. \tag{5.7}$$

The operator U_- intertwines A_- with the outgoing shift on $\ell_-^2(\mathfrak{N}_-)$ in the sense that

$$U_- A_- = S_-^* U_-. \tag{5.8}$$

Analogously we define $\mathfrak{N}_+ = \mathcal{N}\left(A_+^*\right)$. Then $\mathfrak{U}_\Sigma^\dagger = \oplus_{n=0}^\infty A_+^n \mathfrak{N}_+$, and this makes it possible to define a unitary map $U_+ \colon \mathfrak{U}_\Sigma^\dagger \to \ell_+^2(\mathfrak{N}_+)$ by

$$U_+ x = \{P_{\mathfrak{N}_+} (A_+^*)^k\}_{k=0}^\infty, \quad x \in \mathfrak{U}_\Sigma^\dagger. \tag{5.9}$$

The operator U_+ intertwines A_+ with the outgoing shift on $\ell_+^2(\mathfrak{N}_+)$ in the sense that

$$U_+ A_+ = S_+ U_+. \tag{5.10}$$

In the case of the system $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ it is possible to give more explicit formulas for the operators $A_\pm$ defined in Lemma 5.1 and their adjoints.

Lemma 5.6. *In the case of the canonical simple conservative system* $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ *with behavior* $\mathfrak{W}$ *and past/future map* $\Gamma_{\mathfrak{W}}$ *the operators* $A_\mp$

and their adjoints are given by the following formulas:

$$\begin{aligned}
A_-Qw &= QS^{-1}w, && w \in \mathcal{L}(\mathfrak{W}),\ \pi_+w = 0,\\
A_-^*Qw &= \begin{bmatrix} 0 \\ Q_- - \Gamma_{\mathfrak{W}}^*Q_+ \end{bmatrix} Sw, && w \in \mathcal{L}(\mathfrak{W}),\ \pi_+w = 0,\\
A_+Qw &= QSw, && w \in \mathcal{L}(\mathfrak{W}),\ \pi_-w = 0,\\
A_+^*Qw &= \begin{bmatrix} Q_+ - \Gamma_{\mathfrak{W}}Q_- \\ 0 \end{bmatrix} S^{-1}w, && w \in \mathcal{L}(\mathfrak{W}),\ \pi_-w = 0.
\end{aligned} \tag{5.11}$$

Thus, the defect subspaces $\mathfrak{N}_-$ and $\mathfrak{N}_+$ for the system $\Sigma_{\rm sc}$ are given by

$$\begin{aligned}
\mathfrak{N}_- &= \bigl\{Qw \mid w \in \mathcal{L}(\mathfrak{W}) \text{ with } \pi_+w = 0 \text{ and } Q_-Sw = \Gamma_{\mathfrak{W}}^*Q_+Sw\bigr\},\\
\mathfrak{N}_+ &= \bigl\{Qw \mid w \in \mathcal{L}(\mathfrak{W}) \text{ with } \pi_-w = 0 \text{ and } Q_+S^{-1}w = \Gamma_{\mathfrak{W}}Q_-S^{-1}w\bigr\}.
\end{aligned} \tag{5.12}$$

Proof. Below we only prove the formulas for A_- and A_-^*, and leave the proof of the formulas for A_+ and A_+^* to the reader.

By the definition of A_-, for all $x_0 \in \mathfrak{U}$ we have $A_-x_0 = x_1$ where $\left[\begin{smallmatrix} x_1 \\ x_0 \\ 0 \end{smallmatrix}\right] \in V_{\rm sc}$. Since $x_0 \in \mathfrak{U}$ it follows from (3.16) that $x_0 = Qw$ for some $w \in \mathcal{L}(\mathfrak{W})$ satisfying $\pi_+w = 0$. By the definition of $V_{\rm sc}$ and the fact that the first component of $V_{\rm sc}$ is determined uniquely by the last two components we have $x_1 = QS^{-1}w$. This proves the first equation in (5.11).

To compute A_-^* we let $x_0 \in \mathfrak{U}$, and choose some representatives w such that $x_0 = Qw$ with $w \in \mathcal{L}(\mathfrak{W})$ with $\pi_+w = 0$. By Theorem 5.5 and (3.17) , $A_-^*x_0 = P_{\mathfrak{U}}x_{-1} = (\Pi_- - \Gamma_{\mathfrak{W}}^*\Pi_+)x_{-1}$, where $\left[\begin{smallmatrix} x_0 \\ x_{-1} \\ w_{-1} \end{smallmatrix}\right] \in V_{\rm sc}$ for some $w_{-1} \in \mathcal{W}$. One such vector is $\left[\begin{smallmatrix} x_0 \\ x_{-1} \\ w_{-1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} Qw \\ QSw \\ w(-1) \end{smallmatrix}\right]$. Thus,

$$A_-^* = (\Pi_- - \Gamma_{\mathfrak{W}}^*\Pi_+)QSw = Q_-Sw - \Gamma_{\mathfrak{W}}^*Q_+Sw.$$

This proves the second equation in (5.11). □

In the case of the system $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ it is possible to give alternative descriptions of the unobservable and backward unobservable subspaces.

Lemma 5.7. *Let $\Sigma_{\rm sc}^{\mathfrak{W}} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ be the canonical model of a simple conservative s/s system with passive full behavior $\mathfrak{W}$ and past/future map $\Gamma_{\mathfrak{W}}$. Let $\mathfrak{U}$ and $\mathfrak{U}^\dagger$ be the unobservable and backward unobservable subspaces of $\Sigma_{\rm sc}^{\mathfrak{W}}$. Finally, let $\mathcal{H}(\Gamma_{\mathfrak{W}})$ and $\mathcal{H}(\Gamma_{\mathfrak{W}}^*)$ be the de Branges complementary spaces of the contractive operators $\Gamma_{\mathfrak{W}}$ and $\Gamma_{\mathfrak{W}}^*$. Then the following claims hold.*

1) $\mathfrak{U}$ *is given by*

$$\mathfrak{U} = \left\{x \in \mathcal{D}(\mathfrak{W}) \;\middle|\; \Pi_+ x = 0\right\} = \left\{\left[\begin{smallmatrix} 0 \\ x_- \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{smallmatrix}\right] \;\middle|\; x_- \in \mathcal{H}(\Gamma_{\mathfrak{W}}^*)\right\}, \tag{5.13}$$

and

$$\|x\|_{\mathcal{D}(\mathfrak{W})} = \|\Pi_- x\|_{\mathcal{H}(\Gamma_{\mathfrak{W}}^*)}, \qquad x \in \mathfrak{U}. \tag{5.14}$$

Thus, $\Pi_-|_{\mathfrak{U}}$ *is a unitary map from* $\mathfrak{U}$ *onto* $\mathcal{H}(\Gamma_{\mathfrak{W}}^*)$*.*

2) $\mathfrak{U}^\dagger$ *is given by*

$$\mathfrak{U}^\dagger = \left\{x \in \mathcal{D}(\mathfrak{W}) \;\middle|\; \Pi_- x = 0\right\} = \left\{\left[\begin{smallmatrix} x_+ \\ 0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{H}_+ \\ \mathcal{H}_- \end{smallmatrix}\right] \;\middle|\; x_+ \in \mathcal{H}(\Gamma_{\mathfrak{W}})\right\}, \tag{5.15}$$

and

$$\|x\|_{\mathcal{D}(\mathfrak{W})} = \|\Pi_+ x\|_{\mathcal{H}(\Gamma_{\mathfrak{W}}^*)}, \qquad x \in \mathfrak{U}^\dagger. \tag{5.16}$$

Thus, $\Pi_+|_{\mathfrak{U}^\dagger}$ *is a unitary map from* $\mathfrak{U}^\dagger$ *onto* $\mathcal{H}(\Gamma_{\mathfrak{W}})$*.*

3) *The s/s system* $\Sigma_{\rm sc}$ *is observable if and only if* $\Gamma_{\mathfrak{W}}$ *is isometric, or equivalently, if and only if* $\Pi_-|_{\mathcal{D}(\mathfrak{W})} = \Gamma_{\mathfrak{W}}^* \Pi_+|_{\mathcal{D}(\mathfrak{W})}$*.*

4) *The s/s system* $\Sigma_{\rm sc}$ *is controllable if and only if* $\Gamma_{\mathfrak{W}}$ *is co-isometric, or equivalently, if and only if* $\Pi_+|_{\mathcal{D}(\mathfrak{W})} = \Gamma_{\mathfrak{W}} \Pi_-|_{\mathcal{D}(\mathfrak{W})}$*.*

5) *The s/s system* $\Sigma_{\rm sc}$ *is minimal if and only if* $\Gamma_{\mathfrak{W}}$ *is unitary.*

Proof. The first equalities in (5.13) and (5.15) follow from (3.16).

Define

$$\Delta_{\mathfrak{W}} = 1_{\mathcal{H}_-} - \Gamma_{\mathfrak{W}}^* \Gamma_{\mathfrak{W}}, \tag{5.17}$$

and let $x_- = \Delta_{\mathfrak{W}} x_-'$ with $x_-' \in \mathcal{H}_-$. Then

$$\begin{bmatrix} 0 \\ x_- \end{bmatrix} = \begin{bmatrix} 0 \\ x_-' - \Gamma_{\mathfrak{W}}^* \Gamma_{\mathfrak{W}} x_-' \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{H}_+} & \Gamma_{\mathfrak{W}} \\ \Gamma_{\mathfrak{W}}^* & 1_{\mathcal{H}_-} \end{bmatrix} \begin{bmatrix} -\Gamma_{\mathfrak{W}} x_-' \\ x_-' \end{bmatrix} = A_{\mathfrak{W}} \begin{bmatrix} -\Gamma_{\mathfrak{W}} x_-' \\ x_-' \end{bmatrix},$$

where $A_{\mathfrak{W}}$ is the operator in (3.3). Consequently, $\left[\begin{smallmatrix} 0 \\ x_- \end{smallmatrix}\right] \in \mathcal{R}\left(A_{\mathfrak{W}}\right) \subset \mathcal{D}(\mathfrak{W})$, and

$$\begin{aligned}
\left\| \begin{bmatrix} 0 \\ x_- \end{bmatrix} \right\|_{\mathcal{D}(\mathfrak{W})}^2 &= \left\| A_{\mathfrak{W}}^{1/2} \begin{bmatrix} -\Gamma_{\mathfrak{W}} x_- \\ x_- \end{bmatrix} \right\|_{\mathcal{H}_+ \oplus \mathcal{H}_-}^2 \\
&= \left(\begin{bmatrix} -\Gamma_{\mathfrak{W}} x_- \\ x_- \end{bmatrix}, A_{\mathfrak{W}} \begin{bmatrix} -\Gamma_{\mathfrak{W}} x_- \\ x_- \end{bmatrix} \right)_{\mathcal{H}_+ \oplus \mathcal{H}_-} \\
&= \left(\begin{bmatrix} -\Gamma_{\mathfrak{W}} x_- \\ x_- \end{bmatrix}, \begin{bmatrix} 0 \\ x_- \end{bmatrix} \right)_{\mathcal{H}_+ \oplus \mathcal{H}_-} \\
&= (x_-, x_-)_{\mathcal{H}_-} = \left(x_-', \Delta_{\mathfrak{W}} x_-'\right)_{\mathcal{H}_-} = \|x_-\|_{\mathcal{H}(\Gamma_{\mathfrak{W}}^*)}^2.
\end{aligned}$$

Thus,

$$\left\{ \left[\begin{smallmatrix} 0 \\ x_- \end{smallmatrix}\right] \;\middle|\; x_- \in \mathcal{R}\,(\Delta_{\mathfrak{W}}) \right\} \subset \mathcal{D}(\mathfrak{W}) \cap (0 \oplus \mathcal{H}_-) = \mathfrak{U},$$

and (5.14) holds for $x_- \in \mathcal{R}\,(\Delta_{\mathfrak{W}})$. Since $\mathcal{R}\,(\Delta_{\mathfrak{W}})$ is a dense subspace of $\mathcal{H}(\Gamma^*_{\mathfrak{W}})$, we find that (5.14) holds all $x_- \in \mathcal{H}(\Gamma^*_{\mathfrak{W}})$, and that $\mathfrak{U}_0 := \left\{ \left[\begin{smallmatrix} 0 \\ x_- \end{smallmatrix}\right] \;\middle|\; x_- \in \mathcal{H}(\Gamma^*_{\mathfrak{W}}) \right\}$ is a closed subspace of $\mathfrak{U}$. To prove that $\mathfrak{U}_0 = \mathfrak{U}$ we let $\left[\begin{smallmatrix} 0 \\ x_- \end{smallmatrix}\right] \in \mathfrak{U}$ be orthogonal to $\mathfrak{U}_0$ in $\mathcal{D}(\mathfrak{W})$. Then, for all $x_- \in \mathcal{H}_-$,

$$0 = \left\langle \begin{bmatrix} 0 \\ x_- \end{bmatrix}, A_{\mathfrak{W}} \begin{bmatrix} -\Gamma_{\mathfrak{W}} x'_- \\ x'_- \end{bmatrix}_{\mathcal{D}(\mathfrak{W})} \right\rangle = ((x_-, x'_-)_{\mathcal{H}_-},$$

which implies that $x_- = 0$. This proves assertion 1).

Assertion 2) may be proved in an analogous way, with $\Gamma_{\mathfrak{W}}$ replaced by $\Gamma^*_{\mathfrak{W}}$. Assertions 3) and 4) follows from assertions 1) and 2) and (3.17), and assertion 5) follows from assertions 3) and 4). □

For use in some subsequent work we record the following fact.

Proposition 5.8. *Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a simple conservative s/s system for which both the incoming inner channel $(A^*_+; \mathfrak{U}^\dagger)$ and the outgoing inner channel $(A_1; \mathfrak{U})$ are nontrivial. Let $\Gamma \in \mathcal{B}(\mathfrak{U}_\Sigma; \mathfrak{U}^\dagger_\Sigma)$ be the operator*

$$\Gamma = P_{\mathfrak{U}^\dagger_\Sigma}|_{\mathfrak{U}_\Sigma}. \tag{5.18}$$

Then

$$\Gamma A_- = A^*_+ \Gamma \tag{5.19}$$

and

$$\Gamma^*\Gamma < 1_{\mathfrak{U}_\Sigma}, \qquad \Gamma\Gamma^* < 1_{\mathfrak{U}^\dagger_\Sigma}. \tag{5.20}$$

Proof. That Γ has property (5.20) follows from its definition (5.18) and the fact that $\mathfrak{U}_\Sigma \cap \mathfrak{U}^\dagger_\Sigma = 0$. To check the relation (5.19) we take some arbitrary $x_- \in \mathfrak{U}_\Sigma$ and $x_+ \in \mathfrak{U}^\dagger_\Sigma$. Since both $\left[\begin{smallmatrix} A_- x_- \\ x_- \\ 0 \end{smallmatrix}\right] \in V$ and $\left[\begin{smallmatrix} x_+ \\ A_+ x_+ \\ 0 \end{smallmatrix}\right] \in V$ and $V = V^{[\perp]}$, we have

$$(A_- x_-, x_+)_{\mathcal{X}} = (x_-, A_+ x_+)_{\mathcal{X}}.$$

Thus,

$$\begin{aligned}(\Gamma A_- x_-, x_+)_{\mathcal{X}} &= (P_{\mathfrak{U}^\dagger_\Sigma}|_{\mathfrak{U}_\Sigma} A_- x_-, x_+)_{\mathcal{X}} = (A_- x_-, x_+)_{\mathcal{X}} = (x_-, A_+ x_+)_{\mathcal{X}} \\ &= (x_-, P_{\mathfrak{U}_\Sigma}|_{\mathfrak{U}^\dagger_\Sigma} A_+ x_+)_{\mathcal{X}} = (x_-, \Gamma^* A_+ x_+)_{\mathcal{X}}.\end{aligned}$$

This proves (5.19). □

An operator $\Gamma \in \mathcal{B}(\mathfrak{U}_\Sigma; \mathfrak{U}_\Sigma^\dagger)$ satisfying the intertwining condition (5.19) with respect to the isometric completely non-unitary operator A_- and the co-isometric completely non-unitary operator A_+^* is usually called a *Hankel operator*. By (5.20), the Hankel operator Γ defined in (5.19) is a contraction which does not have any singular numbers on the unit circle.

6 Alternative Characterizations of $\mathcal{L}(\mathfrak{W})$ and $\mathfrak{D}(\mathfrak{W})$.

Let $\mathfrak{W}$ be a full passive behavior on the Kreĭn signal space $\mathcal{W}$, let $\mathfrak{W}_+ = \mathfrak{W} \cap k_+^2(\mathcal{W})$ and $\mathfrak{W}_- = \pi_- \mathfrak{W}$ be the corresponding future and past passive behaviors, and denote $\mathcal{H}_+ = \mathcal{H}(\mathfrak{W}_+)$ and $\mathcal{H}_- = \mathcal{H}(\mathfrak{W}_-^{[\perp]})$.

For each $n \in \mathbb{Z}^+$ we define

$$\mathcal{L}_n^-(\mathfrak{W}) := \{w(\cdot) \in k^2(\mathcal{W}) \mid Q_- S^{-n} w \in \mathcal{H}_-\}, \tag{6.1}$$

$$\rho_n^-(w) = \|Q_- S^{-n} w\|_{\mathcal{H}_-}^2 - \sum_{k=0}^{n-1} [w(k), w(k)]_{\mathcal{W}}, \quad w \in \mathcal{L}_n^-(\mathfrak{W}). \tag{6.2}$$

If $w \in \mathcal{L}(\mathfrak{W})$, then by Lemma 3.4, $Q_- S^{-n} w \in \mathcal{H}_-$ for all $n \in \mathbb{Z}^+$, and consequently $\mathcal{L}(\mathfrak{W}) \subset \mathcal{L}_n^-(\mathfrak{W})$ for all $n \in \mathbb{Z}^+$.

Theorem 6.1. *Let $\mathfrak{W}$ be a passive full behavior on $\mathcal{W}$.*

1) *A sequence $w \in k^2(\mathcal{W})$ belongs to $\mathcal{L}(\mathfrak{W})$ if and only if $Q_- S^{-n} w \in \mathcal{H}_-$ for all $n \in \mathbb{Z}^+$ and*

$$\sup_{n \in \mathbb{Z}^+} \|Q_- S^{-n} w\|_{\mathcal{H}_-} < \infty. \tag{6.3}$$

2) *If $w \in \mathcal{L}(\mathfrak{W})$, then the sequence $\rho_n^-(w)$ is nonnegative, nondecreasing and bounded, and*

$$\begin{aligned} \|Qw\|_{\mathcal{D}(\mathfrak{W})}^2 = \|w\|_{\mathcal{L}(\mathfrak{W})}^2 &= \sup_{n \in \mathbb{Z}^+} \rho_n^-(w) = \lim_{n \to \infty} \rho_n^-(w) \\ &= \lim_{n \to \infty} \|Q_- S^{-n} w\|_{\mathcal{H}_-}^2 - [\pi_+ w, \pi_+ w]_{k_+^2(\mathcal{W})}. \end{aligned} \tag{6.4}$$

Proof. Step 1: If (6.3) *holds, then $w \in \mathcal{L}(\mathfrak{W})$.* For each $n \in \mathbb{Z}^+$, the vector $y_n := \left[\begin{smallmatrix} \Gamma_{\mathfrak{M}} \\ 1_{\mathcal{H}_-} \end{smallmatrix}\right] Q_- S^{-n} w$ belongs to $\mathcal{D}(\mathfrak{W})$, and as the sequence $Q_- S^{-n} w \in \mathcal{H}_-$ is assumed to be uniformly bounded in $\mathcal{H}_-$, the sequence y_n is uniformly bounded in $\mathcal{D}(\mathfrak{W})$. Let R_+ be a right-inverse of the quotient map Q_+, and define w_n, $n \geq 1$, by

$$\begin{aligned} w_n &= S^n\big(R_+\Gamma_{\mathfrak{W}}Q_-S^{-n}w + \pi_-S^{-n}w\big) \\ &= S^nR_+\Gamma_{\mathfrak{W}}Q_-S^{-n} + P_{k^2((-\infty,n-1];\mathcal{W})}w. \end{aligned}$$

Then $w_n(k) = w(k)$ for $k \leq n-1$, w_n is uniformly bounded in $k^2(\mathcal{W})$, and $y_n = QS^{-n}w_n$. Finally, define $x_n = Qw_n$. Then, by Lemma 3.4,

$$\begin{aligned} \|x_n\|^2_{\mathcal{D}(\mathfrak{W})} = \|Qw_n\|^2_{\mathcal{D}(\mathfrak{W})} &= \|QS^{-n}w_n\|^2_{\mathcal{D}(\mathfrak{W})} - \sum_{k=0}^{n-1}[w_n(k), w_n(k)]_{\mathcal{W}} \\ &= \|y_n\|^2_{\mathcal{D}(\mathfrak{W})} - \sum_{k=0}^{n-1}[w(k), w(k)]_{\mathcal{W}}, \end{aligned}$$

and hence the sequence x_n is uniformly bounded in $\mathcal{D}(\mathfrak{W})$. Since the unit ball in $\mathcal{D}(\mathfrak{W})$ is weakly compact, we can without loss of generality (by passing to a subsequence) suppose that x_n tends weakly to a limit $x \in \mathcal{D}(\mathfrak{W})$, and hence also in $k^2(\mathcal{W})/(\mathfrak{W}_+ + \mathfrak{W}_-^{[\perp]})$. Since $w_n(k) = w(k)$ for $k \leq n$, it is also true that w_n tends weakly to w in $k^2(\mathcal{W})$ as $n \to \infty$. Let $R\colon k^2(\mathcal{W})/(\mathfrak{W}_+ + \mathfrak{W}_-^{[\perp]}) \to k^2(\mathcal{W})$ be a bounded right-inverse of Q. Then, on one hand, RQw_n tends weakly to RQw in $k^2(\mathcal{W})$, and on the other hand,

$$RQw_n = Rx_n \to Rx \text{ weakly as } n \to \infty.$$

Therefore, $RQw = Rx$. Since R is injective, this implies that $Qw = x \in \mathcal{D}(\mathfrak{W})$, and consequently, $x \in \mathcal{L}(\mathfrak{W})$.

Step 2: If $w \in \mathcal{L}(\mathfrak{W})$, then (6.3) *and* (6.4) *hold.* It follows from Lemma 3.4 that if $w \in \mathcal{L}(\mathfrak{W})$, then $QS^{-n}w \in \mathcal{L}(\mathfrak{W})$, and since $\mathcal{D}(\mathfrak{W})$ is continuously contained in $\mathcal{H}_+ \oplus \mathcal{H}_-$, this implies that $Q_-S^{-n}w \in \mathcal{H}_-$ for all $n \in \mathbb{Z}^+$.

Let$(x(\cdot), w(\cdot))$ be the (unique) stable full trajectory of Σ_{sc} whose signal part is w, i.e., $x(n) = QS^{-n}w$, $n \in \mathbb{Z}$. By the conservativity of Σ_{sc},

$$\|x(n)\|^2_{\mathcal{D}(\mathfrak{W})} = \|x(0)\|^2_{\mathcal{D}(\mathfrak{W})} + \sum_{n=0}^{n}[w(n), w(n)]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+.$$

Write

$$\|x(n)\|^2_{\mathcal{D}(\mathfrak{W})} = \|P_{\mathfrak{R}}x(n)\|^2_{\mathcal{D}(\mathfrak{W})} + \|P_{\mathfrak{U}^\dagger}x(n)\|^2_{\mathcal{D}(\mathfrak{W})},$$

where

$$\|P_{\mathfrak{R}}x(n)\|^2_{\mathcal{D}(\mathfrak{W})} = \|P_{\mathfrak{R}}QS^{-n}w\|^2_{\mathcal{D}(\mathfrak{W})} = \left\| \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} \Pi_-QS^{-n}w \right\|^2_{\mathcal{D}(\mathfrak{W})} = \left\|Q_-S^{-n}w\right\|^2_{\mathcal{H}_-}.$$

Thus,

$$\begin{aligned}\rho_n^-(w) &= \|Q_-S^{-n}w\|^2_{\mathcal{H}_-} - \sum_{k=0}^{n-1}[w(k),w(k)]_{\mathcal{W}} \\ &= \|P_{\mathfrak{R}}x(n)\|^2_{\mathcal{D}(\mathfrak{W})} - \sum_{k=0}^{n-1}[w(k),w(k)]_{\mathcal{W}} \\ &= \|x(0)\|^2_{\mathcal{D}(\mathfrak{W})} - \|P_{\mathfrak{U}^\dagger}x(n)\|^2_{\mathcal{D}(\mathfrak{W})} \\ &= \|w\|^2_{\mathcal{L}(\mathfrak{W})} - \|P_{\mathfrak{U}^\dagger}x(n)\|^2_{\mathcal{D}(\mathfrak{W})}.\end{aligned}$$

Thus $\rho_n^-(w) \geq 0$ since $\|P_{\mathfrak{U}^\dagger}x(n)\|_{\mathcal{D}(\mathfrak{W})} \leq \|w\|_{\mathcal{L}(\mathfrak{W})}$. By Theorem 5.5, the sequence $\|P_{\mathfrak{U}^\dagger}x(n)\|^2_{\mathcal{D}(\mathfrak{W})}$ is nonincreasing and tends to zero as $n \to \infty$, and hence the sequence $\rho_n^-(w)$ is nondecreasing and tends to $\|w\|^2_{\mathcal{L}(\mathfrak{W})}$ as $n \to \infty$. □

Above we looked at the behavior of the sequence $Q_-S^{-n}w$ in $\mathcal{H}_-$ as $n \to \infty$, and related this to the condition $w \in \mathcal{L}(\mathfrak{W})$. It is also possible to instead look at how the sequence Q_+S^nw behaves in $\mathcal{H}_+$ as $n \to \infty$. For each $n \in \mathbb{Z}^+$ we define

$$\mathcal{L}_n^+(\mathfrak{W}) := \{w(\cdot) \in k^2(\mathcal{W}) \mid Q_+S^nw \in \mathcal{H}_+\}, \tag{6.5}$$

$$\rho_n^+(w) := \|Q_+S^nw\|^2_{\mathcal{H}_+} - \sum_{k=-n}^{-1}[w(k),w(k)]_{\mathcal{W}}, \quad w \in \mathcal{L}_n^+(\mathfrak{W}). \tag{6.6}$$

If $w \in \mathcal{L}(\mathfrak{W})$, then by Lemma 3.4, $Q_+S^nw \in \mathcal{H}_+$ for all $n \in \mathbb{Z}^+$, and consequently $\mathcal{L}(\mathfrak{W}) \subset \mathcal{L}_n^+(\mathfrak{W})$ for all $n \in \mathbb{Z}^+$.

Theorem 6.2. *Let $\mathfrak{W}$ be a passive full behavior on $\mathcal{W}$.*

1) *A sequence $w \in k^2(\mathcal{W})$ belongs to $\mathcal{L}(\mathfrak{W})$ if and only if $Q_+S^nw \in \mathcal{H}_+$ for all $n \in \mathbb{Z}^+$ and*
$$\sup_{n\in\mathbb{Z}^+}\|Q_+S^nw\|_{\mathcal{H}_+} < \infty. \tag{6.7}$$

2) *If $w \in \mathcal{L}(\mathfrak{W})$, then the sequence $\rho_n^+(w)$ is nonnegative, nondecreasing and bounded, and*
$$\begin{aligned}\|Qw\|^2_{\mathcal{D}(\mathfrak{W})} = \|w\|^2_{\mathcal{L}(\mathfrak{W})} &= \sup_{n\in\mathbb{Z}^+}\rho_n^+(w) = \lim_{n\to\infty}\rho_n^+(w) \\ &= \lim_{n\to\infty}\|Q_+S^nw\|^2_{\mathcal{H}_+} + [\pi_-w,\pi_-w]_{k^2_-(\mathcal{W})}.\end{aligned} \tag{6.8}$$

Proof. This proof is analogous to the proof of Theorem 6.1. One throughout interchanges $k^2_-(\mathcal{W})$ and $k^2_+(\mathcal{W})$, π_- and π_+, Q_- and Q_+, S^{-1} and S, and $\mathcal{H}_-$ and $\mathcal{H}_+$. (However, $\mathfrak{W}$ and $\mathfrak{W}^{[\perp]}$ should *not* be interchanged.) □

Lemma 6.3. *Let* $w \in \mathcal{L}(\mathfrak{W})$.

1) *The following conditions are equivalent:*

   (a) $w \in \mathfrak{W}$;

   (b) $\|w\|^2_{\mathcal{L}(\mathfrak{W})} = [\pi_- w, \pi_- w]_{k^2_-(\mathcal{W})}$;

   (c) $\lim_{n\to\infty} \|Q_- S^{-n} w\|^2_{\mathcal{H}_-} = [w, w]_{k^2(\mathcal{W})}$;

   (d) $\lim_{n\to\infty} \|Q_+ S^n w\|^2_{\mathcal{H}_+} = 0$.

2) *The following conditions are equivalent:*

   (e) $w \in \mathfrak{W}^{[\perp]}$;

   (f) $\lim_{n\to\infty} \|Q_- S^{-n} w\|^2_{\mathcal{H}_-} = 0$;

   (g) $\lim_{n\to\infty} \|Q_+ S^n w\|^2_{\mathcal{H}_+} = [w, w]_{k^2(\mathcal{W})}$;

   (h) $\|w\|^2_{\mathcal{L}(\mathfrak{W})} = -[\pi_+ w, \pi_+ w]_{k^2_+(\mathcal{W})}$.

Proof. (a) $\Rightarrow$ *(b):* This follows from Lemma 3.3.

(b) $\Leftrightarrow$ *(c):* This follows from Theorem 6.1.

(b) $\Leftrightarrow$ *(d):* This follows from Theorem 6.2.

(d) $\Rightarrow$ *(a):* Let $(x(\cdot), w(\cdot))$ be the unique stable full trajectory of $\Sigma_{\rm sc}$ with signal part $w(\cdot)$, i.e., $x(n) = QS^{-n}w$ for all $n \in \mathbb{Z}$. We decompose $x(n)$ in two orthogonal components, $x(n) = P_{\mathfrak{R}^\dagger} x(n) + P_{\mathfrak{U}} x(n)$. By Theorem 5.5, $P_{\mathfrak{U}} x(n) \to 0$ as $n \to -\infty$, and by (3.17), $P_{\mathfrak{R}^\dagger} x(n) = \begin{bmatrix} Q_+ \\ \Gamma^*_{\mathfrak{W}} Q_+ \end{bmatrix} S^{-n} w$, which tends to zero as $n \to -\infty$ if (d) holds. Thus, $(x(\cdot), w(\cdot))$ is an externally generated trajectory of $\Sigma_{\rm sc}$, and so $w \in \mathfrak{W}$.

Proof of 2). This proof is analogous to the one above. □

7 Forward and Backward Conservative Compressions of the Conservative Model.

In Section 2.4 we presented two additional canonical models, namely the controllable backward passive model $\Sigma^{\mathfrak{W}_-}_{\rm cfc}$, and the observable backward passive model $\Sigma^{\mathfrak{W}_+}_{\rm obc}$, which were originally obtained in [AS09b]. Here we shall study the relationships between these two models and the model $\Sigma_{\rm sc}$ presented in Section 3. As we shall see, the two models in Section 2.4 can be obtained from $\Sigma_{\rm sc}$ by first performing an orthogonal compression, and then applying a unitary similarity transform.

We recall from [AS09b] that the s/s system $\widetilde{\Sigma} = (\widetilde{V}; \widetilde{\mathcal{X}}, \mathcal{W})$ is called an *orthogonal dilation* of the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and Σ is called an *orthogonal compression onto* $\mathcal{X}$ of $\widetilde{\Sigma}$, if $\mathcal{X} \subset \widetilde{\mathcal{X}}$ and

$$V = \begin{bmatrix} P_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \left(\widetilde{V} \cap \begin{bmatrix} \widetilde{\mathcal{X}} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \right), \tag{7.1}$$

If, in addition,

$$\begin{bmatrix} P_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \left(\widetilde{V} \cap \begin{bmatrix} \widetilde{\mathcal{X}} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \right) = \begin{bmatrix} P_{\mathcal{X}} & 0 & 0 \\ 0 & P_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \widetilde{V}, \tag{7.2}$$

then $\widetilde{\Sigma}$ is called an *outgoing* dilation of Σ and Σ is called an outgoing compression of $\widetilde{\Sigma}$. If instead, in addition to (7.1), we have

$$\begin{bmatrix} P_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \left(\widetilde{V} \cap \begin{bmatrix} \widetilde{\mathcal{X}} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix} \right) = \widetilde{V} \cap \begin{bmatrix} \widetilde{\mathcal{X}} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}, \tag{7.3}$$

then $\widetilde{\Sigma}$ is called an *incoming* dilation of Σ and Σ is called an incoming compression of $\widetilde{\Sigma}$. The orthogonal compression of a passive s/s system is passive, and every passive system has a conservative orthogonal s/s dilation. A passive s/s system and its orthogonal passive s/s dilation have the same past, full, and future behaviors.

By compressing $\Sigma_{\rm sc}$ orthogonally onto $\mathfrak{R}$ we get a controllable forward conservative s/s system $\Sigma_{\rm cfc} = (V_{\rm cfc}; \mathfrak{R}, \mathcal{W})$, and by compressing $\Sigma_{\rm sc}$ orthogonally onto $\mathfrak{R}^\dagger$ we get an observable backward conservative s/s $\Sigma_{\rm obc} = (V_{\rm obc}; \mathfrak{R}^\dagger, \mathcal{W})$, both of which have the same future, full, and past behaviors as $\Sigma_{\rm sc}$.

The generating subspace $V_{\rm cfc}$ of the compression $\Sigma_{\rm cfc}$ of $\Sigma_{\rm sc}$ to $\mathfrak{R}$ is given by

$$\begin{aligned} V_{\rm cfc} &= \begin{bmatrix} P_{\mathfrak{R}} & 0 & 0 \\ 0 & 1_{\mathfrak{R}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \left(V_{\rm sc} \cap \begin{bmatrix} \mathcal{D}(\mathfrak{W}) \\ \mathfrak{R} \\ \mathcal{W} \end{bmatrix} \right) = V_{\rm sc} \cap \begin{bmatrix} \mathcal{D}(\mathfrak{W}) \\ \mathfrak{R} \\ \mathcal{W} \end{bmatrix} \\ &= \left\{ \begin{bmatrix} QS^{-1}w \\ Qw \\ w(0) \end{bmatrix} \,\middle|\, w \in \mathcal{L}(\mathfrak{W}),\ Q_+ w = \Gamma_{\mathfrak{W}} Q_- w \right\}. \end{aligned} \tag{7.4}$$

Thus, this is an incoming compression of $\Sigma_{\rm sc}$. That the two different formulas for $V_{\rm cfc}$ given above are the same follows from the fact $\mathfrak{R}$ is strongly invariant in the sense that $x_1 \in \mathfrak{R}$ whenever $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V$ and $x_0 \in \mathfrak{R}$. The subspace $V_{\rm cfc}$ can be compared to the generating subspace $V_{\rm cfc}^{\mathfrak{W}-}$ of the canonical controllable and forward conservative model $\Sigma_{\rm cfc}^{\mathfrak{W}-} = (V_{\rm cfc}^{\mathfrak{W}-}; \mathcal{H}_-, \mathcal{W})$ described in Theorem 2.16. These two systems are unitarily similar, with the similarity operator $\mathfrak{B}_{\Sigma_{\rm sc}} = \begin{bmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}_-} \end{bmatrix} : \mathcal{H}_- \to \mathfrak{R}$, whose inverse is $\Pi_-|_{\mathfrak{R}}$.

The generating subspace V_{obc} of the compression Σ_{obc} of Σ_{sc} to $\mathfrak{R}^\dagger$ is given by

$$\begin{aligned} V_{\mathrm{obc}} &= \begin{bmatrix} P_{\mathfrak{R}^\dagger} & 0 & 0 \\ 0 & 1_{\mathfrak{R}^\dagger} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \left(V_{\mathrm{sc}} \cap \begin{bmatrix} \mathcal{D}(\mathfrak{W}) \\ \mathfrak{R}^\dagger \\ \mathcal{W} \end{bmatrix} \right) = \begin{bmatrix} P_{\mathfrak{R}^\dagger} & 0 & 0 \\ 0 & P_{\mathfrak{R}^\dagger} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_{\mathrm{sc}} \\ &= \left\{ \begin{bmatrix} \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix} Q_+ S^{-1} w \\ \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix} Q_+ w \\ w(0) \end{bmatrix} \middle| \, w \in \mathcal{L}(\mathfrak{W}) \right\}. \end{aligned} \tag{7.5}$$

Thus, this is an outgoing compression. That the two different formulas for V_{obc} are equivalent follows from the fact that $x_1 \in \mathfrak{U} = (\mathfrak{R}^\dagger)^\perp$ whenever $\begin{bmatrix} x_1 \\ x_0 \\ 0 \end{bmatrix} \in V_{\mathrm{sc}}$ and $x_0 \in \mathfrak{U}$; cf. Lemma 5.1. Here the right-hand side depends only on the projection $w_+ := \pi_+ w$ of w. By Theorem 6.2, $w_+ \in \mathcal{K}(\mathcal{W}_+)$. Conversely, if $w_+ \in \mathcal{K}(\mathcal{W}_+)$, the w_+ can be written in the form $w_+ = \pi_+ w$ where $w \in \mathcal{L}(\mathfrak{W})$ is an arbitrary sequence satisfying $Qw = \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix} Q_+ w_+$ and $\pi_+ w = w_+$; that such a sequence exists follows from the fact that $\mathfrak{C}_{\Sigma_{\mathrm{sc}}}^* = \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix}$ maps $\mathcal{H}_+$ into $\mathcal{D}(\mathfrak{W})$. Therefore we can rewrite (7.5) in the form

$$V_{\mathrm{obc}} = \left\{ \begin{bmatrix} \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix} Q_+ S_+^* w_+ \\ \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix} Q_+ w_+ \\ w_+(0) \end{bmatrix} \middle| \, w_+ \in \mathcal{K}(\mathfrak{W}_+) \right\}. \tag{7.6}$$

This can be compared to the generating subspace $V_{\mathrm{obc}}^{\mathfrak{W}_+}$ of the canonical observable and backward passive model $\Sigma_{\mathrm{obc}}^{\mathfrak{W}_+} = (V_{\mathrm{obc}}^{\mathfrak{W}_+}; \mathcal{H}_+, \mathcal{W})$ presented in Theorem 2.11. These two systems are unitarily similar, with the similarity operator $\mathfrak{C}_{\Sigma_{\mathrm{sc}}}^* = \begin{bmatrix} 1_{\mathcal{H}_+} \\ \Gamma_{\mathfrak{W}}^* \end{bmatrix} : \mathcal{H}_+ \to \mathfrak{R}^\dagger$, whose inverse is $\Pi_+|_{\mathfrak{R}^\dagger}$.

We now want to investigate the connections between Σ_{sc} and the two compressions defined above in more detail. Here the results described in Section 5 again become important. These results describe the part of the dynamics which stays in the unobservable subspace $\mathfrak{U}$ or the backward unobservable subspace $\mathfrak{U}^\dagger$. To get a complete picture we also have to describe the part of the dynamics that crosses over between these subspaces and the reachable subspace $\mathfrak{R}$ or the null controllable subspace $\mathfrak{R}^\dagger$, respectively. Here we only directly look at the simple conservative canonical model, but the results can easily be adapted to an arbitrary simple conservative system by using the unitary similarity described in Part 7) of Theorem 4.1.

Lemma 7.1. *Define* V_{obc} *by* (7.5) *let* $\mathfrak{N}_- = \mathcal{N}\left(A_-^*\right)$. *Then the formula*

$$X_o \begin{bmatrix} P_{\mathfrak{R}^\dagger} Q S^{-1} w \\ Qw \\ w(0) \end{bmatrix} = P_{\mathfrak{U}} Q S^{-1} w, \;\; w \in \mathcal{L}(\mathfrak{W}), \;\; Qw \in \mathfrak{R}^\dagger, \tag{7.7}$$

defines a bounded linear operator X_o from V_{obc} onto $\mathfrak{N}_-$ with

$$\mathcal{N}\left(X_o\right) = \begin{bmatrix} 1_{\mathfrak{R}^\dagger} & 0 & 0 \\ 0 & R_{\mathfrak{R}^\dagger} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \left(V_{\mathrm{sc}} \cap \begin{bmatrix} \mathfrak{R}^\dagger \\ \mathcal{D}(\mathfrak{W}) \\ \mathcal{W} \end{bmatrix} \right) = V_{\mathrm{sc}} \cap \begin{bmatrix} \mathfrak{R}^\dagger \\ \mathcal{D}(\mathfrak{W}) \\ \mathcal{W} \end{bmatrix}. \tag{7.8}$$

This operator is isometric with respect to the inner product that V_{obc} inherits from $\mathfrak{K}_o := -\mathfrak{R}^\dagger \,[\dot{+}]\, \mathfrak{R}^\dagger \,[\dot{+}]\, \mathcal{W}$.

Proof. Let $x_0 \in \mathfrak{R}^\dagger$, and choose some $w \in \mathcal{L}(\mathfrak{W})$ such that $Qw = x_0$. By Theorem 3.5,

$$\begin{bmatrix} QS^{-1}w \\ x_0 \\ w(0) \end{bmatrix} \in V_{\mathrm{sc}}$$

and hence by the conservativity of Σ_{sc},

$$-\|QS^{-1}w\|^2_{\mathcal{D}(\mathfrak{W})} + \|x_0\|^2_{\mathcal{D}(\mathfrak{W})} + [w(0), w(0)]_{\mathcal{W}} = 0.$$

Here we split $QS^{-1}w$ into two orthogonal components $QS^{-1}w = x_1 + z_1$, where

$$x_1 := P_{\mathfrak{R}^\dagger} QS^{-1}w \in \mathfrak{R}^\dagger, \qquad z_1 := P_{\mathfrak{U}} QS^{-1}w \in \mathfrak{U}.$$

This gives

$$\begin{aligned} \|z_1\|^2_{\mathcal{D}(\mathfrak{W})} &= \|x_0\|^2_{\mathcal{D}(\mathfrak{W})} + [w(0), w(0)]_{\mathcal{W}} - \|x_1\|^2_{\mathcal{D}(\mathfrak{W})} \\ &= \left[\begin{bmatrix} x_1 \\ x_0 \\ w(0) \end{bmatrix}, \begin{bmatrix} x_1 \\ x_0 \\ w(0) \end{bmatrix} \right]_{-\mathfrak{R}^\dagger [\dot{+}] \mathfrak{R}^\dagger [\dot{+}] \mathcal{W}}. \end{aligned}$$

By (7.5), $\begin{bmatrix} x_1 \\ x_0 \\ w(0) \end{bmatrix} \in V_{\mathrm{obc}}$. This shows that (7.7) defines an isometric map X_o from V_{obc} into $\mathfrak{U}$ whose kernel is the maximal neutral subspace of V_{obc}. This subspace is equal to the orthogonal complement to V_{obc} in $\mathfrak{K}_o$ since Σ_{obc} is backward conservative, and it is not difficult to show that it is explicitly given by (7.8). That the two different expressions for $\mathcal{N}\left(X_o\right)$ are equivalent follows from the fact that $x_0 \in \mathfrak{R}^\dagger$ whenever $\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V_{\mathrm{sc}}$ and $x_1 \in \mathfrak{R}^\dagger$.

It remains to show that the $\mathcal{R}\left(X_o\right) = \mathfrak{N}_-$. Let $z_1 \in \mathcal{R}\left(X_o\right)$. Then by the definition of X_o, there exists $w_0 \in \mathcal{W}$, $x_0 \in \mathfrak{R}^\dagger$, and $x_1 \in \mathfrak{R}^\dagger$ such that $\begin{bmatrix} z_1 + x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V_{\mathrm{sc}}$. By Lemma 5.6,

$$0 = P_{\mathfrak{U}} x_0 = A_-^* P_{\mathfrak{U}}(z_1 + x_1) = A_-^* z_1.$$

Thus, $z_1 \in \mathcal{N}\left(A_-^*\right) = \mathfrak{N}_-$. Conversely, suppose that $z_1 \in \mathfrak{N}_-$, i.e., that $z_1 \in \mathfrak{U}$ and $A_-^* z_1 = 0$. Since Σ_{sc} is well-posed in the backward time direction it is possible to find some $x_0 \in \mathcal{D}(\mathfrak{W})$ and $w_0 \in \mathfrak{W}$ such that $\begin{bmatrix} z_1 \\ x_0 \\ w_0 \end{bmatrix} \in V_{\mathrm{sc}}$.

By Lemma 5.6, $P_{\mathfrak{U}} x_0 = A_-^* P_{\mathfrak{U}} z_1 = A_-^* z_1 = 0$, and so $x_0 \in \mathfrak{R}^\dagger$. Thus, by the definition of X_o, $z_1 = X_o \left[\begin{smallmatrix} 0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$, and so $z_1 \in \mathcal{R}(X_o)$. This proves that $\mathcal{R}(X_o) = \mathfrak{N}_-$. □

Theorem 7.2. *Let $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ be the canonical model of a simple conservative s/s system with passive full behavior $\mathfrak{W}$, and let $\Sigma_{\rm obc} = (V_{\rm obc}; \mathfrak{R}^\dagger, \mathcal{W})$ be the orthogonal compression of $\Sigma_{\rm sc}$ onto the null controllable subspace $\mathfrak{R}^\dagger$ of $\Sigma_{\rm sc}$. Let $(A_-; \mathfrak{U})$ be the outgoing inner channel of $\Sigma_{\rm sc}$, and define the isometric operator X_o from $V_{\rm obc}$ onto $\mathfrak{N}_- = \mathcal{N}(A_-^*)$ by (7.7). Then*

$$V_{\rm sc} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] + \begin{bmatrix} X_o \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \\ 0 \\ 0 \end{bmatrix} + \left[\begin{smallmatrix} A_- z_0 \\ z_0 \\ 0 \end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V_{\rm obc},\ z_0 \in \mathfrak{U} \right\}. \tag{7.9}$$

Proof. Let $\left[\begin{smallmatrix} x_1+z_1 \\ x_0+z_0 \\ w_0 \end{smallmatrix}\right] \in V_{\rm sc}$, where $x_i \in \mathfrak{R}^\dagger$ and $z_i \in \mathfrak{U}$ for $i = 1, 2$. Then

$$\left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V_{\rm obc}, \quad \left[\begin{smallmatrix} A_- z_0 \\ z_0 \\ 0 \end{smallmatrix}\right] \in V_{\rm sc}, \text{ and } \left[\begin{smallmatrix} x_1+z_1-A_- z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V_{\rm sc}.$$

Moreover, $z_1 - A_- z_0 = X_o \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right]$ as was shown in the proof of Lemma 7.1. Thus,

$$\left[\begin{smallmatrix} x_1+z_1 \\ x_0+z_0 \\ w_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} z_1 \\ z_0 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] + \begin{bmatrix} X_o \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \\ 0 \\ 0 \end{bmatrix} + \left[\begin{smallmatrix} A_- z_0 \\ z_0 \\ 0 \end{smallmatrix}\right]$$

Conversely, if $\left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right]$ is an arbitrary vector in $V_{\rm obc}$ and z_0 is an arbitrary vector in $\mathfrak{U}$, then the above sum belongs to $V_{\rm sc}$, as can be seen from the proof of Lemma 7.1. □

Lemma 7.3. *Define $V_{\rm cfc}$ by (7.4). Then the formula*

$$X_c \begin{bmatrix} QS^{-1}w \\ P_{\mathfrak{R}} Qw \\ w(0) \end{bmatrix} = P_{\mathfrak{U}^\dagger} Qw, \ w \in \mathcal{L}(\mathfrak{W}), \ QS^{-1}w \in \mathfrak{R}, \tag{7.10}$$

defines a bounded linear operator X_c from

$$(V_{\rm cfc})^{[\perp]} = \begin{bmatrix} P_{\mathfrak{R}} & 0 & 0 \\ 0 & P_{\mathfrak{R}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_{\rm sc} = \left\{ \left[\begin{smallmatrix} P_{\mathfrak{R}} QS^{-1}w \\ P_{\mathfrak{R}} Qw \\ w(0) \end{smallmatrix}\right] \,\middle|\, w \in \mathcal{L}(\mathfrak{W}) \right\},$$

onto $\mathfrak{N}_+$ with $\mathcal{N}(X_c) = V_{\rm cfc}$. This operator is isometric with respect to the inner product that $(V_{\rm cfc})^{[\perp]}$ inherits from $-(-\mathfrak{R} [\dot{+}] \mathfrak{R} [\dot{+}] \mathcal{W})$.

Proof. The proof of this lemma is analogous to the proof of Lemma 7.1, where one replaces $\Sigma_{\rm obc}$ by the anti-causal dual of $\Sigma_{\rm cfc}$. □

Corollary 7.4. *Let $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ be the canonical model of a simple conservative s/s system with passive full behavior $\mathfrak{W}$, and let $\Sigma_{\rm sc}^c = (V_{\rm sc}^c; \mathfrak{R}, \mathcal{W})$ be the orthogonal compression of $\Sigma_{\rm sc}$ onto the reachable subspace $\mathfrak{R}$ of $\Sigma_{\rm sc}$. Let $(A_+^*; \mathfrak{U}^\dagger)$ be the incoming inner channel of $\Sigma_{\rm sc}$, and define the isometric operator X_c from $-(V_{\rm sc}^c)^{[\perp]}$ to $\mathfrak{N}_+ = \mathcal{N}\left(A_+^*\right)$ by (7.10). Then*

$$V_{\rm sc} = \left\{ \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} + \begin{bmatrix} 0 \\ X_c \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \\ 0 \end{bmatrix} + \begin{bmatrix} A_+^* z_0 \\ z_0 \\ 0 \end{bmatrix} \,\middle|\, \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in (V_{\rm sc}^c)^{[\perp]},\ z_0 \in \mathfrak{U}^\dagger \right\}. \tag{7.11}$$

Proof. The proof is analogous to the proof of Theorem 7.2, where the past and the future have interchanged places, and Lemma 7.1 has been replaced by Lemma 7.3 □

Lemma 7.5. *The subspaces $\mathfrak{N}_\pm$ have the following representations:*

$$\begin{aligned} \mathfrak{N}_- &= \left\{ \begin{bmatrix} 0 \\ Q_- - \Gamma_{\mathfrak{W}}^* Q_+ \end{bmatrix} S^{-1} w \,\middle|\, \begin{matrix} w \in \mathcal{L}(\mathfrak{W}) \text{ and} \\ Q_- w = \Gamma_{\mathfrak{W}}^* Q_+ w \end{matrix} \right\}, \\ \mathfrak{N}_+ &= \left\{ \begin{bmatrix} Q_+ - \Gamma_{\mathfrak{W}} Q_- \\ 0 \end{bmatrix} S w \,\middle|\, \begin{matrix} w \in \mathcal{L}(\mathfrak{W}) \text{ and} \\ Q_+ w = \Gamma_{\mathfrak{W}} Q_- w \end{matrix} \right\}. \end{aligned} \tag{7.12}$$

Proof. Formula (7.12) holds since $\mathfrak{N}_- = \mathcal{R}(X_o)$ and $\mathfrak{N}_+ = \mathcal{R}(X_c)$, where X_o and X_c are the operator defined in Lemmas 7.1 and 7.3. In this formula we have also used (3.16) and substituted the values of $P_{\mathfrak{R}}$, $P_{\mathfrak{U}}$, $P_{\mathfrak{R}^\dagger}$ and $P_{\mathfrak{U}^\dagger}$ given in (3.17). □

Theorem 7.6. *Let $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ be the canonical model of a simple conservative s/s system with passive full behavior $\mathfrak{W}$, and let $\Sigma_{\rm cfc} = (V_{\rm cfc}; \mathfrak{R}, \mathcal{W})$ be the orthogonal compression of $\Sigma_{\rm sc}$ onto the reachable subspace $\mathfrak{R}$ of $\Sigma_{\rm sc}$. Let $(A_+^*; \mathfrak{U}^\dagger)$ be the incoming inner channel of $\Sigma_{\rm sc}$, and define $\mathfrak{N}_+ := \mathcal{N}\left(A_+^*\right)$. Then*

$$\begin{aligned} &V_{\rm sc} = V_{\rm cfc} + V_1 + V_0, \text{ where} \\ &V_1 = V_{\rm sc} \cap \begin{bmatrix} \mathfrak{R} \\ \mathfrak{N}_+ \\ \mathcal{W} \end{bmatrix}, \\ &V_0 = \left\{ \begin{bmatrix} z_1 \\ A_+ z_1 \\ 0 \end{bmatrix} \,\middle|\, z_1 \in \mathfrak{U}^\dagger \right\}. \end{aligned} \tag{7.13}$$

All of $V_{\rm cfc}$, V_0, and V_1 are subspaces of $V_{\rm sc}$, and

$$V_{\rm cfc} \cap V_0 = \{0\}, \quad V_1 \cap V_0 = \{0\}, \quad V_{\rm cfc} \cap V_1 = V_{\rm sc} \cap \begin{bmatrix} \mathcal{D}(\mathfrak{W}) \\ 0 \\ \mathcal{W} \end{bmatrix}. \tag{7.14}$$

The interpretation of the above splitting is the following: The subspace V_{cfc} represents the dynamics of the forward conservative controllable compression of Σ_{sc}, the subspace V_0 represents the internal dynamics in $\mathfrak{U}^\dagger$, and the subspace V_1 describes the part of the dynamics that crosses over from $\mathfrak{N}_+ \subset \mathfrak{U}^\dagger$ to $\mathfrak{R}$.

Proof of Theorem 7.6. It is clear that each of V_{cfc}, V_0, and V_1 are subspaces of V_{sc}. Conversely, let $\left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right]$ be an arbitrary vector in V_{sc}. Let $x_{01} = P_{\mathfrak{R}} x_0$ and $z_1 = P_{\mathfrak{U}^\dagger} x_1$. By (7.4), it is possible to find $x_{11} \in \mathfrak{R}$ and $w_{01} \in \mathcal{W}$ such that $\left[\begin{smallmatrix} x_{11} \\ x_{01} \\ w_{01} \end{smallmatrix}\right] \in V_{\mathrm{cfc}} \subset V_{\mathrm{sc}}$, and by Lemma 5.1, $\left[\begin{smallmatrix} z_1 \\ A_+ z_1 \\ 0 \end{smallmatrix}\right] \in V_{\mathrm{sc}}$. Thus, also, also the difference $\left[\begin{smallmatrix} x_{12} \\ x_{02} \\ w_{02} \end{smallmatrix}\right] := \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_{11} \\ x_{01} \\ w_{01} \end{smallmatrix}\right] - \left[\begin{smallmatrix} z_1 \\ A_+ z_1 \\ 0 \end{smallmatrix}\right]$ belongs to V_{sc}. We have now decomposed $\left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right]$ into

$$\begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} x_{11} \\ x_{01} \\ w_{01} \end{bmatrix} + \begin{bmatrix} x_{12} \\ x_{02} \\ w_{02} \end{bmatrix} + \begin{bmatrix} z_1 \\ A_+ z_1 \\ 0 \end{bmatrix},$$

where each term belongs to V_{sc}, the first term belongs to V_{cfc}, and the last term belongs to V_0. The two top components of the middle term satisfies $x_{12} \in \mathfrak{R}$ and $x_{02} \in \mathfrak{U}^\dagger$, and hence by Lemma 5.4, $A_+^* x_{02} = P_{\mathfrak{U}^\dagger} x_{12} = 0$. Thus $x_{02} \in \mathfrak{N}_+$, and we conclude that the middle term belongs to V_1.

The splitting of x_0 into $x_0 = x_{01} + x_{02} + A_+ z_1$ is orthogonal, since $x_{01} \in \mathfrak{R}$, $x_{01} \in \mathfrak{N}_+$, and $A_+ z_1 \in \mathcal{R}(A_+) = \mathfrak{U}^\dagger \ominus \mathfrak{N}_+$. This together with (7.4) implies (7.14). □

8 Conservative Dilations of the Forward and Backward Conservative Canonical Models.

In the preceding section we described how to obtain the forward and backward conservative models from the simple conservative model Σ_{sc} by first performing an orthogonal compression, and then applying a unitary similarity transform. Here we shall proceed in the opposite direction and show how to construct a simple conservative model by a dilation from a forward or backward conservative model. We begin with a central lemma, which is related to Lemma 7.1.

Lemma 8.1. *Let V be a maximal nonnegative subspace of a Kreĭn space $\mathcal{K}$ satisfying $V^{[\perp]} \subset V$, $V^{[\perp]} \neq V$. Let $\mathfrak{N}_0$ be the quotient $\mathfrak{N}_0 = V/V^{[\perp]}$. Then $\mathfrak{N}_0$ is a Hilbert space with the inner product inherited from $\mathcal{K}$. Let Π be the* *quotient map* $V \mapsto V/V^{[\perp]}$, *and define*

$$V_{\rm ext} = \left\{ \begin{bmatrix} \kappa \\ \Pi\kappa \end{bmatrix} \in \begin{bmatrix} \mathcal{K} \\ \mathfrak{N}_0 \end{bmatrix} \middle| \kappa \in V \right\}.$$

Then $V_{\rm ext}$ *is a Lagrangian subspace of* $\mathcal{K}\,[\dotplus]\,-\mathfrak{N}_0$.

Proof. If $\kappa \in V$, then

$$\left[\begin{bmatrix} \kappa \\ \Pi\kappa \end{bmatrix}, \begin{bmatrix} \kappa \\ \Pi\kappa \end{bmatrix} \right]_{\mathcal{K}[\dotplus]-\mathfrak{N}_0} = [v,v]_{\mathcal{K}} - [\Pi v, \Pi v]_{\mathfrak{N}_0} = 0.$$

Thus, $V_{\rm ext}$ is a neutral subspace of $\mathcal{K}\,[\dotplus]\,-\mathfrak{N}_0$

Next suppose that $\kappa_1,\ \kappa_2 \in V$, and that $\left[\begin{smallmatrix} \kappa_1 \\ \Pi\kappa_2 \end{smallmatrix}\right]$ is orthogonal to $V_{\rm ext}$ in $\mathcal{K}\,[\dotplus]\,-\mathfrak{N}_0$. Then, for all $\kappa \in V$,

$$\begin{aligned} 0 &= \left[\begin{bmatrix} \kappa_1 \\ \Pi\kappa_2 \end{bmatrix}, \begin{bmatrix} \kappa \\ \Pi\kappa \end{bmatrix} \right]_{\mathcal{K}[\dotplus]-\mathfrak{N}_0} = [\kappa_1, \kappa]_{\mathcal{K}} - [\Pi\kappa_2, \Pi\kappa]_{\mathfrak{N}_0} \\ &= [\kappa_1, \kappa]_{\mathcal{K}} - [\kappa_2, \kappa]_{\mathcal{K}} = [\kappa_1 - \kappa_2, \kappa]_{\mathcal{K}}. \end{aligned}$$

Thus $\kappa_1 - \kappa_2 \subset V^{[\perp]} \subset V$. Since $\kappa_2 \in V$, this implies that $\kappa_1 \in V$, and that $\Pi\kappa_1 = \Pi\kappa_2$, and consequently $\left[\begin{smallmatrix} \kappa_1 \\ \Pi\kappa_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \kappa_1 \\ \Pi\kappa_1 \end{smallmatrix}\right] \in V_{\rm ext}$. □

Theorem 8.2. *Let* $\Sigma^o = (V^o; \mathcal{X}^o, \mathcal{W})$ *be an observable backward conservative passive s/s system which is not conservative. Then the quotient space* $\mathfrak{N}^o := V^o/(V^o)^{[\perp]}$ *is a Hilbert space with the inner product inherited from the node space* $\mathfrak{K}^o = \left[\begin{smallmatrix} -\mathcal{X}^o \\ \mathcal{X}^o \\ \mathcal{W} \end{smallmatrix}\right]$. *Let* $X^o \colon V^o \to V^o/(V^o)^{[\perp]}$ *be the quotient map* $X^o\kappa := \kappa + (V^o)^{\perp}$, $\kappa \in V^o$, *let* $\mathcal{Z}^o := \ell^2_-(\mathfrak{N}^o)$ *and let* $\mathcal{X}^o_{\rm ext} := \mathcal{X}^o \oplus \mathcal{Z}^o$. *Define* $V^o_{\rm ext}$ *by*

$$V^o_{\rm ext} = \left\{ \begin{bmatrix} \left[\begin{smallmatrix} x_1 \\ z_1 \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} x_0 \\ 0 \end{smallmatrix}\right] \\ w_0 \end{bmatrix} + \begin{bmatrix} \left[\begin{smallmatrix} 0 \\ S_-^* z_0 \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} 0 \\ z_0 \end{smallmatrix}\right] \\ 0 \end{bmatrix} \middle| \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V^o,\ z_0 \in \mathcal{Z}^o,\ z_1 = d_{-1}(\cdot) X^o \left[\begin{smallmatrix} x_1 \\ x_0 \\ w_0 \end{smallmatrix}\right] \right\}, \tag{8.1}$$

where $d_{-1}(\cdot)$ *is the scalar sequence defined by* $d_{-1}(-1) = 1$, $d_{-1}(k) = 0$ *for* $k < -1$. *Then* $V^o_{\rm ext}$ *is the generating subspace of a simple conservative s/s system* $\Sigma^o_{\rm ext} = (V^o_{\rm ext}; \mathcal{X}^o_{\rm ext}, \mathcal{W})$ *which is an outgoing dilation of* Σ *with outgoing inner channel* $(S_-, \mathcal{Z}^o)$.

Proof. Define

$$\begin{aligned} \mathcal{Z}^o_0 &= \{ w_- \in \ell^2_-(\mathfrak{N}_-) \mid w_-(k) = 0 \text{ for } k < -1 \}, \\ \mathcal{Z}^o_1 &= \{ w_- \in \ell^2_-(\mathfrak{N}_-) \mid w_-(-1) = 0 \}. \end{aligned}$$

The node space $\mathfrak{K}^o_{\rm ext} = -\mathcal{X}^o_{\rm ext} \,[\dotplus]\, \mathcal{X}^o_{\rm ext} \,[\dotplus]\, \mathcal{W}$ can be written as the orthogonal sum of two Kreĭn spaces $\mathfrak{K}^o_{\rm ext} = \mathfrak{K}_1 \,[\dotplus]\, \mathfrak{K}_0$, where

$$\mathfrak{K}_0 = \begin{bmatrix} -\mathcal{Z}^o_1 \\ \mathcal{Z}^o \\ 0 \end{bmatrix}, \qquad \mathfrak{K}_1 = \begin{bmatrix} -(\mathcal{X}^o \oplus \mathcal{Z}^o_0) \\ \mathcal{X}^o \\ \mathcal{W} \end{bmatrix}.$$

We decompose $V^o_{\rm ext}$ in the same way to get $V^o_{\rm ext} = V_0 \,[\dotplus]\, V_1$, where

$$V_0 = \left\{ \begin{bmatrix} S^*_+ z_0 \\ z_0 \\ 0 \end{bmatrix} \,\middle|\, z_0 \in \mathcal{Z}^o \right\},$$
$$V_1 = \left\{ \begin{bmatrix} \begin{bmatrix} x_1 \\ z_1 \end{bmatrix} \\ \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \\ w_0 \end{bmatrix} \,\middle|\, \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in V^o,\ z_1 = d_{-1} X^o \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \right\}.$$

Here V_0 is a Lagrangian subspace of $\mathfrak{K}_0$, and by Lemma 8.1, V_1 is a Lagrangian subspace of $\mathfrak{K}_1$. Thus, $V^o_{\rm ext}$ is a Lagrangian subspace of $\mathfrak{K}^o_{\rm ext} = \mathfrak{K}_1 \,[\dotplus]\, \mathfrak{K}_0$. This means that $V^o_{\rm ext}$ is the generating subspace of a conservative s/s system $\Sigma^o_{\rm ext} = (V^o_{\rm ext}; \mathcal{X}^o_{\rm ext}, \mathcal{W})$. It can easily be seen that $\Sigma^o_{\rm ext}$ is an outgoing dilation of Σ^o with outgoing inner channel $(S_-; \mathcal{Z}^o)$. That $\Sigma^o_{\rm ext}$ is simple follows from Theorem 5.3 and the fact that S_- is completely non-unitary. □

Theorem 8.3. *Let $\Sigma^c = (V^c; \mathcal{X}^c, \mathcal{W})$ be a controllable forward conservative passive s/s system. Then the quotient space $\mathfrak{N}^c := (V^c)^{[\perp]}/V^c$ is a Hilbert space with the inner product inherited from $-\mathfrak{K}^c = \begin{bmatrix} \mathcal{X}^c \\ -\mathcal{X}^c \\ -\mathcal{W} \end{bmatrix}$. Let $X^c \colon V^c \to (V^c)^{[\perp]}/V^c$ be the quotient map $X^c \kappa := \kappa + V^c$, $\kappa \in (V^c)^\perp$, let $\mathcal{Z}^c := \ell^2_+(\mathfrak{N}_c)$ and $\mathcal{X}^c_{\rm ext} := \mathcal{X}^c \oplus \mathcal{Z}^c$, and define $V^c_{\rm ext}$ by*

$$V^c_{\rm ext} = \left\{ \begin{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} \\ w_0 \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} 0 \\ z_1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ S_+ z_1 \end{bmatrix} \\ 0 \end{bmatrix} \,\middle|\, \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \in (V^c)^{[\perp]},\ z_1 \in \mathcal{Z}^c,\ z_0 = d_0 X^c \begin{bmatrix} x_1 \\ x_0 \\ w_0 \end{bmatrix} \right\}, \tag{8.2}$$

where $d_0(\cdot)$ is the scalar sequence defined by $d_0(0) = 1$, $\delta_0(k) = 0$ for $k > 1$. Then $V^c_{\rm ext}$ is the generating subspace of a simple conservative s/s system $\Sigma^c_{\rm ext} = (V^c_{\rm ext}; \mathcal{X}^c_{\rm ext}, \mathcal{W})$ which is an outgoing dilation of Σ^c with incoming inner channel $(S^*_+; \mathcal{Z}^c)$.

Proof. The proof of this theorem is analogous to the proof of 8.2. □

By applying Theorems 8.2 and 8.3 to the canonical backward conservative and forward conservative models $\Sigma^{\mathfrak{W}_+}_{\rm obc}$ and $\Sigma^{\mathfrak{W}_-}_{\rm cfc}$ presented in Section 2.4 we can construct two additional non-symmetrical models of a simple conservative s/s system with a given passive behavior $\mathfrak{W}$.

The state space in the model that we get by applying Theorem 8.2 to the observable backward conservative model $\Sigma^{\mathfrak{W}_+}_{\rm obc}$ is $\mathcal{H}(\mathfrak{W}_+) \oplus \ell^2_-(\mathfrak{N}^o)$, where $\mathfrak{N}^o = V^{\mathfrak{W}^+}_{\rm obc}/(V^{\mathfrak{W}^+}_{\rm obc})^{[\perp]}$ with the inner product constructed in Lemma 8.1. This model is unitarily similar to the symmetrical model $\Sigma_{\rm sc}$. If we decompose the state space $\mathcal{D}(\mathfrak{W})$ of $\Sigma_{\rm sc}$ into $\mathcal{D}(\mathfrak{W}) = \mathfrak{R}^\dagger \oplus \mathfrak{U}$, then with respect to this decomposition the similarity operator U_o between these two models is block diagonal, i.e., it is of the form $U_o = \left[\begin{smallmatrix} U_{\mathfrak{R}^\dagger} & 0 \\ 0 & U_{\mathfrak{U}} \end{smallmatrix}\right]$. Here $U_{\mathfrak{R}^\dagger} = \left[\begin{smallmatrix} 1_{\mathcal{H}(\mathfrak{W}_+)} \\ \Gamma^*_{\mathfrak{W}} \end{smallmatrix}\right]$ and $U_{\mathfrak{R}^\dagger}^{-1} = \Pi_+$. To compute $U_{\mathfrak{U}}$ we first investigate the restriction of $U_{\mathfrak{U}}|_{\mathfrak{N}^o}$, which maps $\mathfrak{N}^o$ unitarily onto $\mathfrak{N}_-$. Recall that $\mathfrak{N}^o = V_{\rm obc}^{\mathfrak{W}+}/(V_{\rm obc}^{\mathfrak{W}+})^{[\perp]}$. The operator $\left[\begin{smallmatrix} U_{\mathfrak{R}^\dagger} & 0 & 0 \\ 0 & U_{\mathfrak{R}^\dagger} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{smallmatrix}\right]$ is a unitary map of $\mathfrak{K}_{\rm obc}^{\mathfrak{W}_+} := -\mathcal{H}(\mathfrak{W}_+)[\dotplus]\mathcal{H}(\mathfrak{W}_+)[\dotplus]\mathcal{W}$ onto $\mathfrak{K}_{\rm obc} := -\mathfrak{R}^\dagger \,[\dotplus]\, \mathfrak{R}^\dagger \,[\dotplus]\, \mathcal{W}$, and hence it induces a unitary map of $V_{\rm obc}^{\mathfrak{W}+}/(V_{\rm obc}^{\mathfrak{W}+})^{[\perp]}$ onto $V_{\rm obc}/\mathcal{N}\,(X_o)$, where X_o is the operator defined in (7.7). By composing this unitary map with the operator X_o in (7.7) we get the unitary map $U_{\mathfrak{U}}|_{\mathfrak{N}^o} \colon \mathfrak{N}^o \to \mathfrak{N}_-$. The space $\ell^2_-(\mathfrak{N}_-)$ can be mapped unitarily onto $\mathfrak{U}$ by means of the inverse of the operator U_- in (5.8), and by combining this map with the earlier described unitary similarity from $\mathfrak{N}^o$ onto $\mathfrak{N}_-$ we get the full formula for $U_{\mathfrak{U}}$.

The state space in the model that we get by applying Theorem 8.3 to the controllable forward conservative model $\Sigma_{\rm cfc}^{\mathfrak{W}-}$ is $\mathcal{H}(\mathfrak{W}_-^{[\perp]}) \oplus \ell^2_+(\mathfrak{N}^c)$, where $\mathfrak{N}^c = (V_{\rm cfc}^{\mathfrak{W}-})^{[\perp]}/V_{\rm cfc}^{\mathfrak{W}-}$ with the inner product constructed in Lemma 8.1, with the node space $\mathfrak{K}_{\rm cfc}^{\mathfrak{W}-}$ replaced by its anti-space. This model is unitarily similar to the symmetrical model $\Sigma_{\rm sc}$. If we decompose the state space $\mathcal{D}(\mathfrak{W})$ of $\Sigma_{\rm sc}$ into $\mathcal{D}(\mathfrak{W}) = \mathfrak{R} \oplus \mathfrak{U}^\dagger$, then with respect to this decomposition the similarity operator U_c between these two models is block diagonal, i.e., it is of the form $U_c = \left[\begin{smallmatrix} U_{\mathfrak{R}} & 0 \\ 0 & U_{\mathfrak{U}^\dagger} \end{smallmatrix}\right]$. Here $U_{\mathfrak{R}} = \left[\begin{smallmatrix} \Gamma_{\mathfrak{W}} \\ 1_{\mathcal{H}(\mathfrak{W}_-^{[\perp]})} \end{smallmatrix}\right]$ and $U_{\mathfrak{R}}^{-1} = \Pi_-$. To compute $U_{\mathfrak{U}^\dagger}$ we first investigate the restriction of $U_{\mathfrak{U}^\dagger}|_{\mathfrak{N}^c}$, which maps $\mathfrak{N}^c$ unitarily onto $\mathfrak{N}_+$. Recall that $\mathfrak{N}^c = (V_{\rm cfc}^{\mathfrak{W}-})^{[\perp]}/V_{\rm cfc}^{\mathfrak{W}-}$. The operator $\left[\begin{smallmatrix} U_{\mathfrak{R}} & 0 & 0 \\ 0 & U_{\mathfrak{R}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{smallmatrix}\right]$ is a unitary map of $\mathfrak{K}_{\rm cfc}^{\mathfrak{W}-} := -\mathcal{H}(\mathfrak{W}_-^{[\perp]}) \,[\dotplus]\, \mathcal{H}(\mathfrak{W}_-^{[\perp]}) \,[\dotplus]\, \mathfrak{W}$ onto $\mathfrak{K}_{\rm cfc} := -\mathfrak{R} \,[\dotplus]\, \mathfrak{R} \,[\dotplus]\, \mathcal{W}$, and hence it induces a unitary map of $(V_{\rm cfc}^{\mathfrak{W}-})^{[\perp]}/V_{\rm cfc}^{\mathfrak{W}-}$ onto $V_{\rm cfc}/\mathcal{N}\,(X_c)$, where X_c is the operator defined in (7.10). By composing this unitary map with the operator X_c in (7.10) we get the unitary map $U_{\mathfrak{U}^\dagger}|_{\mathfrak{N}^c} \colon \mathfrak{N}^c \to \mathfrak{N}_+$. The space $\ell^2_+(\mathfrak{N}_+)$ can be mapped unitarily onto $\mathfrak{U}^\dagger$ by means of the inverse of the operator U_+ in (5.10), and by combining this map with the earlier described unitary similarity from $\mathfrak{N}^c$ onto $\mathfrak{N}_+$ we get the full formula for $U_{\mathfrak{U}^\dagger}$.

9 Passive Realizations of Frequency Domain Behaviors

The Fourier Transform. Up to now we have throughout worked in the *time domain*, and formulated all our results in terms of sequences in $k^2(I;\mathcal{W})$, where I is a discrete time interval. It is also possible to work in the frequency domain instead, replacing all the signal sequences $w(\cdot)$ by their Fourier transforms. In this section we assume, for simplicity, that the signal space $\mathcal{W}$ is separable.

As is well-known, for each Hilbert space $\mathcal{X}$, the Fourier transform $\mathcal{F}$, formally defined by $(\mathcal{F}w)(z) := \hat{w}(z) = \sum_{n=-\infty}^{\infty} w(n)z^n$ is a unitary map from $\ell^2(\mathcal{X})$ onto the Lebesgue space $L^2(\mathcal{X}) := L^2(\mathbb{T};\mathcal{X})$, where $\mathbb{T} := \{\xi \in \mathbb{C} \mid |\xi| = 1\}$. The restrictions $\mathcal{F}_\pm = \mathcal{F}|_{\ell^2_\pm(\mathcal{X})}$ of $\mathcal{F}$ to $\ell^2_\pm(\mathcal{X})$ are unitary maps from $\ell^2_\pm(\mathcal{X})$ onto the Hardy spaces $H^2_\pm(\mathcal{X}) := H^2(\mathbb{D}_\pm;\mathcal{X})$, where

$$\mathbb{D}_+ := \{z \in \mathbb{Z} \mid |z| < 1\}, \ \mathbb{D}_- := \{\zeta \in \mathbb{Z} \mid |\zeta| > 1\} \cup \{\infty\}.$$

Functions in $H^2_\pm(\mathcal{X})$ are analytic in $\mathbb{D}_\pm$, they have nontangential boundary values in the strong sense a.e. on $\mathbb{T}$, and the boundary function belongs to $L^2(\mathcal{X})$. The norms in $L^2(\mathcal{X})$ and $H^2_\pm(\mathcal{X})$ are given by the same formula

$$\|\widehat{w}(\cdot)\|^2_{L^2(\mathcal{X})} = \frac{1}{2\pi}\oint_{\xi\in\mathbb{T}} \|\widehat{w}(\xi)\|^2_{\mathcal{X}}\,|\mathrm{d}\xi| = \frac{1}{2\pi i}\oint_{\xi\in\mathbb{T}} \|\widehat{w}(\xi)\|^2_{\mathcal{X}}\frac{\mathrm{d}\xi}{\xi}, \tag{9.1}$$

and $L^2(\mathcal{X}) = H^2_-(\mathcal{X}) \oplus H^2_-(\mathcal{X})$. We denote the orthogonal projections of $L^2(\mathcal{X})$ onto $\mathcal{H}^2_\pm(\mathcal{X})$ by $\hat{\pi}_\pm$. They are explicitly given by

$$\begin{aligned}(\hat{\pi}_+\widehat{w})(z) &= \frac{1}{2\pi i}\oint_{\xi\in\mathbb{T}} \frac{\widehat{w}(\xi)}{\xi - z}\,\mathrm{d}\xi, &&\widehat{w} \in L^2(\mathcal{W}),\quad z \in \mathbb{D}_+,\\ (\hat{\pi}_-\widehat{w})(\zeta) &= -\frac{1}{2\pi i}\oint_{\xi\in\mathbb{T}} \frac{\widehat{w}(\xi)}{\xi - \zeta}\,\mathrm{d}\xi, &&\widehat{w} \in L^2(\mathcal{W}),\quad \zeta \in \mathbb{D}_-.\end{aligned} \tag{9.2}$$

If we denote the inverse Fourier transform of $\widehat{w}$ by w, then $w(0) = (\widehat{\pi}_+\widehat{w})(0)$, and it can be computed from the first formula in (9.2) with $z = 0$.

Above we discussed the situation where $\mathcal{X}$ is a Hilbert space, and these considerations can be extended to the case where $\mathcal{X}$ is replaced by a Kreĭn space $\mathcal{W}$. We denote the images of $k^2(\mathcal{W})$ and $k^2_\pm(\mathcal{W})$ under the Fourier transform by $K^2(\mathcal{W}) := K^2(\mathbb{T};\mathcal{W})$ and $K^2_\pm(\mathcal{W}) := K^2(\mathbb{D}_\pm;\mathcal{W})$, respectively, and define the indefinite inner products in these spaces so that the Fourier transform is a unitary operator in each case. This means that, if we fix some admissible Hilbert space inner product in $\mathcal{W}$, then the spaces $K^2(\mathcal{W})$ and $K^2_\pm(\mathcal{W})$ coincide with $L^2(\mathcal{W})$ and $H^2_\pm(\mathcal{W})$, respectively, and that the inner product in $K^2(\mathcal{W})$ and $K^2_\pm(\mathcal{W})$ are given by the same formula

$$\begin{aligned}[\widehat{w}_1(\cdot),\widehat{w}_2(\cdot)]_{K^2(\mathcal{W})} &= \frac{1}{2\pi}\oint_{\xi\in\mathbb{T}} [\widehat{w}_1(\xi),\widehat{w}_2(\xi)]_{\mathcal{W}}\,|\mathrm{d}\xi|\\ &= \frac{1}{2\pi i}\oint_{\xi\in\mathbb{T}} [\widehat{w}_1(\xi),\widehat{w}_2(\xi)]_{\mathcal{W}}\frac{\mathrm{d}\xi}{\xi}.\end{aligned} \tag{9.3}$$

Every fundamental decomposition $\mathcal{W} = -\mathcal{Y} \, [\dot{+}] \, \mathcal{U}$ of the signal space gives rise to the fundamental decompositions

$$K^2(\mathcal{W}) = -L^2(\mathcal{Y}) \, [\dot{+}] \, L^2(\mathcal{U}), \; K^2_\pm(\mathcal{W}) = -H^2_\pm(\mathcal{Y}) \, [\dot{+}] \, H^2_\pm(\mathcal{U}).$$

Under the Fourier transform the three shift operators S_+, S, and S_- and their adjoints are mapped into the frequency domain shift operators

$$\begin{aligned}
&(\widehat{S}_+\widehat{w})(z) := z\widehat{w}(z), \; (\widehat{S}_+^*\widehat{w})(z) := \tfrac{1}{z}(\widehat{w}(z) - \widehat{w}(0)), && \widehat{w}(\cdot) \in K^2_+(\mathcal{W}),\\
&(\widehat{S}\,\widehat{w})(\xi) := \xi\widehat{w}(\xi), \; (\widehat{S}^{-1}\widehat{w})(\xi) := \tfrac{1}{\xi}\widehat{w}(\xi), && \widehat{w}(\cdot) \in K^2(\mathcal{W}),\\
&(\widehat{S}_-\widehat{w})(\zeta) := \zeta\widehat{w}(\zeta) - \lim_{\zeta\to\infty}\zeta\widehat{w}(\zeta), \; (\widehat{S}_-^*\widehat{w})(\zeta) := \tfrac{1}{\zeta}\widehat{w}(\zeta), && \widehat{w}(\cdot) \in K^2_-(\mathcal{W}).
\end{aligned} \tag{9.4}$$

Frequency Domain Behaviors. Under the Fourier transform the class of all passive future behaviors $\mathfrak{W}_+$ on $\mathcal{W}$ is mapped onto the class of all maximal nonnegative $\widehat{S}_+$-invariant subspaces $\widehat{\mathfrak{W}}_+$ of $K^2_+(\mathcal{W})$, the class of all passive past behaviors $\mathfrak{W}_-$ on $\mathcal{W}$ is mapped onto the class of all maximal nonnegative $\widehat{S}_-$-invariant subspaces $\widehat{\mathfrak{W}}_-$ of $K^2_-(\mathcal{W})$, and the class of all passive full behaviors $\mathfrak{W}$ is mapped onto the class of all maximal nonnegative $\widehat{S}$-reducing causal subspaces $\widehat{\mathfrak{W}}$ of $K^2(\mathcal{W})$. The definition of causality in the frequency domain is analogous to the definition of causality in time domain, i.e., a $\widehat{S}$-reducing maximal nonnegative subspace $\widehat{\mathfrak{W}}$ is causal if it is true that $\widehat{\mathfrak{W}}_- := \hat{\pi}_-\widehat{\mathfrak{W}}$ is a maximal nonnegative subspace of $K^2_-(\mathcal{W})$, or equivalently, that $\widehat{\mathfrak{W}}_+ := \widehat{\mathfrak{W}} \cap K^2_+(\mathcal{W})$ is a maximal nonnegative subspace of $K^2_+(\mathcal{W})$.

All our earlier results on passive realizations of passive (time domain) behaviors can be reformulated in frequency domain terms. In particular, the frequency domain analogues of (2.29)–(2.31) are

$$\begin{aligned}
&\widehat{\mathfrak{W}}_- = \hat{\pi}_-\widehat{\mathfrak{W}}, \quad \widehat{\mathfrak{W}}_+ = \widehat{\mathfrak{W}} \cap K^2_+(\mathcal{W}), \quad \widehat{\mathfrak{W}} = \bigvee_{n\in\mathbb{Z}^+} \widehat{S}^{-n}\widehat{\mathfrak{W}}_+,\\
&\widehat{\mathfrak{W}} = \bigcap_{n\in\mathbb{Z}^+} \left\{\widehat{w}(\cdot) \in K^2(\mathcal{W}) \;\middle|\; \hat{\pi}_-\widehat{S}^{-n}\widehat{w} \in \widehat{\mathfrak{W}}_-\right\}.
\end{aligned} \tag{9.5}$$

The Fourier transforms of $\mathfrak{W}^\Sigma$ and $\mathfrak{W}^\Sigma_\pm$ give the respective frequency domain behaviors $\widehat{\mathfrak{W}}^\Sigma$ and $\widehat{\mathfrak{W}}^\Sigma_\pm$.

Frequency Domain Versions of the Canonical Models. The frequency domain analogue of the space $\mathcal{H}(\mathfrak{W}_+)$ is the Hilbert space $\mathcal{H}(\widehat{\mathfrak{W}}_+)$, where $\widehat{\mathfrak{W}}_+$ is a maximal nonnegative $\widehat{S}_+$-invariant subspace of $K^2_+(\mathcal{W})$, and the frequency domain analogue of the space $\mathcal{H}(\mathfrak{W}_-^{[\perp]})$ is the Hilbert space $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})$, where $\widehat{\mathfrak{W}}_-$ is a maximal nonnegative $\widehat{S}_-$-invariant subspace of $K^2_-(\mathcal{W})$. These spaces are defined in the same way as in Section 2.4, with

$k^2_\pm(\mathcal{W})$ replaced by $K^2_\pm(\mathcal{W})$ and with $\mathfrak{M}_\pm$ replaced by $\widehat{\mathfrak{W}}_\pm$. Since the $\mathcal{F}_\pm$ is a unitary map of $k^2_\pm(\mathcal{W})$ onto $K^2_\pm(\mathcal{W})$, and since the frequency domain constructions are identical to the time domain constructions, the Fourier transform induces two unitary maps $\mathcal{H}(\mathfrak{W}_\pm) \to \mathcal{H}(\widehat{\mathfrak{W}}_\pm)$ which map $\mathcal{H}^0(\mathfrak{W}_\pm)$ isometrically onto $\mathcal{H}^0(\widehat{\mathfrak{W}}_\pm)$. We shall use the same notation $\mathcal{F}_\pm$ for these two unitary maps.

We denote the frequency domain analogue of the past/future map $\Gamma_{\mathfrak{W}}$ by $\Gamma_{\widehat{\mathfrak{W}}} = \mathcal{F}_+\Gamma_{\mathfrak{W}}\mathcal{F}_-^{-1}$. Thus, if $\mathfrak{W}$ is a passive full behavior on $\mathcal{W}$ with the corresponding passive future and past behaviors $\mathfrak{W}_+$ and $\mathfrak{W}_-$, then $\Gamma_{\widehat{\mathfrak{W}}}$ is the unique linear contraction $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]}) \to \mathcal{H}(\widehat{\mathfrak{W}}_+)$, which is defined by the relation

$$\widehat{Q}_+\widehat{w} = \Gamma_{\widehat{\mathfrak{W}}}\widehat{Q}_-\widehat{w}, \qquad \widehat{w} \in \widehat{\mathfrak{W}},$$

on the dense subspace $\mathcal{H}^0(\widehat{\mathfrak{W}}_-^{[\perp]}) := \widehat{Q}_-\widehat{\mathfrak{W}}$ of $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})$ and then extended to $\mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]})$ by continuity.

Frequency domain versions of the two canonical backward and forward conservative models presented in Theorems 2.11 and 2.16 were given in [AS09b, Section 10], and there it was also shown how to derive the respective de Branges–Rovnyak canonical scattering models from our backward and forward conservative canonical models. Below we shall carry out the same program for simple conservative canonical model presented in Theorem 3.5.

The frequency domain analogue of the space $\mathcal{D}(\mathfrak{W})$ is the Hilbert space $\mathcal{D}(\widehat{\mathfrak{W}})$, which is the range space of the operator $A_{\widehat{\mathfrak{W}}}^{1/2}$, where

$$A_{\widehat{\mathfrak{W}}} := \begin{bmatrix} 1_{\widehat{\mathcal{H}}_+} & \Gamma_{\widehat{\mathfrak{W}}} \\ \Gamma_{\widehat{\mathfrak{W}}}^* & 1_{\widehat{\mathcal{H}}_-} \end{bmatrix}, \qquad \widehat{\mathcal{H}}_+ := \mathcal{H}(\widehat{\mathfrak{W}}_+), \qquad \widehat{\mathcal{H}}_- := \mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]}). \tag{9.6}$$

The frequency domain analogues of the quotient maps Q and $Q_\pm$ are the quotient maps $\widehat{Q}$ and $\widehat{Q}_\pm$ given by

$$\widehat{Q}\widehat{w} = \widehat{w} + \widehat{\mathfrak{W}}_+ + \widehat{\mathfrak{W}}_-^{[\perp]},\ \widehat{Q}_+\widehat{w} = \hat{\pi}_+\widehat{w} + \widehat{\mathfrak{W}}_+,\ \widehat{Q}_-\widehat{w} = \hat{\pi}_-\widehat{w} + \widehat{\mathfrak{W}}_-^{[\perp]},$$

for $\widehat{w} \in K^2(\mathcal{W})$. The frequency domain analogue of the subspace $\mathcal{L}(\mathfrak{W})$ of $k^2(\mathcal{W})$ is the subspace $\mathcal{L}(\widehat{\mathfrak{W}})$ of $K^2(\mathcal{W})$ defined by

$$\mathcal{L}(\widehat{\mathfrak{W}}) = \left\{\widehat{w} \in K^2(\mathcal{W}) \;\middle|\; \widehat{Q}\widehat{w} \in \mathcal{D}(\widehat{\mathfrak{W}})\right\}.$$

The frequency domain analogue of the canonical model $\Sigma_{\rm sc} = (V_{\rm sc}; \mathcal{D}(\mathfrak{W}), \mathcal{W})$ is the canonical model $\widehat{\Sigma}_{\rm sc} = (\widehat{V}_{\rm sc}; \mathcal{D}(\widehat{\mathfrak{W}}), \mathcal{W})$, where

$$\widehat{V}_{\rm sc} := \left\{ \begin{bmatrix} \widehat{Q}\widehat{S}^{-1}\widehat{w} \\ \widehat{Q}\widehat{w} \\ \widehat{w}(0) \end{bmatrix} \;\middle|\; \widehat{w} \in \mathcal{L}(\widehat{\mathfrak{W}}) \right\}. \tag{9.7}$$

The input map of $\widehat{\Sigma}_{\rm sc}$ is $\mathfrak{B}_{\widehat{\Sigma}_{\rm sc}} = \begin{bmatrix} \Gamma_{\widehat{\mathfrak{W}}} \\ 1_{\widehat{\mathcal{H}}_-} \end{bmatrix}$ with $\widehat{B}^*_{\widehat{\Sigma}_{\rm sc}} = \widehat{\Pi}_-|_{\mathcal{D}(\widehat{\mathfrak{W}})}$, and the output map of $\Sigma_{\rm sc}$ is $\widehat{\mathfrak{C}}_{\widehat{\Sigma}_{\rm sc}} = \widehat{\Pi}_+|_{\mathcal{D}(\widehat{\mathfrak{W}})}$ with $\widehat{\mathfrak{C}}^*_{\widehat{\Sigma}_{\rm sc}} = \begin{bmatrix} 1_{\widehat{\mathcal{H}}_+} \\ \Gamma_{\widehat{\mathfrak{W}}} \end{bmatrix}$, where $\widehat{\Pi}_\pm$ are the orthoprojections from $\widehat{\mathcal{H}}_+ \oplus \widehat{\mathcal{H}}_-$ onto $\widehat{\mathcal{H}}_\pm$.

10 The deBranges–Rovnyak Conservative I/S/O Model.

In this final section we shall use our simple conservative frequency domain canonical s/s model $\widehat{\Sigma}_{\rm sc}$ to derive the classical de Branges–Rovnyak model of a simple conservative scattering realization of a given Schur function, originally developed in [dBR66a, dBR66b].

Graph Representations of Frequency Domain Behaviors. Let $\mathfrak{W}$ be a passive full behavior on the Kreĭn signal space $\mathcal{W}$, let $\mathfrak{W}_+ = \mathfrak{W} \cap k^2_+(\mathcal{W})$ and $\mathfrak{W}_- = \pi_- \mathfrak{W}$ be the corresponding passive future and past behaviors, and let $\widehat{\mathfrak{W}}$ and $\widehat{\mathfrak{W}}_\pm$ be the corresponding frequency domain behaviors. Let $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$ be a fundamental decomposition of $\mathcal{W}$, with the corresponding fundamental decompositions $K^2(\mathcal{W}) = -L^2(\mathcal{Y}) \,[\dotplus]\, L^2(\mathcal{U})$ and $K^2_\pm(\mathcal{W}) = -H^2_\pm(\mathcal{Y}) \,[\dotplus]\, H^2_\pm(\mathcal{U})$ of $K^2(\mathcal{W})$ and $K^2_\pm(\mathcal{W})$, respectively. Since $\widehat{\mathfrak{W}}$ and $\widehat{\mathfrak{W}}_\pm$ are maximal nonnegative subspaces of $K^2(\mathcal{W})$ and $K^2_\pm(\mathcal{W})$, respectively, it follows from assertion 1) of Proposition 2.1 that they have the graph representations

$$\begin{aligned} \widehat{\mathfrak{W}} &= \left\{ \widehat{w} = \begin{bmatrix} \widehat{\mathfrak{D}}\hat{u} \\ \hat{u} \end{bmatrix} \;\middle|\; \hat{u} \in L^2(\mathcal{U}) \right\}, \\ \widehat{\mathfrak{W}}_\pm &= \left\{ \widehat{w}_\pm = \begin{bmatrix} \widehat{\mathfrak{D}}_\pm \hat{u}_\pm \\ \hat{u}_\pm \end{bmatrix} \;\middle|\; \hat{u}_\pm \in H^2_\pm(\mathcal{U}) \right\}, \end{aligned} \tag{10.1}$$

where $\widehat{\mathfrak{D}} \in \mathcal{B}(L^2(\mathcal{U}); L^2(\mathcal{Y}))$ and $\widehat{\mathfrak{D}}_\pm \in \mathcal{B}(H^2_\pm(\mathcal{U}); H^2_\pm(\mathcal{Y}))$ are contractions. It follows from (9.5) that $\widehat{\mathfrak{D}}_+ = \widehat{\mathfrak{D}}|_{H^2_+(\mathcal{U})}$ and $\widehat{\mathfrak{D}}_- = \hat{\pi}_- \widehat{\mathfrak{D}}|_{H^2_-(\mathcal{U})}$. As was shown in [AS09b, Section 9], the operators $\widehat{\mathfrak{D}}$ and $\widehat{\mathfrak{D}}_\pm$ are (Laurent) operators of the type

$$\begin{aligned} (\widehat{\mathfrak{D}}\hat{u})(\xi) &= \Phi(\xi)\hat{u}(\xi), & \hat{u} &\in L^2(\mathcal{U}), \quad \xi \in \mathbb{T}, \\ (\widehat{\mathfrak{D}}_+ \hat{u}_+)(z) &= \Phi(z)\hat{u}_+(z), & \hat{u}_+ &\in H^2_+(\mathcal{U}), \quad z \in \mathbb{D}_+, \\ (\widehat{\mathfrak{D}}_- \hat{u}_-)(\zeta) &= -\frac{1}{2\pi i} \oint_{\xi \in \mathbb{T}} \frac{\Phi(\xi)\hat{u}_-(\xi)}{\xi - \zeta}\, {\rm d}\xi, & \hat{u}_- &\in H^2_-(\mathcal{U}), \quad \zeta \in \mathbb{D}_-, \end{aligned} \tag{10.2}$$

whose symbol Φ is a function in the Schur class $S(\mathcal{U}, \mathcal{Y})$, i.e., Φ is an analytic $\mathcal{B}(\mathcal{U}; \mathcal{Y})$-valued function in $\mathbb{D}_+$ satisfying $\|\Phi(z)\|_{\mathcal{B}(\mathcal{U},\mathcal{Y})} \le 1$, $z \in \mathbb{D}_+$. Such a function has a strong nontangential limit $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$. The boundary function belongs to $L^\infty(\mathbb{T})$, $\|\Phi(\zeta)\|_{\mathcal{B}(\mathcal{U},\mathcal{Y})} \le 1$ for almost all $\zeta \in \mathbb{T}$, and

$$\|\Phi\|_{H^\infty(\mathbb{D}_+)} = \sup_{z\in\mathbb{D}_+} \|\Phi(z)\|_{\mathcal{B}(\mathcal{U},\mathcal{Y})} = \sup_{\zeta\in\mathbb{T}} \|\Phi(\zeta)\|_{\mathcal{B}(\mathcal{U},\mathcal{Y})} = \|\Phi\|_{L^\infty(\mathbb{T})}.$$

The orthogonal companions of $\widehat{\mathfrak{W}}$ and $\widehat{\mathfrak{W}}_\pm$ have the representations

$$\begin{aligned}\widehat{\mathfrak{W}}^{[\perp]} &= \Big\{\widehat{w} = \begin{bmatrix} \hat{y} \\ \widehat{\mathfrak{D}}^* \hat{y}\end{bmatrix} \;\Big|\; \hat{y} \in L^2(\mathcal{Y})\Big\},\\ \widehat{\mathfrak{W}}^{[\perp]}_\pm &= \Big\{\widehat{w}_\pm = \begin{bmatrix} \hat{y}_\pm \\ \widehat{\mathfrak{D}}^*_\pm \hat{y}_\pm\end{bmatrix} \;\Big|\; \hat{y}_\pm \in H^2_\pm(\mathcal{Y})\Big\}.\end{aligned} \tag{10.3}$$

The adjoint $\widehat{\mathfrak{D}}^*$ of $\widehat{\mathfrak{D}}$ is the Laurent operator whose symbol is $\Phi^*(\zeta)$, $\zeta \in \mathbb{T}$, and $\widehat{\mathfrak{D}}^*_+$ and $\widehat{\mathfrak{D}}^*_+$ are the appropriate compressions of $\widehat{\mathfrak{D}}^*$. More precisely,

$$\begin{aligned}(\widehat{\mathfrak{D}}^*\hat{y})(\xi) &= \Phi(\xi)^*\hat{y}(\xi), & \hat{y} &\in L^2(\mathcal{Y}), \quad \xi \in \mathbb{T},\\ (\widehat{\mathfrak{D}}^*_+\hat{y}_+)(z) &= \frac{1}{2\pi i}\oint_{\zeta\in\mathbb{T}} \frac{\Phi(\xi)^*\hat{y}_+(\xi)}{\xi - z}\,\mathrm{d}\xi, & \hat{y}_+ &\in H^2_+(\mathcal{Y}), \quad z \in \mathbb{D}_+,\\ (\widehat{\mathfrak{D}}^*_-\hat{y}_-)(\zeta) &= \Phi(1/\overline{\zeta})^*\hat{y}_-(\zeta), & \hat{y}_- &\in H^2_-(\mathcal{Y}), \quad \zeta \in \mathbb{D}_-,\end{aligned} \tag{10.4}$$

The de Branges Complementary Spaces $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ and $\mathcal{H}(\widehat{\mathfrak{D}}^*_-)$. The connection between the Hilbert space $\mathcal{H}(\mathcal{Z})$, where $\mathcal{Z}$ is a maximal nonnegative subspace of a Kreĭn space $\mathcal{K}$, and the de Branges complementary space $\mathcal{H}(A)$, where A is a contraction between two Hilbert spaces, was explained in Section 2.1. We shall now use this connection with the following two sets of substitutions:

1) $\mathcal{Z} \to \widehat{\mathfrak{W}}_+$, $\mathcal{K} \to \mathcal{K}^2_+(\mathcal{W})$, $\mathcal{U} \to H^2_+(\mathcal{U})$, $\mathcal{Y} \to H^2_+(\mathcal{Y})$, $A \to \widehat{\mathfrak{D}}_+$, and $T \to \widehat{T}_+$,
2) $\mathcal{Z} \to \widehat{\mathfrak{W}}^{[\perp]}_-$, $\mathcal{K} \to -\mathcal{K}^2_-(\mathcal{W})$, $\mathcal{U} \to H^2_-(\mathcal{Y})$, $\mathcal{Y} \to H^2_-(\mathcal{U})$, $A \to \widehat{\mathfrak{D}}^*_-$, and $T \to \widehat{T}_-$.

Here $\widehat{T}_+ \colon \mathcal{H}(\widehat{\mathfrak{W}}_+) \to \mathcal{H}(\widehat{\mathfrak{D}}_+)$ and $\widehat{T}_- \colon \mathcal{H}(\widehat{\mathfrak{W}}^{[\perp]}_-) \to \mathcal{H}(\widehat{\mathfrak{D}}_-)$ are the unitary operators that we get by carrying out the above substitutions in (2.21), and they are explicitly given by

$$\begin{aligned}\widehat{T}_+\widehat{Q}_+ \begin{bmatrix} \hat{y}_+ \\ \hat{u}_+\end{bmatrix} &= \hat{y}_+ - \widehat{\mathfrak{D}}_+\hat{u}_+, & \begin{bmatrix} \hat{y}_+ \\ \hat{u}_+\end{bmatrix} &\in \mathcal{K}(\widehat{\mathfrak{W}}_+),\\ \widehat{T}_-\widehat{Q}_- \begin{bmatrix} \hat{y}_- \\ \hat{u}_-\end{bmatrix} &= \hat{u}_- - \widehat{\mathfrak{D}}^*_-\hat{y}_-, & \begin{bmatrix} \hat{y}_- \\ \hat{u}_-\end{bmatrix} &\in \mathcal{K}(\widehat{\mathfrak{W}}^{[\perp]}_-),\\ \widehat{T}^{-1}_+\hat{y}_+ &= \widehat{Q}_+ \begin{bmatrix} \hat{y}_+ \\ 0\end{bmatrix}, & \hat{y}_+ &\in \mathcal{H}(\mathfrak{W}_+),\\ \widehat{T}^{-1}_-\hat{u}_- &= \widehat{Q}_- \begin{bmatrix} 0 \\ \hat{u}_-\end{bmatrix}, & \hat{u}_- &\in \mathcal{H}(\mathfrak{W}^{[\perp]}_-).\end{aligned} \tag{10.5}$$

The Past/Future Map From $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ to $\mathcal{H}(\widehat{\mathfrak{D}}_+)$. By using the unitary maps $\widehat{T}_-\colon \mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]}) \to \mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ and $\widehat{T}_+\colon \mathcal{H}(\widehat{\mathfrak{W}}_+) \to \mathcal{H}(\widehat{\mathfrak{D}}_+)$ we can define a version of the past/future map $\Gamma_{\mathfrak{W}}$ of a passive full behavior which is a contraction from $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ to $\mathcal{H}(\widehat{\mathfrak{D}}_+)$, namely

$$\Gamma_{(\widehat{\mathfrak{D}}_-^*,\widehat{\mathfrak{D}}_+)} := \widehat{T}_+\Gamma_{\widehat{\mathfrak{W}}}\widehat{T}_-^{-1} = \widehat{T}_+\mathcal{F}_+\Gamma_{\mathfrak{W}}\mathcal{F}_-^{-1}\widehat{T}_-^{-1}.$$

This map is related to but not identical with the *Hankel operator*

$$\Gamma_{\widehat{\mathfrak{D}}} := \hat{\pi}_+\widehat{\mathfrak{D}}|_{H^2_-(\mathcal{U})}\colon H^2_-(\mathcal{U}) \to H^2_+(\mathcal{Y})$$

induced by $\widehat{\mathfrak{D}}$.

We recall the following results from [AS09b].

Lemma 10.1 ([AS09b, Lemma 9.1])**.** *Let $\widehat{w} \in \widehat{\mathfrak{W}}$, and write $\widehat{w}$ in the form $\widehat{w} = \left[\begin{smallmatrix}\widehat{\mathfrak{D}}\hat{u}\\ \hat{u}\end{smallmatrix}\right] \in \widehat{\mathfrak{W}}$ where $\hat{u} = P_{L^2(\mathcal{U})}\widehat{w} \in L^2(\mathcal{U})$ (cf. (10.1)). Then*

$$\begin{aligned}\widehat{T}_-(\hat{\pi}_-\widehat{w} + \widehat{\mathfrak{W}}_-^{[\perp]}) &= (1_{H^2_-(\mathcal{U})} - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-)\hat{u}_-,\\ \widehat{T}_+(\hat{\pi}_+\widehat{w} + \widehat{\mathfrak{W}}_+) &= \Gamma_{\widehat{\mathfrak{D}}}\hat{u}_-,\end{aligned} \tag{10.6}$$

where $\hat{u}_- = \hat{\pi}_- u \in H^2_-(\mathcal{U})$.

Lemma 10.2 ([AS09b, Lemma 9.2])**.** *The operator $\Gamma_{(\widehat{\mathfrak{D}}_-^*,\widehat{\mathfrak{D}}_+)}$ is the unique linear contraction $\mathcal{H}(\widehat{\mathfrak{D}}_-^*) \to \mathcal{H}(\widehat{\mathfrak{D}}_+)$, which is defined by the relation*

$$\Gamma_{(\widehat{\mathfrak{D}}_-^*,\widehat{\mathfrak{D}}_+)} = \Gamma_{\widehat{\mathfrak{D}}}\big(1_{H^2_-(\mathcal{U})} - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)^{[-1]}, \tag{10.7}$$

on the dense subspace $\mathcal{H}^0(\widehat{\mathfrak{D}}_-^*) = \mathcal{R}\big(1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\big)$ of $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ and then extended to $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ by continuity.

Lemma 10.3 ([AS09b, Lemma 9.3])**.** *The adjoint of the inclusion map $\widehat{\mathcal{I}}_-\colon \mathcal{H}(\widehat{\mathfrak{D}}_-^*) \hookrightarrow H^2_-(\mathcal{U})$ is the operator $\widehat{\mathcal{I}}_-^* = 1_{H^2_-(\mathcal{U})} - \widehat{\mathfrak{D}}_-^*\widehat{\mathfrak{D}}_-\colon H^2_-(\mathcal{U}) \to \mathcal{H}(\widehat{\mathfrak{D}}_-^*)$.*

We shall also need the dual versions of Lemmas 10.1–10.3, which read as follows, and which may be proved in the same way as Lemmas 10.1–10.3.

Lemma 10.4. *Let $\widehat{w}^\dagger \in \widehat{\mathfrak{W}}^{[\perp]}$, and write $\widehat{w}^\dagger$ in the form $\widehat{w}^\dagger = \left[\begin{smallmatrix}\hat{y}\\ \widehat{\mathfrak{D}}^*\hat{y}\end{smallmatrix}\right]$ where $\hat{y} = P_{L^2(\mathcal{Y})}\widehat{w}^\dagger \in L^2(\mathcal{Y})$ (cf. (10.3)). Then*

$$\begin{aligned}\widehat{T}_-(\hat{\pi}_-\widehat{w}^\dagger + \widehat{\mathfrak{W}}_-^{[\perp]}) &= \Gamma_{\widehat{\mathfrak{D}}}^*\hat{y}_+,\\ \widehat{T}_+(\hat{\pi}_+\widehat{w}^\dagger + \widehat{\mathfrak{W}}_+) &= (1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_+\widehat{\mathfrak{D}}_+^*)\hat{y}_+,\end{aligned} \tag{10.8}$$

where $\hat{y}_+ = \hat{\pi}_+\hat{y} \in H^2_+(\mathcal{Y})$.

Lemma 10.5. *The operator* $\Gamma^*_{(\widehat{\mathfrak{D}}^*_-, \widehat{\mathfrak{D}}_+)}$ *is the unique linear contraction* $\mathcal{H}(\widehat{\mathfrak{D}}_+) \to \mathcal{H}(\widehat{\mathfrak{D}}^*_-)$*, which is defined by the relation*

$$\Gamma^*_{(\widehat{\mathfrak{D}}^*_-, \widehat{\mathfrak{D}}_+)} = \Gamma^*_{\widehat{\mathfrak{D}}}\big(1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_+\widehat{\mathfrak{D}}^*_+\big)^{[-1]}, \tag{10.9}$$

on the dense subspace $\mathcal{H}^0(\widehat{\mathfrak{D}}_+) = \mathcal{R}\big(1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_+\widehat{\mathfrak{D}}^*_+\big)$ *of* $\mathcal{H}(\widehat{\mathfrak{D}}^*_+)$ *and then extended to* $\mathcal{H}(\widehat{\mathfrak{D}}^*_+)$ *by continuity.*

Lemma 10.6. *The adjoint of the inclusion map* $\widehat{\mathcal{I}}_+ : \mathcal{H}(\widehat{\mathfrak{D}}_+) \hookrightarrow H^2_+(\mathcal{Y})$ *is the operator* $\widehat{\mathcal{I}}^*_+ = 1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_+\widehat{\mathfrak{D}}^*_+ : H^2_+(\mathcal{Y}) \to \mathcal{H}(\widehat{\mathfrak{D}}_+)$.

The Spaces $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ and $\mathcal{H}(\widehat{\mathfrak{D}}^*_-)$ as Reproducing Kernel Hilbert Spaces. We begin by showing that our space $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ is equal to the standard de Branges reproducing kernel Hilbert space $\mathcal{H}(\Phi)$, where Φ is the symbol of $\widehat{\mathfrak{D}}_+$, as defined in, e.g., [ADRdS97, Definition 2.1.1].

Let $E_{H^2_+(\mathcal{Y})}(z) : H^2_+(\mathcal{Y}) \to \mathcal{Y}$ be the point evaluation operator $E_{H^2_+(\mathcal{Y})}(z)\hat{y}_+ = \hat{y}_+(z)$, $z \in \mathbb{D}_+$. Since $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ is continuously contained in $H^2_+(\mathcal{Y})$, the restriction $E_+(z) = E_{H^2_+(\mathcal{Y})}(z)|_{\mathcal{H}(\widehat{\mathfrak{D}}_+)}$ is a bounded linear operator $\mathcal{H}(\widehat{\mathfrak{D}}_+) \to \mathcal{Y}$ given by the same formula $E_+(z)\hat{y}_+ = \hat{y}_+(z)$, $z \in \mathbb{D}_+$. Since each $E_+(z)$ is a bounded linear operator, and since each $\hat{y}_+ \in \mathcal{H}(\widehat{\mathfrak{D}}_+)$ is determined uniquely by its values in $\mathbb{D}_+$, it follows from [ADRdS97, Theorem 1.1.2] that $\mathcal{H}(\widehat{\mathfrak{D}}_+)$ is a reproducing kernel Hilbert space, with the reproducing kernel $K_{\widehat{\mathfrak{D}}_+}(z, z_*) = E_+(z)E_+(z_*)^*$ on $\mathbb{D}_+ \times \mathbb{D}_+$. Let $\widehat{\mathcal{I}}_+$ be the inclusion map $\mathcal{H}(\widehat{\mathfrak{D}}_+) \hookrightarrow H^2_+(\mathcal{Y})$. Then $E_+(z_*)^* = \widehat{\mathcal{I}}^*_+ E_{H^2_+(\mathcal{Y})}(z_*)^*$. By Lemma 10.6, $\widehat{\mathcal{I}}^*_+ = 1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_+\mathfrak{D}^*_+$. A direct computation shows that

$$\begin{aligned}
E_{H^2_+(\mathcal{Y})}(z_*)^* y_0 &= \left(z \mapsto \frac{y_0}{1 - z\overline{z}_*}\right), \quad y_0 \in \mathcal{Y}, \\
\widehat{\mathcal{I}}^*_+ E_{H^2_+(\mathcal{Y})}(z_*)^* y_0 &= \big(1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_+\mathfrak{D}^*_+\big) E_{H^2_+(\mathcal{Y})}(z_*)^* y_0 \\
&= \left(z \mapsto \frac{y_0 - \Phi(z)\Phi(z_*)^* y_0}{1 - z\overline{z}_*}\right), \quad y_0 \in \mathcal{Y}, \\
K_{\widehat{\mathfrak{D}}_+}(z, z_*) &= E_{H^2_+(\mathcal{Y})}(z)\widehat{\mathcal{I}}^*_+ E_{H^2_+(\mathcal{Y})}(z_*)^* \\
&= \frac{1_{\mathcal{Y}} - \Phi(z)\Phi(z_*)^*}{1 - z\overline{z}_*}, \quad (z, z_*) \in \mathbb{D}_+ \times \mathbb{D}_+.
\end{aligned}$$

This is the reproducing kernel of the standard de Branges space $\mathcal{H}(\Phi)$ (see, e.g., [ADRdS97, Definition 2.1.1]). Thus, we conclude that $\mathcal{H}(\widehat{\mathfrak{D}}_+) = \mathcal{H}(\Phi)$.

A similar result can be derived for our space $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$. This is a reproducing kernel Hilbert space of analytic $\mathcal{U}$-valued functions defined on $\mathbb{D}_-$ and continuously contained in $H^2_-(\mathcal{U})$. A computation similar to the one above shows that the reproducing kernel of this space is given by

$$K_{\widehat{\mathfrak{D}}_-^*}(\zeta,\zeta_*) = E_-(\zeta)E_-(\zeta_*)^* = E_{H^2_-(\mathcal{U})}(\zeta)\widehat{\mathcal{I}}_-^* E_{H^2_-(\mathcal{U})}(\zeta_*)^*$$

where $E_-(\zeta)$ and $E_{H^2_-(\mathcal{U})}(\zeta)$ are the point evaluation operator in $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ and $H^2_-(\mathcal{U})$, respectively, at the point $\zeta \in \mathbb{D}_-$, and $\widehat{\mathcal{I}}_-$ is the inclusion map $\mathcal{H}(\widehat{\mathfrak{D}}_-^*) \hookrightarrow H^2_-(\mathcal{U})$. A direct computation shows that

$$\begin{aligned} E_{H^2_-(\mathcal{U})}(\zeta_*)^* u_0 &= \left(\zeta \mapsto \frac{u_0}{\zeta\overline{\zeta}_* - 1}\right), \quad u_0 \in \mathcal{U}, \\ K_{\widehat{\mathfrak{D}}_-^*}(\zeta,\zeta_*) &= E_{H^2_-(\mathcal{U})}(\zeta)\widehat{\mathcal{I}}_-^* E_{H^2_-(\mathcal{U})}(\zeta_*)^* \\ &= \frac{1_{\mathcal{U}} - \Phi(1/\overline{\zeta})^*\Phi(1/\overline{\zeta}_*)}{\zeta\overline{\zeta}_* - 1}, \quad (\zeta,\zeta_*) \in \mathbb{D}_- \times \mathbb{D}_-. \end{aligned}$$

Let R be the reflection operator which maps $\widehat{u}_- \in H^2_-(\mathcal{U})$ onto the function $(R\widehat{u}_-)(z) = (1/z)\widehat{u}_-(1/z) \in H^2_+(\mathcal{U})$, and define $\widehat{\mathfrak{D}}_+^\dagger = R\widehat{\mathfrak{D}}_-^* R^{-1}$. Then $\widehat{\mathfrak{D}}_+^\dagger$ is a causal convolution operator whose symbol is the Schur function $\Phi^\dagger(z) = \Phi(\overline{z})^*$. The operator R is a unitary map from $H^2_-(\mathcal{U})$ onto $H^2_+(\mathcal{U})$, and this implies that $R|_{\mathcal{H}(\widehat{\mathfrak{D}}_-^*)}$ is a unitary map of $\mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ onto $\mathcal{H}(\widehat{\mathfrak{D}}_+^\dagger)$, as can easily be seen from the definition of these two spaces. By comparing the above reproducing kernel to the reproducing kernel of the de Branges space $\mathcal{H}(\Phi^\dagger)$ (see, e.g., [ADRdS97, Definition 2.1.1]) we conclude that $\mathcal{H}(\widehat{\mathfrak{D}}_-^*) = R^{-1}\mathcal{H}(\Phi^\dagger)$.

The Space $\mathcal{D}(\widehat{\mathfrak{D}})$ as a Reproducing Kernel Hilbert Space. Since $\mathcal{D}(\widehat{\mathfrak{D}})$ is continuously contained in $H^2_+(\mathcal{Y}) \oplus H^2_-(\mathcal{U})$ it is true that the point evaluation operators $\begin{bmatrix} \hat{y}_+ \\ \hat{u}_- \end{bmatrix} \mapsto \begin{bmatrix} \hat{y}_+(z) \\ \hat{u}_-(\zeta) \end{bmatrix}$ are continuous for all $(z,\zeta) \in \mathbb{D}_+ \times \mathbb{D}_-$. We can apply [ADRdS97, Theorem 1.1.2] to the space $\mathcal{D}(\widehat{\mathfrak{D}})$ by interpreting each vector in $\mathcal{D}(\widehat{\mathfrak{D}})$ as a function defined on $\Omega = \mathbb{D}_+ \times \mathbb{D}_-$, so that the point evaluation $E_{\mathcal{D}(\widehat{\mathfrak{D}})}$ is given by $E_{\mathcal{D}(\widehat{\mathfrak{D}})}(z,\zeta)\begin{bmatrix} \hat{y}_+ \\ \hat{u}_- \end{bmatrix} = \begin{bmatrix} \hat{y}_+(z) \\ \hat{u}_-(\zeta) \end{bmatrix}$, $\begin{bmatrix} \hat{y}_+ \\ \hat{u}_- \end{bmatrix} \in \mathcal{D}(\widehat{\mathfrak{D}})$. We claim that $\mathcal{D}(\widehat{\mathfrak{D}}) = \begin{bmatrix} 1_{H^2_+(\mathcal{Y})} & 0 \\ 0 & R^{-1} \end{bmatrix}\mathcal{D}(\Phi)$, where R is the reflection operator defined above and $\mathcal{D}(\Phi)$ is the standard de Branges space induced by the symbol Φ of $\widehat{\mathfrak{D}}$. This space was introduced in [dBR66a] and [dBR66b] as the state space in the de Branges–Rovnyak canonical model of a simple conservative i/s/o scattering system with scattering matrix Φ. The same space is characterized in [ADRdS97] as a reproducing kernel Hilbert space. See the cited references for details, as well as [Sar94].

Arguing above we find that the reproducing kernel of the reproducing kernel Hilbert space $\mathcal{D}(\widehat{\mathfrak{D}})$ of $(\mathcal{Y}\times\mathcal{U})$-valued functions defined on $\Omega = \mathbb{D}_+ \times \mathbb{D}_-$ is given by

$$K(z,\zeta;z_*,\zeta_*) = E_{H^2_+(\mathcal{Y})\oplus H^2_-(\mathcal{U})}(z,\zeta)\widehat{\mathcal{I}}^* E^*_{H^2_+(\mathcal{Y})\oplus H^2_-(\mathcal{U})}(z_*,\zeta_*),$$

where $E_{H^2_+(\mathcal{Y})\oplus H^2_-(\mathcal{U})}(z,\zeta)$ is the point evaluation operator in $H^2_+(\mathcal{Y})\oplus H^2_-(\mathcal{U})$ at the point $(z,\zeta)\in\mathbb{D}_+\times\mathbb{D}_-$, and $\widehat{\mathcal{I}}$ is the inclusion map $\mathcal{D}(\widehat{\mathfrak{D}})\hookrightarrow H^2_+(\mathcal{Y})\oplus H^2_-(\mathcal{U})$. To compute the adjoint of $\widehat{\mathcal{I}}$ we factor it into $\widehat{\mathcal{I}} = \begin{bmatrix}\widehat{\mathcal{I}}_+ & 0\\ 0 & \widehat{\mathcal{I}}_-\end{bmatrix}\widehat{\mathcal{I}}_{\mathcal{D}(\widehat{\mathfrak{D}})}$, where $\widehat{\mathcal{I}}_{\mathcal{D}(\widehat{\mathfrak{D}})}$ is the inclusion map $\mathcal{D}(\widehat{\mathfrak{D}})\hookrightarrow\mathcal{H}(\widehat{\mathfrak{D}}_+)\oplus\mathcal{H}(\widehat{\mathfrak{D}}^*_-)$, and $\begin{bmatrix}\widehat{\mathcal{I}}_+ & 0\\ 0 & \widehat{\mathcal{I}}_-\end{bmatrix}$ is the inclusion map $\mathcal{H}(\widehat{\mathfrak{D}}_+)\oplus\mathcal{H}(\widetilde{\mathfrak{D}}_-)\hookrightarrow H^2_+(\mathcal{Y}\oplus\mathcal{U})$. Thus, $\widehat{\mathcal{I}}^* = \widehat{\mathcal{I}}^*_{\mathcal{D}(\widehat{\mathfrak{D}})}\begin{bmatrix}\widehat{\mathcal{I}}_+ & 0\\ 0 & \widehat{\mathcal{I}}_-\end{bmatrix}^*$. By Lemma 3.2, $\widehat{\mathcal{I}}^*_{\mathcal{D}(\widehat{\mathfrak{D}})} = A_{\widehat{\mathfrak{D}}}$, where

$$A_{\widehat{\mathfrak{D}}} := \begin{bmatrix} 1_{\widehat{\mathcal{H}}(\widehat{\mathfrak{D}}_+)} & \Gamma_{(\widehat{\mathfrak{D}}^*_-,\widehat{\mathfrak{D}}_+)} \\ \Gamma^*_{(\widehat{\mathfrak{D}}^*_-,\widehat{\mathfrak{D}}_+)} & 1_{\widehat{\mathcal{H}}(\widehat{\mathfrak{D}}^{[\perp]}_-)}\end{bmatrix}, \tag{10.10}$$

and by Lemmas 10.3 and 10.6,

$$\begin{bmatrix}\widehat{\mathcal{I}}_+ & 0\\ 0 & \widehat{\mathcal{I}}_-\end{bmatrix}^* = \begin{bmatrix} 1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_+\widehat{\mathfrak{D}}^*_+ & 0\\ 0 & 1_{H^2_-(\mathcal{U})} - \widehat{\mathfrak{D}}^*_-\widehat{\mathfrak{D}}_-\end{bmatrix}.$$

These identities together with (10.7) and (10.9) imply that

$$\widehat{\mathcal{I}}^* = B_{\widehat{\mathfrak{D}}} := \begin{bmatrix} 1_{H^2_+(\mathcal{Y})} - \widehat{\mathfrak{D}}_+\widehat{\mathfrak{D}}^*_+ & \Gamma_{\widehat{\mathfrak{D}}}\\ \Gamma^*_{\widehat{\mathfrak{D}}} & 1_{H^2_-(\mathcal{U})} - \widehat{\mathfrak{D}}^*_-\widehat{\mathfrak{D}}_-\end{bmatrix}. \tag{10.11}$$

A direct computation shows that

$$E_{H^2(\mathcal{Y}\oplus\mathcal{U})}(\zeta)^*\begin{bmatrix}y_0\\ u_0\end{bmatrix} = \left(z\mapsto\begin{bmatrix}\frac{y_0}{1-z\overline{z}_*}\\ \frac{u_0}{\zeta\overline{\zeta}_*-1}\end{bmatrix}\right),\quad \begin{bmatrix}y_0\\ u_0\end{bmatrix}\in\begin{bmatrix}\mathcal{Y}\\ \mathcal{U}\end{bmatrix},$$

$$\begin{aligned} K(z,\zeta;z_*,\zeta_*) &= E_{H^2_+(\mathcal{Y})\oplus H^2_-(\mathcal{U})}(z,\zeta)\widehat{\mathcal{I}}^* E^*_{H^2_+(\mathcal{Y})\oplus H^2_-(\mathcal{U})}(z_*,\zeta_*)\\ &= \begin{bmatrix}\frac{1_{\mathcal{Y}}-\Phi(z)\Phi(z_*)^*}{1-z\overline{z}_*} & \frac{\Phi(z)-\Phi(1/\overline{\zeta}_*)}{z\overline{\zeta}_*-1}\\ \frac{\Phi(1/\overline{\zeta})^*-\Phi(z_*)^*}{1-\zeta\overline{z}_*} & \frac{1_{\mathcal{U}}-\Phi(1/\overline{\zeta})^*\Phi(1/\overline{\zeta}_*)}{\zeta\overline{\zeta}_*-1}\end{bmatrix},\\ &\qquad (z,\zeta;z_*,\zeta_*)\in(\mathbb{D}_+\times\mathbb{D}_+)\times(\mathbb{D}_-\times\mathbb{D}_-).\end{aligned}$$

This differs from the reproducing kernel of the standard de Branges space $\mathcal{D}(\Phi)$ (see, e.g., [ADRdS97, Definition 2.1.1]) only by a reflection in the second component. Thus, we conclude that $\mathcal{D}(\widehat{\mathfrak{D}}) = \left[\begin{smallmatrix} 1_{H^2_+(\mathcal{Y})} & 0 \\ 0 & R^{-1} \end{smallmatrix}\right] \mathcal{D}(\Phi)$.

Above we defined $B_{\widehat{\mathfrak{D}}}$ to be the adjoint of the inclusion map $\widehat{\mathcal{I}}_+ \colon \mathcal{D}(\widehat{\mathfrak{D}}) \hookrightarrow H^2_+(\mathcal{Y}) \oplus H^2_-(\mathcal{U})$. We can also interpret $B_{\widehat{\mathfrak{D}}}$ as an operator mapping $H^2_+(\mathcal{Y}) \oplus H^2_-(\mathcal{U})$ into itself by multiplying $B_{\widehat{\mathfrak{D}}}$ to the left by $\widehat{\mathcal{I}}_+$, after which it becomes equal to $\widehat{\mathcal{I}}_+ A_{\widehat{\mathfrak{D}}} \widehat{\mathcal{I}}_+^*$. Here $A_{\widehat{\mathfrak{D}}} = \widehat{T} A_{\widehat{\mathfrak{W}}} \widehat{T}^{-1}$, where

$$\widehat{T} := \left[\begin{smallmatrix} \widehat{T}_+ & 0 \\ 0 & \widehat{T}_- \end{smallmatrix}\right] : \mathcal{H}(\widehat{\mathfrak{W}}_+) \oplus \mathcal{H}(\widehat{\mathfrak{W}}_-^{[\perp]}) \to \mathcal{H}(\widehat{\mathfrak{D}}_+) \oplus \mathcal{H}(\widehat{\mathfrak{D}}_-^*). \tag{10.12}$$

The operator $A_{\widehat{\mathfrak{D}}}$ can be interpreted as a nonnegative operator on $\mathcal{H}(\widehat{\mathfrak{D}}_+) \oplus \mathcal{H}(\widehat{\mathfrak{D}}_-^*)$. Thus, with this interpretation $B_{\widehat{\mathfrak{D}}}$ becomes a nonnegative operator on $H^2_+(\mathcal{Y}) \oplus H^2_-(\mathcal{U})$. Moreover, $A_{\widehat{\mathfrak{D}}}^{1/2} = \widehat{T} A_{\widehat{\mathfrak{W}}}^{1/2} \widehat{T}^{-1}$. Thus the range space $\widehat{\mathcal{D}}(\widehat{\mathfrak{D}}) = \mathcal{R}\left(A_{\widehat{\mathfrak{D}}}^{1/2}\right)$ is the unitary image under the operator $\widehat{T}|_{\widehat{\mathcal{D}}(\widehat{\mathfrak{W}})}$ of the range space $\mathcal{D}(\widehat{\mathfrak{W}})$.

Lemma 10.7. *With the above definitions, $\mathcal{R}\left(B_{\widehat{\mathfrak{D}}}^{1/2}\right) = \mathcal{R}\left(A_{\widehat{\mathfrak{D}}}^{1/2}\right)$, with equality of range norms. Thus, the Hilbert space $\mathcal{D}(\widehat{\mathfrak{D}})$ is the range space of the operator $B_{\widehat{\mathfrak{D}}}^{1/2}$ in $\mathcal{H}_+(\mathcal{Y}) \oplus \mathcal{H}_-(\mathcal{U})$ as well as the range space of the operator $A_{\widehat{\mathfrak{D}}}^{1/2}$ in $\mathcal{H}(\widehat{\mathfrak{D}}_+) \oplus \mathcal{H}(\widehat{\mathfrak{D}}_-^*)$.*

Proof. Clearly $\mathcal{R}\left(B_{\widehat{\mathfrak{D}}}\right) \subset \mathcal{R}\left(A_{\widehat{\mathfrak{D}}}\right) \subset \mathcal{R}\left(A_{\widehat{\mathfrak{D}}}^{1/2}\right)$. Moreover, $\mathcal{R}\left(B_{\widehat{\mathfrak{D}}}\right)$ is dense in $\mathcal{R}\left(B_{\widehat{\mathfrak{D}}}^{1/2}\right)$ and $\mathcal{R}\left(A_{\widehat{\mathfrak{D}}}\widehat{\mathcal{I}}^*\right)$ is dense in $\mathcal{R}\left(A_{\widehat{\mathfrak{D}}}^{1/2}\right)$, so to prove the lemma it suffices to show that for all $x = B_{\widehat{\mathfrak{D}}} y = A_{\widehat{\mathfrak{D}}} \widehat{\mathcal{I}}^* y \in \mathcal{R}\left(B_{\widehat{\mathfrak{D}}}\right)$ we have

$$\|B_{\widehat{\mathfrak{D}}} y\|^2_{\mathcal{R}\left(B_{\widehat{\mathfrak{D}}}^{1/2}\right)} = \|A_{\widehat{\mathfrak{D}}} \widehat{\mathcal{I}}^* y\|^2_{\mathcal{R}\left(A_{\widehat{\mathfrak{D}}}^{1/2}\right)}.$$

But this follows from the fact that

$$\begin{aligned} \|B_{\widehat{\mathfrak{D}}} y\|^2_{\mathcal{R}\left(B_{\widehat{\mathfrak{D}}}^{1/2}\right)} &= \|B_{\widehat{\mathfrak{D}}}^{1/2} y\|^2_{H^2_+(\mathcal{Y}) \oplus H^2_-(\mathcal{U})} = (y, B_{\widehat{\mathfrak{D}}} y)_{H^2_+(\mathcal{Y}) \oplus H^2_-(\mathcal{U})} \\ &= (y, \widehat{\mathcal{I}} A_{\widehat{\mathfrak{D}}} \widehat{\mathcal{I}}^* y)_{H^2_+(\mathcal{Y}) \oplus H^2_-(\mathcal{U})} = (\widehat{\mathcal{I}}^* y, A_{\widehat{\mathfrak{D}}} \widehat{\mathcal{I}}^* y)_{\mathcal{H}_+(\widehat{\mathfrak{D}}_-) \oplus \mathcal{H}_-(\widehat{\mathfrak{D}}_-^*)} \\ &= \|A_{\widehat{\mathfrak{D}}} \widehat{\mathcal{I}}^* y\|^2_{\mathcal{R}\left(A_{\widehat{\mathfrak{D}}}^{1/2}\right)}. \quad \square \end{aligned}$$

Remark 10.8. The characterization in Lemma 10.7 of $\mathcal{D}(\widehat{\mathfrak{D}})$ as the range of the operator $A_{\widehat{\mathfrak{D}}}^{1/2}$ in the space $\mathcal{H}(\widehat{\mathfrak{D}}_+) \oplus \mathcal{H}(\widehat{\mathfrak{D}}_-^*)$ is equivalent to the one given in [ADRdS97, Theorem 3.4.3]. The operator Λ appearing in that theorem is given by $\Lambda = R^{-1} \Gamma^*_{(\widehat{\mathfrak{D}}_-^*, \widehat{\mathfrak{D}}_+)}$.

Scattering I/S/O Representations of a Passive S/S System Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a passive s/s system with future, full, and past behaviors $\mathfrak{W}_+$, $\mathfrak{W}$, and $\mathfrak{W}_-$, respectively. Let $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ e a fundamental decomposition of $\mathcal{W}$, and let $\mathfrak{D}$ and $\mathfrak{D}_\pm$ be the operators in the graph representations of $\mathfrak{W}$ and $\mathfrak{W}_\pm$. The Kreĭn nodes space $\mathfrak{K} = -\mathcal{X}\,[\dotplus]\,\mathcal{X}\,[\dotplus]\,\mathcal{W}$ has the fundamental decomposition $\mathfrak{K} = -(\mathcal{X} \oplus \mathcal{Y})\,[\dotplus]\,(\mathcal{X} \oplus \mathcal{U})$. By assertion 1) of Proposition 2.1, V has the graph representation

$$V = \left\{ \left[\begin{smallmatrix} A\hat{x}_0 + Bu_0 \\ \hat{x}_0 \\ C\hat{x}_0 + Du_0 + u_0 \end{smallmatrix}\right] \in \mathfrak{K} \,\middle|\, x_0 \in \mathcal{X},\ u_0 \in \mathcal{U} \right\}, \tag{10.13}$$

where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a contraction $\mathcal{X} \oplus \mathcal{U} \to \mathcal{X} \oplus \mathcal{Y}$. This means that Σ has i/s/o representation $\Sigma_{i/s/o} = \big(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\big)$ where the state space $\mathcal{X}$, input space $\mathcal{U}$, and output space $\mathcal{Y}$ are Hilbert spaces. The set of trajectories of $\Sigma_{i/s/o}$ on an interval I consists of triples $(x(\cdot), u(\cdot), y(\cdot))$ satisfying

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Du(n), \end{aligned} \qquad n \in I. \tag{10.14}$$

The i/s/o system $\Sigma_{i/s/o}$ defined above is called a *scattering representation* of the passive s/s system Σ. The transfer function, which is also called the *scattering matrix*, of this i/s/o representation is given by

$$\Phi(z) = zC(1_{\mathcal{X}} - zA)^{-1}B + D, \tag{10.15}$$

and it is a Schur function in $\mathbb{D}_+$.

A scattering representation of a s/s system is controllable, or observable, or simple, or minimal, or forward conservative, or backward conservative, or conservative if the corresponding s/s system has the corresponding property. More details about scattering representations of passive s/s systems can be found in, e.g., [AS07a] and [AS09b].

Scattering Representations of the Frequency Domain Versions of the Canonical S/S Model. We continue by developing a description of the i/s/o representation of $\widehat{\Sigma}_{\rm sc}$ corresponding to a fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of the signal space $\mathcal{W}$. This description contains the unitary operator $\widehat{T}$ defined in (10.12). The operator $\widehat{T}$ and its inverse are explicitly given by

$$\begin{aligned} \widehat{T}\widehat{Q}\begin{bmatrix} \widehat{y} \\ \widehat{u} \end{bmatrix} &= \begin{bmatrix} \widehat{\pi}_+ & -\widehat{\mathfrak{D}}_+\widehat{\pi}_+ \\ -\widehat{\mathfrak{D}}_-^*\widehat{\pi}_- & \widehat{\pi}_- \end{bmatrix} \begin{bmatrix} \widehat{y} \\ \widehat{u} \end{bmatrix}, \qquad \begin{bmatrix} \widehat{y} \\ \widehat{u} \end{bmatrix} \in \mathcal{L}(\widehat{\mathfrak{W}}), \\ \widehat{T}^{-1}\begin{bmatrix} \widehat{y}_+ \\ \widehat{u}_- \end{bmatrix} &= \widehat{Q}\begin{bmatrix} \hat{y}_+ \\ \hat{u}_- \end{bmatrix}, \qquad \begin{bmatrix} \hat{y}_+ \\ \hat{u}_- \end{bmatrix} \in \mathcal{D}(\widehat{\mathfrak{W}}). \end{aligned} \tag{10.16}$$

We begin by applying the unitary similarity transform $\widehat{T}$ to $\widehat{\Sigma}_{\rm sc}$ in order to replace the state space $\mathcal{D}(\widehat{\mathfrak{W}})$ of $\widehat{\Sigma}_{\rm sc}$ by the state space $\mathcal{D}(\widehat{\mathfrak{D}})$ of the new system $\Sigma_{\rm sc}^{\widehat{\mathfrak{D}}} = (V_{\rm sc}^{\widehat{\mathfrak{D}}}; \mathcal{D}(\widehat{\mathfrak{D}}), \mathcal{W})$ with generating subspace

$$V_{\rm sc}^{\widehat{\mathfrak{D}}} := \begin{bmatrix} \widehat{T} & 0 & 0 \\ 0 & \widehat{T} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V_{\rm sc}^{\widehat{\mathfrak{W}}}. \tag{10.17}$$

The fundamental decomposition $\mathcal{W} = -\mathcal{Y} \,[\dotplus]\, \mathcal{U}$ of $\mathcal{W}$ is admissible for $\Sigma_{\rm sc}^{\widehat{\mathfrak{D}}}$, and the corresponding i/s/o representation $\Sigma_{i/s/o}^{\widehat{\mathfrak{D}}} = \left(\left[\begin{smallmatrix} A_{\rm sc} & B_{\rm sc} \\ C_{\rm sc} & D_{\rm sc} \end{smallmatrix} \right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ is a simple conservative scattering system with scattering matrix Φ.

Explicit formulas for the operators $A_{\rm sc}$, $B_{\rm sc}$, $C_{\rm sc}$, and $D_{\rm sc}$ can be computed in the following way. Let $\left[\begin{smallmatrix} \hat{y}_+ \\ \hat{u}_- \end{smallmatrix} \right] \in \mathcal{D}(\widehat{\mathfrak{D}})$ be the initial state of $\Sigma_{\rm sc}^{\widehat{\mathfrak{D}}}$. Then $\widehat{T}^{-1} \left[\begin{smallmatrix} \hat{y}_+ \\ \hat{u}_- \end{smallmatrix} \right]$ is the corresponding initial state of $\widehat{\Sigma}_{\rm sc}$. By (10.16), this initial state can be written in the form

$$\widehat{T}^{-1} \begin{bmatrix} \hat{y}_+ \\ \hat{u}_- \end{bmatrix} = \begin{bmatrix} \hat{y}_+ \\ \hat{u}_- \end{bmatrix} + \widehat{\mathfrak{W}}_+ + \widehat{\mathfrak{W}}_-^{[\perp]},$$

and hence $T^{-1} \left[\begin{smallmatrix} \hat{y}_+ \\ \hat{u}_- \end{smallmatrix} \right] = \widehat{Q}\widehat{w}$, where

$$\widehat{w} = \begin{bmatrix} \hat{y}_+ \\ \hat{u}_- \end{bmatrix} + \begin{bmatrix} \widehat{\mathfrak{D}}_+ \hat{u}_- \\ \hat{u}_- \end{bmatrix} + \begin{bmatrix} \hat{y}_- \\ \widehat{\mathfrak{D}}_-^* \hat{y}_- \end{bmatrix}, \tag{10.18}$$

and $\hat{u}_+$ and $\hat{y}_-$ are free parameters in $H^2_+(\mathcal{U})$ and $H^2_-(\mathcal{Y})$, respectively.

By (9.7) and (10.17),

$$V_{\rm sc}^{\widehat{\mathfrak{D}}} = \left\{ \left[\begin{smallmatrix} \widehat{T}\widehat{Q}\widehat{S}^{-1}\widehat{w} \\ \widehat{T}\widehat{Q}\widehat{w} \\ \widehat{w}(0) \end{smallmatrix} \right] \,\middle|\, \widehat{w} \in \mathcal{L}(\widehat{\mathfrak{W}}) \right\}.$$

Here $\widehat{T}\widehat{Q}\widehat{w} = \left[\begin{smallmatrix} \hat{y}_+ \\ \hat{u}_- \end{smallmatrix} \right]$ and

$$\widehat{w}(0) = \begin{bmatrix} \hat{y}_+(0) \\ 0 \end{bmatrix} + \begin{bmatrix} \widehat{\mathfrak{D}}_+ \hat{u}_- \\ \hat{u}_- \end{bmatrix}(0) = \begin{bmatrix} \hat{y}_+(0) \\ 0 \end{bmatrix} + \begin{bmatrix} \Phi(0)\hat{u}_+(0) \\ \hat{u}_+(0) \end{bmatrix}.$$

In order to compute $\widehat{T}\widehat{Q}\widehat{S}^{-1}\widehat{w}$ we apply $\widehat{T}\widehat{Q}\widehat{S}^{-1}$ to each of the components in (10.18), and get

$$
\begin{aligned}
\widehat{T}\widehat{Q}\widehat{S}^{-1}\begin{bmatrix}\hat{y}_+\\ \hat{u}_-\end{bmatrix} &= \begin{bmatrix}\widehat{\pi}_+\widehat{S}^{-1} & 0\\ -\widehat{\mathfrak{D}}_-^*\widehat{\pi}_-\widehat{S}^{-1} & \widehat{\pi}_-\widehat{S}^{-1}\end{bmatrix}\begin{bmatrix}\hat{y}_+\\ \hat{u}_-\end{bmatrix},\\
\widehat{T}\widehat{Q}\widehat{S}^{-1}\begin{bmatrix}\mathfrak{D}_+\hat{u}_+\\ \hat{u}_+\end{bmatrix} &= \begin{bmatrix}\widehat{S}_+^*\widehat{\mathfrak{D}}_+ - \widehat{\mathfrak{D}}_+\widehat{S}_+^*\\ \widehat{\pi}_+\widehat{S}^{-1} - \widehat{\mathfrak{D}}_-^*\widehat{\pi}_-\widehat{S}^{-1}\widehat{\mathfrak{D}}_+\end{bmatrix}\hat{u}_+,\\
\widehat{T}\widehat{Q}\widehat{S}^{-1}\begin{bmatrix}\hat{y}_-\\ \widehat{\mathfrak{D}}_-^*\hat{y}_-\end{bmatrix} &= 0.
\end{aligned}
\tag{10.19}
$$

The operators above can be computed explicitly by means of (9.2), (9.4), (10.2), and (10.4), and they turn out to be

$$
\begin{aligned}
\begin{bmatrix}\widehat{\pi}_+\widehat{S}^{-1} & 0\\ -\widehat{\mathfrak{D}}_-^*\widehat{\pi}_-\widehat{S}^{-1} & \widehat{\pi}_-\widehat{S}^{-1}\end{bmatrix}\begin{bmatrix}\hat{y}_+\\ \hat{u}_-\end{bmatrix} &= \left((z,\zeta)\mapsto\begin{bmatrix}\frac{1}{z}\big(\hat{y}_+(z)-\hat{y}_+(0)\big)\\ \frac{1}{\zeta}\big(\hat{u}_-(\zeta)-\Phi(1/\overline{\zeta})^*\hat{y}_+(0)\big)\end{bmatrix}\right),\\
\begin{bmatrix}\widehat{S}_+^*\widehat{\mathfrak{D}}_+ - \widehat{\mathfrak{D}}_+\widehat{S}_+^*\\ \widehat{\pi}_+\widehat{S}^{-1} - \widehat{\mathfrak{D}}_-^*\widehat{\pi}_-\widehat{S}^{-1}\widehat{\mathfrak{D}}_+\end{bmatrix}\hat{u}_+ &= \left((z,\zeta)\mapsto\begin{bmatrix}\frac{1}{z}\big(\Phi(z)-\Phi(0)\big)\\ \frac{1}{\zeta}\big(1_{\mathcal{U}}-\Phi(1/\overline{\zeta})^*\Phi(0)\big)\end{bmatrix}\hat{u}_+(0)\right).
\end{aligned}
$$

Thus, we conclude that $V_{\rm sc}^{\widehat{\mathfrak{D}}}$ has the representation

$$
V_{\rm sc}^{\widehat{\mathfrak{D}}} = \left\{\left[\begin{smallmatrix}A_{\rm sc}\hat{x}_0+B_{\rm sc}u_0\\ \hat{x}_0\\ C_{\rm sc}\hat{x}_0+D_{\rm sc}u_0\ \ +u_0\end{smallmatrix}\right]\in\left[\begin{smallmatrix}\mathcal{D}(\widehat{\mathfrak{D}})\\ \mathcal{D}(\widehat{\mathfrak{D}})\\ \mathcal{W}\end{smallmatrix}\right]\,\middle|\,\hat{x}_0\in\mathcal{D}(\widehat{\mathfrak{D}}),\ u_0\in\mathcal{U}\right\},
\tag{10.20}
$$

where

$$
\begin{aligned}
\left(A_{\rm sc}\begin{bmatrix}\hat{y}_+\\ \hat{u}_-\end{bmatrix}\right)(z,\zeta) &= \begin{bmatrix}\frac{1}{z}\big(\hat{y}_+(z)-\hat{y}_+(0)\big)\\ \frac{1}{\zeta}\big(\hat{u}_-(\zeta)-\Phi(1/\overline{\zeta})^*\hat{y}_+(0)\big)\end{bmatrix},\\
(B_{\rm sc}u_0)(z,\zeta) &= \left((z,\zeta)\mapsto\begin{bmatrix}\frac{1}{z}\big(\Phi(z)-\Phi(0)\big)\\ \frac{1}{\zeta}\big(1_{\mathcal{U}}-\Phi(1/\overline{\zeta})^*\Phi(0)\big)\end{bmatrix}u_0\right),\\
C_{\rm sc}\begin{bmatrix}\hat{y}_+\\ \hat{u}_-\end{bmatrix} &= \hat{y}_+(0),\\
D_{\rm sc} &= \Phi(0).
\end{aligned}
\tag{10.21}
$$

Comparing these coefficients $\left[\begin{smallmatrix}A_{\rm sc} & B_{\rm sc}\\ C_{\rm sc} & D_{\rm sc}\end{smallmatrix}\right]$ to those given in, e.g., [ADRdS97] we find that the scattering representation $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix}A_{\rm sc} & B_{\rm sc}\\ C_{\rm sc} & D_{\rm sc}\end{smallmatrix}\right], \mathcal{D}(\widehat{\mathfrak{D}}), \mathcal{U}, \mathcal{Y}\right)$ of $\Sigma_{\rm sc}^{\widehat{\mathfrak{D}}}$ corresponding to the fundamental decomposition $\mathcal{W} = -\mathcal{Y}\,[\dotplus]\,\mathcal{U}$ of $\mathcal{W}$ is unitarily similar with similarity operator $\left[\begin{smallmatrix}1_{H^2_+(\mathcal{Y})} & 0\\ 0 & R^{-1}\end{smallmatrix}\right]$ to the canonical de Branges–Rovnyak model of a simple conservative i/s/o scattering system with scattering matrix Φ.

Scattering representations of the two Conservative Dilations in Section 8. It is possible to apply the Fourier transform to also convert the two models at the end of Section 8 into frequency domain models. The scattering representations of these models that we obtain by applying the same method that have been used earlier in this section coincide with the corresponding models in [ADRdS97, Section 2.4]. We leave the details to the reader.

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