Stabilization by Collocated Feedback

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Abstract

Recently Guo and Luo (and independently Weiss and Tucsnak) were able to prove that the damped second order system

$$\begin{aligned} \ddot{z}(t) + A_0 z(t) &= -\frac{1}{2} C_0^* C_0 \dot{z}(t) + C_0^* u(t), \\ y(t) &= -C_0 \dot{z}(t) + u(t), \end{aligned}$$

can be interpreted as a continuous time (well-posed and stable) scattering conservative system with input u, state $\begin{bmatrix} \sqrt{A_0}\, z \\ \dot{z} \end{bmatrix}$, and output y. Here A_0 is a positive (unbounded) self-adjoint operator on a Hilbert space Z with a bounded inverse, and C_0 is a bounded linear operator from $\mathcal{D}(\sqrt{A_0}\,)$ to another Hilbert space U. We show that this is a special case of the following more general result: if we apply the so called diagonal transform (which is a particular rescaled feedback/feedforward transform) to an arbitrary continuous time impedance conservative system, then we always get a scattering conservative system. In the particular case mentioned above the corresponding impedance conservative system is the undamped system

$$\begin{aligned} \ddot{z}(t) + A_0 z(t) &= \frac{1}{\sqrt{2}}\, C_0^* u(t), \\ y(t) &= \frac{1}{\sqrt{2}}\, C_0 \dot{z}(t), \end{aligned}$$

which may be interpreted as a second order system with collocated actuators and sensors.

Keywords

Scattering, impedance, conservative, passive, compatible, diagonal transform, feed-back, flow-inversion.

1 Introduction

In two recent articles Guo and Luo [1] and Weiss and Tucsnak [15] study the abstract second order system of differential equations

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + A_0 z(t) &= -\frac{1}{2} C_0^* \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t) + C_0^* u(t), \\ y(t) &= -\frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t) + u(t), \end{aligned} \tag{1}$$

with input u, state $\begin{bmatrix} \sqrt{A_0}\, z \\ \dot{z} \end{bmatrix}$, and output y. Here A_0 is an arbitrary positive (unbounded) self-adjoint operator on a Hilbert space Z with a bounded inverse. We define the fractional powers of A_0 in the usual way, and denote $Z_{1/2} = \mathcal{D}(\sqrt{A_0}\,)$ and $Z_{-1/2} = (Z_{1/2})^*$ (where we identify Z with its dual). Thus, $Z_{1/2} \subset Z \subset Z_{-1/2}$, with continuous and dense injections, and A^{-1} maps $Z_{-1/2}$ onto $Z_{1/2}$. The operator C is an arbitrary bounded linear operator from $Z_{1/2}$ to another Hilbert space U. Guo and Luo showed in [1] and Weiss and Tucsnak showed in [15] (independently of each other) that the above system may be interpreted as a continuous time (well-posed and energy stable) *scattering conservative* system with input u, state $x = \begin{bmatrix} \sqrt{A_0}\, z \\ \dot{z} \end{bmatrix}$, and output y. The input and output spaces are both U, and the state space is $X = \begin{bmatrix} Z \\ Z \end{bmatrix} (= Z \times Z)$.

Formally, the system (1) is equivalent to the *diagonally transformed system*

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + A_0 z(t) &= \frac{1}{\sqrt{2}} C_0^* u^\times(t), \\ y^\times(t) &= \frac{1}{\sqrt{2}} \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t), \end{aligned} \tag{2}$$

which we get from (1) by replacing u and y in (1) by $u^\times = \frac{1}{\sqrt{2}}(u + y)$ respectively $y^\times = \frac{1}{\sqrt{2}}(u - y)$. We can formally get back to (1) by repeating the same transform: we replace $u^\times$ and $y^\times$ in (2) by $u = \frac{1}{\sqrt{2}}(u^\times + y^\times)$ respectively $y = \frac{1}{\sqrt{2}}(u^\times - y^\times)$. This transform, drawn in Figure 1, is simply a rescaled feedback/feedforward connection.

The purpose of this article is to show that the above transformations are not just *formal*, but that that they can be mathematically justified, thereby

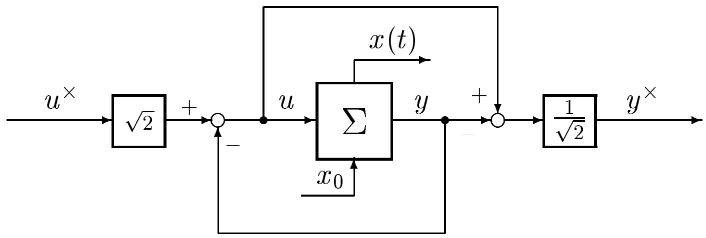

Figure 1: The diagonal transform

giving a positive answer to the question posed in [1, Remark 2]. It follows directly from [8, Theorem 4.7] that (2) is an impedance conservative system of the type introduced in [8]. According to [9, Theorem 8.2], by applying the diagonal transform to this system we get a scattering passive system. As we shall show below, this scattering passive system is exactly the system described by (1).

2 Infinite-Dimensional Linear Systems

Many infinite-dimensional linear time-invariant continuous-time systems can be described by the equations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{3}$$

on a triple of Hilbert spaces, namely, the input space U, the state space X, and the output space Y. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operator A is supposed to be the generator of a strongly continuous semigroup. The operators A, B and C are usually unbounded, but D is bounded.

By modifying this set of equations slightly we get the class of systems which will be used in this article. In the sequel, we think about the block matrix $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as *one single (unbounded) operator* from $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, and write (3) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0. \tag{4}$$

The operator S completely determines the system. Thus, we may identify the system with such an operator, which we call the *node* of the system.

The system nodes that we use have a certain structure which makes it resemble a block matrix operator of the type $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. To describe this structure we need the notion of *rigged Hilbert spaces*. Let A be the generator of a C_0 semigroup on the Hilbert space X. We denote its domain $\mathcal{D}(A)$ by X_1. We identify the dual of X with X itself, and denote $X_{-1} = \mathcal{D}(A^*)^*$. Then $X_1 \subset X \subset X_{-1}$ with continuous and dense injections. The operator A has a unique extension to an operator in $\mathcal{L}(X; X_{-1})$ which we denote by $A_{|X}$ (thereby indicating that the domain of this operator is all of X). This operator is the generator a C_0 semigroup on X_{-1}, whose restriction to X is the semigroup generated by A.

Definition 2.1. We call S a *system node* on the three Hilbert spaces (U, X, Y) if it satisfies condition (S) below:[1]

(S) $S := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \supset \mathcal{D}(S) \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ is a closed linear operator. Here $A\&B$ is the restriction to $\mathcal{D}(S)$ of $\begin{bmatrix} A_{|X} & B \end{bmatrix}$, where A is the *generator of a C_0 semigroup* on X (the notations $A_{|X} \in \mathcal{L}(X; X_{-1})$ and X_{-1} were introduced in the text above). The operator B is an arbitrary operator in $\mathcal{L}(U; X_{-1})$, and $C\&D$ is an arbitrary linear operator from $\mathcal{D}(S)$ to Y. In addition, we require that

$$\mathcal{D}(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \;\middle|\; A_{|X}x + Bu \in X \right\}.$$

We shall use the following names of the different parts of the system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. The operator A is the *main operator* or the *semigroup generator*, B is the *control operator*, $C\&D$ is the *combined observation/feedthrough operator*, and the operator C defined by

$$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in X_1,$$

is the *observation operator* of S.

An easy algebraic computation (see, e.g., [10, Section 4.7] for details) shows that for each $\alpha \in \rho(A) = \rho(A_{|X})$, the operator $\left[\begin{smallmatrix} 1 & (\alpha - A_{|X})^{-1}B \\ 0 & 1 \end{smallmatrix}\right]$ is an boundedly invertible mapping between $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} X_1 \\ U \end{smallmatrix}\right] \to \mathcal{D}(S)$. Since $\left[\begin{smallmatrix} X_1 \\ U \end{smallmatrix}\right]$ is dense in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$, this implies that $\mathcal{D}(S)$ is dense in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. Furthermore, since the second column $\left[\begin{smallmatrix} (\alpha - A_{|X})^{-1}B \\ 1 \end{smallmatrix}\right]$ of this operator maps U into $\mathcal{D}(S)$, we can define

[1] This definition is equivalent to the corresponding definitions used by Smuljan in [6] and by Salamon in [4, 5].

the *transfer function* of S by

$$\widehat{\mathfrak{D}}(s) := C\&D \begin{bmatrix} (s - A_{|X})^{-1}B \\ 1 \end{bmatrix}, \qquad s \in \rho(A), \tag{5}$$

which is a $\mathcal{L}(U;Y)$-valued analytic function on $\rho(A)$. By the resolvent formula, for any two α, $\beta \in \rho(A)$,

$$\begin{aligned} \widehat{\mathfrak{D}}(\alpha) - \widehat{\mathfrak{D}}(\beta) &= C\big[(\alpha - A_{|X})^{-1} - (\beta - A_{|X})^{-1}\big]B \\ &= (\beta - \alpha)C(\alpha - A)^{-1}(\beta - A_{|X})^{-1}B. \end{aligned} \tag{6}$$

Let us finally present the class of *compatible* system nodes, originally introduced by Helton [2]). This class can be defined in several different ways, one of which is the following. We introduce an auxiliary Banach space W satisfying $X_1 \subset W \subset X$, fix some $\alpha \in \rho(A)$, and define $W_{-1} = (\alpha - A_{|X})W$ with $|x|_{W_{-1}} = |(\alpha - A_{|X})^{-1}x|_W$ (defined in this way the norm in W_{-1} depends on α, but the space itself does not). Thus

$$X_1 \subset W \subset X \subset W_{-1} \subset X_{-1}.$$

The embeddings $W \subset X$ and $W_{-1} \subset X_{-1}$ are always dense, but the embeddings $X_1 \subset W$ and $X \subset W_{-1}$ need not be dense. The system is *compatible* if $\mathcal{R}(B) \subset W_{-1}$ and C has an extension to an operator $C_{|W} \in \mathcal{L}(W;Y)$ (this extension is not unique unless the embedding $X_1 \subset W$ is dense). Thus, in this case the operator $C_{|W}(\alpha - A_{|X})^{-1}B \in \mathcal{L}(U;Y)$ for all $\alpha \in \rho(A)$. If we fix some $\alpha \in \rho(A)$ and define

$$D := \widehat{\mathfrak{D}}(\alpha) - C_{|W}(\alpha - A_{|X})^{-1}B,$$

then $D \in \mathcal{L}(U;Y)$, and it follows from (6) that D *does not depend on* α, only on A, B, $C_{|W}$, and $\widehat{\mathfrak{D}}$ (in particular, different extensions of C to W give different operators D). Clearly, the above formula means that $\widehat{\mathfrak{D}}$ can be written in the simple form

$$\widehat{\mathfrak{D}}(s) = C_{|W}(s - A_{|X})^{-1}B + D, \quad s \in \rho(A). \tag{7}$$

Another way of describing compatibility is to say that the system node S can be extended to a bounded linear operator $\left[\begin{smallmatrix} A_{|W} & B \\ C_{|W} & D \end{smallmatrix}\right] \in \mathcal{L}\big(\left[\begin{smallmatrix} W \\ U \end{smallmatrix}\right];\left[\begin{smallmatrix} W_{-1} \\ U \end{smallmatrix}\right]\big)$, where $A_{|W}$ is the restriction of $A_{|X}$ to W. Thus

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} A_{|W} & B \\ C_{|W} & D \end{bmatrix}_{|\mathcal{D}(S)}.$$

We shall refer to the operator $\begin{bmatrix} A_{|W} & B \\ C_{|W} & D \end{bmatrix}$ as a (possibly non-unique) *compatible representation of S over the space W*. There is always a minimal space W which can be used in this representation, namely $W := (\alpha - A)^{-1}W_{-1}$ where $\alpha \in \rho(A)$ and $W_{-1} := (X + BU)$, but it is frequently more convenient to work in some other space W (for example, it may be possible to choose a larger space W for which the embedding $X_1 \subset W$ is dense and the extension is unique).

As shown in [11], the system node S *of a well-posed system* is always compatible, but the converse is not true (an example of a compatible system of the type (2) which is not well-posed is given in [13]).

Every system node induces a 'dynamical system' of a certain type:

Lemma 2.2. *Let S be a system node on (U, X, Y). Then, for each $x_0 \in X$ and $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^+; U)$ with $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$, the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0, \tag{8}$$

has a unique solution (x, y) satisfying $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S)$ for all $t \geq 0$, $x \in C^1(\mathbb{R}^+; X)$, and $y \in C(\mathbb{R}^+; Y)$.

This lemma is proved in [3] (and also in [10]).[2]

So far we have *defined Σ_0^t only for the class of smooth data* given in Lemma 2.2. It is possible to allow arbitrary initial states $x_0 \in X$ and input functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+; U)$ in Lemma 2.2 by allowing the state to take values in the larger space X_{-1} instead of in X, and by allowing y to be a distribution. Rather than presenting this result in its full generality, let us observe the following fact.

Lemma 2.3. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, Y). Let $x_0 \in X$, and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+; U)$, and let x and y be the state trajectory and output of S with initial state x_0, and input function u. If $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+; X)$, then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{D}(S))$, $y \in L^1_{\mathrm{loc}}(\mathbb{R}^+; Y)$, and $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is the unique solution with the above properties of the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \text{ for almost all } t \geq 0, \quad x(0) = x_0. \tag{9}$$

If $u \in C(\mathbb{R}^+; U)$ and $x \in C^1(\mathbb{R}^+; X)$, then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(\mathbb{R}^+; \mathcal{D}(S))$, $y \in C(\mathbb{R}^+; Y)$, and the equation (9) holds for all $t \geq 0$.

[2] Well-posed versions of this lemma (see Definition 2.4) are (implicitly) found in [4] and [6] (and also in [11]). In the well-posed case we need less smoothness of u: it suffices to take $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^+; U)$. In addition y will be smoother: $y \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^+; Y)$.

See [10, Section 4.7] for the proof.

Many system nodes are *well-posed*:

Definition 2.4. A system node S is *well-posed* if, for some $t > 0$, there is a finite constant $K(t)$ such that the solution (x, y) in Lemma 2.2 satisfies

$$|x(t)|^2 + \|y\|^2_{L^2(0,t)} \leq K(t)\big(|x_0|^2 + \|u\|^2_{L^2(0,t)}\big). \qquad \textbf{(WP)}$$

It is *energy stable* if there is some $K < \infty$ so that, for all $t \in \mathbb{R}^+$, the solution (x, y) in Lemma 2.2 satisfies

$$|x(t)|^2 + \|y\|^2_{L^2(0,t)} \leq K\big(|x_0|^2 + \|u\|^2_{L^2(0,t)}\big). \qquad \textbf{(ES)}$$

For more details, explanations and examples we refer the reader to [3] and [7, 8, 9, 10] (and the references therein).

3 Passive and Conservative Scattering and Impedance Systems

The following definitions are slightly modified versions of the definitions in the two classical papers [16, 17] by Willems (although we use a slightly different terminology: our *passive* is the *same as Willems' dissipative*, and we use Willems' *storage function* as the norm in the state space).

Definition 3.1. A system node S is *scattering passive* if, for all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies

$$|x(t)|^2 - |x_0|^2 \leq \|u\|^2_{L^2(0,t)} - \|y\|^2_{L^2(0,t)}. \qquad \textbf{(SP)}$$

It is *scattering energy preserving* if the above inequality holds in the form of an equality: for all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies

$$|x(t)|^2 - |x_0|^2 = \|u\|^2_{L^2(0,t)} - \|y\|^2_{L^2(0,t)}. \qquad \textbf{(SE)}$$

Finally, it is *scattering conservative* if both S and S^* are scattering energy preserving.[3]

Thus, *every scattering passive system is well-posed and energy stable*: the passivity inequality **(SP)** implies the energy stability inequality **(ES)**.

[3] If S is a system node on (U, X, Y), then its adjoint S^* is a system node on (Y, X, U). See, e.g., [3].

Definition 3.2. A system node S on (U, X, U) (note that $Y = U$) is *impedance passive* if, for all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies

$$|x(t)|_X^2 - |x_0|_X^2 \leq 2\int_0^t \Re\langle y(t), u(t)\rangle_U \, dt. \tag{\textbf{IP}}$$

It is *impedance energy preserving* if the above inequality holds in the form of an equality: for all $t > 0$, the solution (x, y) in Lemma 2.2 satisfies

$$|x(t)|_X^2 - |x_0|_X^2 = 2\int_0^t \Re\langle y(t), u(t)\rangle_U \, dt. \tag{\textbf{IE}}$$

Finally, S is *impedance conservative* if both S and the dual system node S^* are impedance energy preserving.

Note that in this case *well-posedness is neither guaranteed, nor relevant.*

Physically, *passivity* means that *there are no internal energy sources.* An energy preserving system has neither any internal energy sources nor any sinks. Other types of passivity have also been considered in the literature; in particular *transmission (or chain scattering)* passive or conservative systems.

Both in the scattering and in the impedance setting, the property of being passive is conserved under the passage from a system node S to its dual. See [8] for details.

The following theorem can be used to test if a system node is impedance passive or energy preserving or conservative:

Theorem 3.3 ([8, Theorems 4.2, 4.6, and 4.7]). *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a system node on* (U, X, U).

(i) *S is impedance passive if and only if the system node* $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is dissipative, i.e, for all* $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}(S)$,

$$\Re\left\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \leq 0. \tag{10}$$

(ii) *S is impedance energy preserving if and only if the system node* $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ *is skew-symmetric, i.e.,* $\mathcal{D}(S) = \mathcal{D}(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]) \subset \mathcal{D}(\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]^*)$, *and*

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = -\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \qquad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathcal{D}(S). \tag{11}$$

(iii) *S is impedance conservative if and only if the system node $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is skew-adjoint, i.e.,*

$$\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}^* = -\begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}. \tag{12}$$

Equivalently, S is impedance conservative if and only $A^* = -A$, $B^* = C$, and $\widehat{\mathfrak{D}}(\alpha) + \widehat{\mathfrak{D}}(-\overline{\alpha})^* = 0$ for some (or equivalently, for all) $\alpha \in \rho(A)$ (in particular, this identity is true for all α with $\Re\alpha \neq 0$).

Many impedance passive systems are well-posed. There is a simple way of characterizing such systems:

Theorem 3.4. *An impedance passive system node is well-posed if and only if its transfer function $\widehat{\mathfrak{D}}$ is bounded on some (or equivalently, on every) vertical line in $\mathbb{C}^+$. When this is the case, the growth bound of the system is zero, and, in particular, $\widehat{\mathfrak{D}}$ is bounded on every right half-plane $\mathbb{C}^+_\epsilon = \{s \in \mathbb{C} \mid \Re s > \epsilon\}$ with $\epsilon > 0$.*

This is [8, Theorem 5.1]. It can be used to show that many systems with *collocated actuators and sensors* are well-posed.

Example 3.5. To get the system described by (2) we take the state to be $x = \left[\begin{smallmatrix} \sqrt{A_0}\, z \\ \dot{z} \end{smallmatrix}\right]$, the input to be u, and the output to be y. The input and output spaces are U, the state space is $\left[\begin{smallmatrix} Z \\ Z \end{smallmatrix}\right]$, and, in compatibility notion with $W = Z_{1/2}$ and $W_{-1/2} = Z_{-1/2}$, the extended system node is given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} 0 & \sqrt{A_0} & 0 \\ -\sqrt{A_0} & 0 & \frac{1}{\sqrt{2}} C_0^* \\ \hline 0 & \frac{1}{\sqrt{2}} C_0 & 0 \end{array}\right]$$

(the first element in the middle row stands for an extended version of $\sqrt{A_0}$). The domain of the system node itself consists of those $\left[\begin{smallmatrix} x_1 \\ x_2 \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} Z \\ Z \\ U \end{smallmatrix}\right]$ which satisfy $x_1 - A_0^{-1/2} C_0^* u \in Z_{1/2}$ and $x_2 \in Z_{1/2}$, and its transfer function is

$$\widehat{\mathfrak{D}}(s) = C_0 \Big(s + \frac{1}{s} A_0\Big)^{-1} C_0^* \qquad \Re s \neq 0$$

(where the inverse maps $Z_{-1/2}$ onto $Z_{1/2}$). By Theorem 3.3, this system node is impedance conservative.

Example 3.6. Also the system described by (1) can be formulated as a system node with the same input, state, and output as in Example 3.5. This time we take the extended system node to be (in the notation below we have anticipated the fact, which will be proved later, that this example is the diagonal transform of Example 3.5) (2))

$$\left[\begin{array}{c|c} A^\times & B^\times \\ \hline C^\times & D^\times \end{array}\right] = \left[\begin{array}{cc|c} 0 & \sqrt{A_0} & 0 \\ -\sqrt{A_0} & \frac{1}{2}\,C_0^*C_0 & C_0^* \\ \hline 0 & -C_0 & 1 \end{array}\right]$$

(again the first element in the middle row stands for an extended version of $\sqrt{A_0}$). The domain of the system node itself consists of those $\left[\begin{smallmatrix} x_1 \\ x_2 \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} Z \\ Z \\ U \end{smallmatrix}\right]$ which satisfy $x_1 - A_0^{-1/2}\big(\frac{1}{2}\,C_0^*C_0x_2 + C_0^*u\big) \in Z_{1/2}$ and $x_2 \in Z_{1/2}$, and its transfer function is

$$\widehat{\mathfrak{D}}(s) = 1 - C_0\Big(s + \frac{1}{2}\,C_0^*C_0 + \frac{1}{s}\,A_0\Big)^{-1}C_0^* \qquad \Re s \neq 0.$$

It is not obvious that Example 3.6 is scattering conservative (hence well-posed and energy stable). That this is, indeed, the case is the main result of [15]. Here we shall rederive that result by a completely different method, appealing to the following general result.

Theorem 3.7 ([9, Theorem 8.2]). *A system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ on (U,X,U) is impedance passive (or energy preserving or conservative) if and only if it is diagonally transformable,*[4] *and the diagonally transformed system node $S^\times = \left[\begin{smallmatrix} [A\&B]^\times \\ [C\&D]^\times \end{smallmatrix}\right]$ is scattering passive (or energy preserving, or conservative) (in particular, it is well-posed and energy stable). The system node $S^\times$ can be determined from its main operator $A^\times$, control operator $B^\times$, observation operator $C^\times$, and transfer function $\widehat{\mathfrak{D}}^\times$, which can be computed from the following*

[4]This notion will be defined in Section 5.

formulas, valid for all $\alpha \in \rho(A) \cap \rho(A^\times)$,[5]

$$\begin{bmatrix} (\alpha - A^\times)^{-1} & \frac{1}{\sqrt{2}}(\alpha - A^\times_{|X})^{-1} B^\times \\ \frac{1}{\sqrt{2}} C^\times (\alpha - A^\times)^{-1} & \frac{1}{2}(1 + \widehat{\mathfrak{D}}^\times(\alpha)) \end{bmatrix}$$
$$= \left(\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix} \right)^{-1}$$
$$= \begin{bmatrix} (\alpha - A)^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} (\alpha - A_{|X})^{-1} B \\ 1 \end{bmatrix} (1 + \widehat{\mathfrak{D}}(\alpha))^{-1} \begin{bmatrix} -C(\alpha - A)^{-1} & 1 \end{bmatrix} \tag{13}$$

In particular, $1 + \widehat{\mathfrak{D}}(\alpha)$ *is invertible and* $\widehat{\mathfrak{D}}^\times(\alpha) = (1 - \widehat{\mathfrak{D}}(\alpha))(1 + \widehat{\mathfrak{D}}(\alpha))^{-1}$ *for all* $\alpha \in \rho(A) \cap \rho(A^\times)$.

Thus, in order to show that Example 3.6 is scattering conservative, it suffices to show that it is the diagonal transform of Example 3.5. This can be achieved via a lengthy computation based on formula (13), but instead of doing this we shall derive an alternative formula to (13) which is valid (only) for *compatible* systems. See Corollary 5.2 and Remark 5.4.

4 Flow-Inversion

In order to get a compatible version of (13) we need to develop a version of the diagonal transform which is more direct than the one presented in [9] (there this transformation was defined as a Cayley transform, followed by a discrete time diagonal transform, followed by an inverse Cayley transform). Instead of using this lengthy chain of transformations we here want to use a (non-well-posed) system node version of the approach used in [8, Section 5]. That approach used the theory of *flow-inversion* of a well-posed system developed in [12], so we have to start by first extending the notion of flow-inversion to a general system node.[6]

Definition 4.1. Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, Y). We call S *flow-invertible* if there exists another system node $S^\times = \left[\begin{smallmatrix} [A\&B]^\times \\ [C\&D]^\times \end{smallmatrix}\right]$ on (Y, X, U)

[5] $A^\times_{|X}$ is the extension of $A^\times$ to an operator in $\mathcal{L}(X; X^\times_{-1})$, where $X^\times_{-1}$ is the analogue of X_{-1} with A replaced by $A^\times$.

[6] Flow-inversion can be interpreted as a special case of output feedback, and conversely, output feedback can be interpreted as a special case of flow-inversion. See [12, Remark 5.5].

which together with S satisfies the following conditions: the operator $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$ maps $\mathcal{D}(S)$ continuously onto $\mathcal{D}(S^\times)$, its inverse is $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$, and

$$\begin{aligned} S^\times &= \begin{bmatrix} [A\&B]^\times \\ [C\&D]^\times \end{bmatrix} = \begin{bmatrix} A\&B \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ C\&D \end{bmatrix}^{-1}, \\ S &= \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} [A\&B]^\times \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ [C\&D]^\times \end{bmatrix}^{-1}. \end{aligned} \tag{14}$$

In this case we call S and $S^\times$ *flow-inverses* of each other.

Obviously, the flow-inverse of a node S in unique (when it exists). Furthermore, by [12, Corollary 5.3], in the well-posed case this notion agrees with the notion of flow-inversion introduced in [12].

The following theorem lists a number of alternative characterizations for the flow-invertibility of a system node.[7]

Theorem 4.2. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, Y), with main operator A, control operator B, observation operator C, and transfer function $\mathfrak{D}$, and let $S^\times = \left[\begin{smallmatrix} [A\&B]^\times \\ [C\&D]^\times \end{smallmatrix}\right]$ be a system node on (Y, X, U), with main operator $A^\times$, control operator $B^\times$, observation operator $C^\times$, and transfer function $\mathfrak{D}^\times$. We denote $\mathcal{D}(A) = X_1$, $(\mathcal{D}(A^*))^* = X_{-1}$, $\mathcal{D}(A^\times) = X_1^\times$, and $(\mathcal{D}((A^\times)^*))^* = X_{-1}$. Then the following conditions are equivalent:*

(i) *S and $S^\times$ are flow-inverses of each other.*

(ii) *The operator $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$ maps $\mathcal{D}(S^\times)$ one-to-one onto $\mathcal{D}(S)$, and*

$$\begin{bmatrix} [A\&B]^\times \\ 0 \quad 1 \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ [C\&D]^\times \end{bmatrix} \quad (\text{on } \mathcal{D}(S^\times)). \tag{15}$$

(iii) *For all $\alpha \in \rho(A^\times)$, the operator $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right] - S$ maps $\mathcal{D}(S)$ one-to-one onto $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, and its (bounded) inverse is given by*

$$\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} - S \right)^{-1} = \begin{bmatrix} (\alpha - A^\times)^{-1} & -(\alpha - A^\times_{|X})^{-1} B^\times \\ C^\times (\alpha - A^\times)^{-1} & -\widehat{\mathfrak{D}}^\times(\alpha) \end{bmatrix}. \tag{16}$$

[7] In this list we have not explicitly included the equivalent discrete time eigenvalue conditions that can be derived from the alternative characterization of continuous time flow-inversion as a Cayley transform, followed by a discrete time flow inversion, followed by an inverse Cayley transform.

(iv) *For some* $\alpha \in \rho(A^\times)$, *the operator* $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right] - S$ *maps* $\mathcal{D}(S)$ *one-to-one onto* $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ *and* (16) *holds.*

(v) *For all* $\alpha \in \rho(A) \cap \rho(A^\times)$, $\widehat{\mathfrak{D}}(\alpha)$ *is invertible and the following operator identity holds in* $\mathcal{L}\big(\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]; \mathcal{D}(S)\big)$:

$$\begin{aligned} \begin{bmatrix} (\alpha - A^\times)^{-1} & -(\alpha - A^\times_{|X})^{-1} B^\times \\ C^\times (\alpha - A^\times)^{-1} & -\widehat{\mathfrak{D}}^\times(\alpha) \end{bmatrix} &= \begin{bmatrix} (\alpha - A)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &- \begin{bmatrix} (\alpha - A_{|X})^{-1} B \\ 1 \end{bmatrix} [\widehat{\mathfrak{D}}(\alpha)]^{-1} \begin{bmatrix} C(\alpha - A)^{-1} & 1 \end{bmatrix}. \end{aligned} \tag{17}$$

(vi) *For some* $\alpha \in \rho(A) \cap \rho(A^\times)$, $\widehat{\mathfrak{D}}(\alpha)$ *is invertible and* (17) *holds.*

When these equivalent conditions hold, then $\left[\begin{smallmatrix} 1 \\ C \end{smallmatrix}\right]$ *maps* X_1 *into* $\mathcal{D}(S^\times)$, $\left[\begin{smallmatrix} 1 \\ C^\times \end{smallmatrix}\right]$ *maps* $X_1^\times$ *into* $\mathcal{D}(S)$, *and*

$$\begin{aligned} A &= A^\times_{|X_1} + B^\times C, & A^\times &= A_{|X_1^\times} + B C^\times, \\ 0 &= [C\&D]^\times \begin{bmatrix} 1 \\ C \end{bmatrix}, & 0 &= C\&D \begin{bmatrix} 1 \\ C^\times \end{bmatrix}. \end{aligned} \tag{18}$$

Proof. We begin by observing that (18), which is equivalent to

$$\begin{bmatrix} [A\&B]^\times \\ [C\&D]^\times \end{bmatrix} \begin{bmatrix} 1 \\ C \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix}, \quad \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} 1 \\ C^\times \end{bmatrix} = \begin{bmatrix} A^\times \\ 0 \end{bmatrix}, \tag{19}$$

follows from (i) and (14) since $\left[\begin{smallmatrix} X_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{D}(S)$ and $\left[\begin{smallmatrix} X_1^\times \\ 0 \end{smallmatrix}\right] \in \mathcal{D}(S_\times)$.

(i) $\Rightarrow$ (ii): This is obvious (see Definition 4.1).

(ii) $\Rightarrow$ (i): Suppose that (ii) holds. Then $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ on $\mathcal{D}(S^\times)$ (since, by assumption, $C\&D \left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right] = \begin{bmatrix} 0 & 1 \end{bmatrix}$, and we always have $\begin{bmatrix} 1 & 0 \end{bmatrix} \left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right] = \begin{bmatrix} 1 & 0 \end{bmatrix}$). Thus, $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$ is a left-inverse of $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$. However, as (by assumption) $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$ is both one-to-one and onto, it is invertible, so the left inverse is also a right inverse, i.e., the inverse of $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$. Multiplying (15) to the right by $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]^{-1}$ we get the second identity in (14). The first identity in (14) can equivalently be written in the form $\left[\begin{smallmatrix} [A\&B]^\times \\ [C\&D]^\times \end{smallmatrix}\right] = \left[\begin{smallmatrix} A\&B \\ 0 & 1 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$. The top part $[A\&B]^\times = A\&B \left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$ of this identity is contained in (15)), and the bottom part $[C\&D]^\times = \begin{bmatrix} 0 & 1 \end{bmatrix} \left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$ is always valid. We conclude that (ii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iii): Let $\alpha \in \mathbb{C}$ be arbitrary. Clearly, (ii) is equivalent to the requirement that $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$ maps $\mathcal{D}(S^\times)$ one-to-one onto $\mathcal{D}(S)$, combined with the identity (note that $\begin{bmatrix} \alpha & 0 \end{bmatrix} \left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right] = \begin{bmatrix} \alpha & 0 \end{bmatrix}$)

$$\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} - S\right) \begin{bmatrix} 1 & 0 \\ [C\&D]^\times \end{bmatrix} = \left(\begin{bmatrix} \alpha & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} [A\&B]^\times \\ 0 \end{bmatrix}\right) \text{ (on } \mathcal{D}(S^\times)).$$

If $\alpha \in \rho(A^\times)$, then $\left[\begin{smallmatrix} (\alpha - A^\times)^{-1} & (\alpha - A^\times_{|X})^{-1} B^\times \\ 0 & 1 \end{smallmatrix}\right]$ maps $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ one-to-one onto $\mathcal{D}(S^\times)$, so we may multiply the above identity by this operator to the right to get the equivalent identity

$$\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} - S\right) \begin{bmatrix} (\alpha - A^\times)^{-1} & (\alpha - A^\times_{|X})^{-1} B^\times \\ C^\times(\alpha - A^\times)^{-1} & \widehat{\mathfrak{D}}^\times(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is now valid on all of $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. This can alternatively be written as (multipy by $\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ to the right)

$$\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} - S\right) \begin{bmatrix} (\alpha - A^\times)^{-1} & -(\alpha - A^\times_{|X})^{-1} B^\times \\ C^\times(\alpha - A^\times)^{-1} & -\widehat{\mathfrak{D}}^\times(\alpha) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By tracing the history of the second factor on the left-hand side we find that it maps $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ one-to-one onto $\mathcal{D}(S)$. Thus, $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right] - S$ is the left-inverse of an invertible operator, hence invertible, and (16) holds.

(iii) $\Rightarrow$ (iv): This is obvious.

(iv) $\Rightarrow$ (ii): This is the same computation that we did in the proof of the implication (ii) $\Rightarrow$ (iii) done backwards, for one particular value of $\alpha \in \rho(A^\times)$. Observe, in particular, that $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^\times \end{smallmatrix}\right]$ maps $\mathcal{D}(S^\times)$ one-to-one onto $\mathcal{D}(S)$ if and only if the operator on the right-hand side of (16) maps $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ one-to-one onto $\mathcal{D}(S)$.

(iii) $\Rightarrow$ (v): This follows from the easily verified identity

$$\begin{aligned} &\left(\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \\ -C(\alpha - A)^{-1} & 1 \end{bmatrix} \begin{bmatrix} \alpha - A & 0 \\ 0 & -\widehat{\mathfrak{D}}(\alpha) \end{bmatrix} \begin{bmatrix} 1 & -(\alpha - A_{|X})^{-1} B \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{20}$$

valid for all $\alpha \in \rho(A)$.

(v) $\Rightarrow$ (vi): This is obvious.

(vi) $\Rightarrow$ (iv): Argue as in the proof of the implication (iii) $\Rightarrow$ (v). $\square$

The original idea behind the flow-inversion of a well-posed system introduced in [12, Section 5] was to interchange the roles of the input and output. A similar interpretation is valid for the flow-inversion of system nodes, too.

Theorem 4.3. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a flow-invertible system node on (Y, X, U), whose flow-inverse $S^\times$ is also a system node (on (U, X, Y)). Let x and y be the state trajectory and output of S with initial state $x_0 \in X$ and input function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^+; U)$, and suppose that $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+; X)$. Then $y \in L^1_{\mathrm{loc}}(\mathbb{R}^+; Y)$, and x and u are the state trajectory and output of $S^\times$ with initial state x_0 and input function y.*

Proof. By Lemma 2.3, $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{D}(S))$, $y \in L^1_{\mathrm{loc}}(\mathbb{R}^+; Y)$, and $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is the unique solution with the above properties of the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \text{ for almost all } t \geq s, \quad x(s) = x_s.$$

Since $\left[\begin{smallmatrix} 1 & 0 \\ \multicolumn{2}{c}{C\&D} \end{smallmatrix}\right]$ maps $\mathcal{D}(S)$ continuously onto $\mathcal{D}(S^\times)$, this implies that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ \multicolumn{2}{c}{C\&D} \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{D}(S^\times))$. Moreover, since $\left[\begin{smallmatrix} 1 & 0 \\ \multicolumn{2}{c}{C\&D} \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ \multicolumn{2}{c}{[C\&D]^\times} \end{smallmatrix}\right]$, we have for almost all $t \geq s$,

$$\begin{aligned} \begin{bmatrix} x'(t) \\ u(t) \end{bmatrix} &= \begin{bmatrix} A\&B \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ [C\&D]^\times \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} [A\&B]^\times \\ [C\&D]^\times \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}. \end{aligned}$$

By Lemma 2.3, this implies that x and u are the state and output function of $S^\times$ with initial time s, initial state x_s, and input function y. $\square$

Our next theorem shows that compatibility is preserved under flow-inversion in most cases.

Theorem 4.4. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a compatible system node on (Y, X, U), and let $\left[\begin{smallmatrix} A_{|W} & B \\ C_{|W} & D \end{smallmatrix}\right] \in \mathcal{L}(\left[\begin{smallmatrix} W \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} W_{-1} \\ Y \end{smallmatrix}\right])$ be a compatible extension of S (here $X_1 \subset W \subset X$ and W_{-1} is defined as in Section 2). Suppose that S if flow-invertible. Denote the flow-inverted system node by $S^\times = \left[\begin{smallmatrix} [A\&B]^\times \\ [C\&D]^\times \end{smallmatrix}\right]$, let $X_1^\times$ and $X_{-1}^\times$ be the analogues of X_1 and X_{-1} for $S^\times$, and let $W_{-1}^\times$ be the analogue of W_{-1} for $S^\times$ (i.e., $W_{-1}^\times = (\alpha - A_{|W}^\times)W$ for some $\alpha \in \rho(A^\times)$).*

(i) *If D has a left inverse $D_{\mathrm{left}}^{-1} \in \mathcal{L}(Y;U)$, then $X_1^\times \subset W$ and $S^\times$ is compatible with extended observation operator $C_{|W}^\times \colon W \to U$ and corresponding feedthrough operator $D^\times$ given by*

$$\begin{aligned} C_{|W}^\times &= -D_{\mathrm{left}}^{-1} C_{|W}, \\ D^\times &= D_{\mathrm{left}}^{-1}, \end{aligned} \tag{21}$$

and the the main operator $A^\times$ of $S^\times$ is given by

$$A^\times = \big(A_{|X} - BD_{\mathrm{left}}^{-1} C_{|W}\big)_{|X_1^\times}.$$

In this case the space W_{-1} can be identified with a closed subspace of $W_{-1}^\times$, so that $X \subset W_{-1} \subset X_{-1} \cap X_{-1}^\times$. With this identification,

$$A_{|W} = A_{|W}^\times + B^\times C_{|W}, \qquad B = B^\times D$$

(where we by $A_{|W}$ and $A_{|W}^\times$ mean the restrictions of $A_{|X}$ and $A_{|X}^\times$ to W).

(ii) *If D is invertible (with a bounded inverse), then $W_{-1} = W_{-1}^\times$, $A^\times W \subset W_{-1}$, $B^\times U \subset W_{-1}$, and the operator $\begin{bmatrix} A_{|W}^\times & B^\times \\ C_{|W}^\times & D^\times \end{bmatrix} \in \mathcal{L}\big(\left[\begin{smallmatrix} W \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} W_{-1} \\ Y \end{smallmatrix}\right]\big)$ defined by*

$$\begin{aligned} \begin{bmatrix} A_{|W}^\times & B^\times \\ C_{|W}^\times & D^\times \end{bmatrix} &= \begin{bmatrix} A_{|W} - BD^{-1}C_{|W} & BD^{-1} \\ -D^{-1}C_{|W} & D^{-1} \end{bmatrix} \\ &= \begin{bmatrix} A_{|W} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B \\ 1 \end{bmatrix} D^{-1} \begin{bmatrix} -C_{|W} & 1 \end{bmatrix} \\ &= \begin{bmatrix} A_{|W} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B \\ 1 \end{bmatrix} \begin{bmatrix} C_{|W}^\times & 1 \end{bmatrix} \\ &= \begin{bmatrix} A_{|W} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B^\times \\ 1 \end{bmatrix} \begin{bmatrix} -C_{|W} & 1 \end{bmatrix} \end{aligned}$$

is a compatible extension of $S^\times$.

Proof. (i) Take $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{D}(S^\times)$, and define $u = [C\&D]^\times \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S)$ and $y = C\&D \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = C_{|W}x + Du$. Multiplying the above identity by D_{left}^{-1} to the left we get for all $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{D}(S^\times)$,

$$u = [C\&D]^\times \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = -D_{\mathrm{left}}^{-1} C_{|W} x + D_{\mathrm{left}}^{-1} y.$$

The right-hand side is defined (and continuous) on all of $W \times Y$. By (17), for all $y \in Y$ and all $\alpha \in \rho(A) \cap \rho(A^\times)$,

$$(\alpha - A^\times_{|X})^{-1} B^\times y = (\alpha - A_{|X})^{-1} B \widehat{\mathfrak{D}}^\times(\alpha) y \in W,$$

so $\mathcal{R}(B^\times) \in W^\times_{-1}$. This implies that $\begin{bmatrix} A^\times_{|W} & B^\times \\ C^\times_{|W} & D^\times \end{bmatrix}$ is a compatible extension of $S^\times$, with $C^\times_{|W} = -D^{-1}_{\text{left}} C_{|W}$ and $D^\times = D^{-1}_{\text{left}}$. By (18), for all $x \in X^\times_1$, we have $A^\times x = (A_{|X} + BC^\times)x = (A_{|X} - BD^{-1}_{\text{left}} C_{|W})x$, as claimed.

Next we construct an embedding operator $J\colon W_{-1} \to W^\times_{-1}$. This operator is required to be one-to-one, and its restriction to X should be the identity operator. We define

$$\begin{aligned} J &= (\alpha - A^\times_{|W} - B^\times C_{|W})(\alpha - A_{|W})^{-1}, \\ J^\times &= (\alpha - A_{|W} - BC^\times_{|W})(\alpha - A^\times_{|W})^{-1}. \end{aligned} \tag{22}$$

The compatibility of S and $S^\times$ implies that $J \in \mathcal{L}(W_{-1}; W^\times_{-1})$ and $J^\times \in \mathcal{L}(W^\times_{-1}; W_{-1})$ and by (18), both J and $J^\times$ reduce to the identity operator on X.

We claim that $J^\times \in \mathcal{L}(W^\times_{-1}; W_{-1})$ is a left inverse of $J \in \mathcal{L}(W_{-1}; W^\times_{-1})$, or equivalently, that $(\alpha - A_{|W})^{-1} J^\times J (\alpha - A_{|W})$ is the identity on W. To see that this is the case we use (22), (21), (17), and (7) (in this order) to compute

$$\begin{aligned}
&(\alpha - A_{|W})^{-1} J^\times J (\alpha - A_{|W}) \\
&\quad = (\alpha - A_{|W})^{-1} (\alpha - A_{|W} - BC^\times_{|W}) \\
&\qquad\quad \times (\alpha - A^\times_{|W})^{-1} (\alpha - A^\times_{|W} - B^\times C_{|W}) \\
&\quad = (1 - (\alpha - A_{|W})^{-1} BC^\times_{|W})(1 - (\alpha - A^\times_{|W})^{-1} B^\times C_{|W}) \\
&\quad = (1 + (\alpha - A_{|W})^{-1} BD^{-1}_{\text{left}} C_{|W})(1 - (\alpha - A_{|W})^{-1} B \widehat{\mathfrak{D}}^{-1}(\alpha) C_{|W}) \\
&\quad = 1 + (\alpha - A_{|W})^{-1} B \big[D^{-1}_{\text{left}} - \widehat{\mathfrak{D}}^{-1}(\alpha) - D^{-1}_{\text{left}} C_{|W} (\alpha - A_{|W})^{-1} B \widehat{\mathfrak{D}}^{-1}(\alpha) \big] C_{|W} \\
&\quad = 1 + (\alpha - A_{|W})^{-1} BD^{-1}_{\text{left}} \big[\widehat{\mathfrak{D}}(\alpha) - D - C_{|W} (\alpha - A_{|W})^{-1} B \big] \widehat{\mathfrak{D}}^{-1}(\alpha) C_{|W} \\
&\quad = 1.
\end{aligned}$$

This implies that the operator J is one-to-one; hence it defines a (not necessarily dense) embedding of W_{-1} into $W^\times_{-1}$. In the sequel we shall identify W_{-1} with the range of J. That W_{-1} is closed in $W^\times_{-1}$ follows from the fact that J has a bounded left inverse.

The identification of W_{-1} with a subspace of $W^\times_{-1}$ means that the embedding operator $J = (\alpha - A^\times_{|W} - B^\times C_{|W})(\alpha - A_{|W})^{-1}$ becomes the identity on

W_{-1}, and hence, with this identification, $(\alpha - A_{|W}) = (\alpha - A^\times_{|W} - B^\times C_{|W})$, or equivalently,

$$A_{|W} = A^\times_{|W} + B^\times C_{|W}.$$

The remaining identity $B = B^\times D$ can verified as follows. By (17) and the fact that $A^\times_{|W} = A_{|W} - B^\times C_{|W}$,

$$\begin{aligned} B^\times \widehat{\mathfrak{D}}(\alpha) &= (\alpha - A^\times_{|W})(\alpha - A_{|W})^{-1}B \\ &= (\alpha - A_{|W} + B^\times C_{|W})(\alpha - A_{|W})^{-1}B \\ &= (B + B^\times C_{|W}(\alpha - A_{|W})^{-1}B) \\ &= (B + B^\times(\widehat{\mathfrak{D}}(\alpha) - D)) \\ &= B^\times \widehat{\mathfrak{D}}(\alpha) + B - B^\times D. \end{aligned}$$

Thus $B = B^\times D$.

(ii) Part (ii) follows from part (i) if we interchange S and $S^\times$. (This will also interchange W_{-1} with $W^\times_{-1}$ and J with $J^\times$.) □

5 The Diagonal Transform

With the theory that we developed in the preceding section at our disposal we can now proceed in the same way as we did in [8, Section 5] to investigate the continuous time diagonal transform. First of all, by comparing (13) and (17) we observe that it is possible to reduce the continuous time diagonal transform to flow-inversion in the following way.

Definition 5.1. Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, U) (note that $Y = U$). We call S *diagonally transformable* if the system node $\widetilde{S} = \left[\begin{smallmatrix} A\&B \\ \widetilde{C\&D} \end{smallmatrix}\right]$ is flow-invertible, where

$$\widetilde{C\&D} = \frac{1}{\sqrt{2}}\Big(C\&D + \begin{bmatrix} 0 & 1 \end{bmatrix}\Big).$$

Denote the flow-inverse of this system node by $\widetilde{S}^\times = \left[\begin{smallmatrix} [A\&B]^\times \\ [\widetilde{C\&D}]^\times \end{smallmatrix}\right]$. Then the *diagonal transform* of S is the system node $S^\times = \left[\begin{smallmatrix} [A\&B]^\times \\ [C\&D]^\times \end{smallmatrix}\right]$, where

$$[C\&D]^\times = \sqrt{2}\,[\widetilde{C\&D}]^\times - \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The diagonal transform can be computed more explicitly as follows.

Corollary 5.2. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a diagonally transformable system node on (U, X, U). Then the diagonal transform $S^\times = \left[\begin{smallmatrix} [A\&B]^\times \\ [C\&D]^\times \end{smallmatrix}\right]$ of S satisfies*

$$S^\times + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} A\&B \\ 0 \quad 1 \end{bmatrix} \left(\begin{bmatrix} 1 \quad 0 \\ C\&D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}.$$

If S is compatible with a compatible extension $\left[\begin{smallmatrix} A_{|W} & B \\ C_{|W} & D \end{smallmatrix}\right] \in \mathcal{L}(\left[\begin{smallmatrix} W \\ U \end{smallmatrix}\right]; \left[\begin{smallmatrix} W_{-1} \\ U \end{smallmatrix}\right])$ where $1 + D$ invertible, then $S^\times$ is also compatible, with the compatible extension (over the same space W)

$$\begin{aligned} \begin{bmatrix} A^\times_{|W} & B^\times \\ C^\times_{|W} & D^\times \end{bmatrix} &= \begin{bmatrix} A_{|W} & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} B \\ \sqrt{2} \end{bmatrix} (1 + D)^{-1} \begin{bmatrix} -C_{|W} & \sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} A_{|W} - B(1 + D)^{-1} C_{|W} & \sqrt{2} B (1 + D)^{-1} \\ -\sqrt{2}(1 + D)^{-1} C_{|W} & (1 - D)(1 + D)^{-1} \end{bmatrix}. \end{aligned} \tag{23}$$

This follows directly from Definition 5.1 and Theorems 4.3 and 4.4.

Corollary 5.3. *Example 3.6 is a scattering conservative system node.*

This follows from Theorem 3.7 and Corollary 5.2.

Remark 5.4. By applying the same theory to other examples of impedance passive or conservative systems we can create many more examples of continuous time scattering passive or conservative systems. One particularly interesting class is the one which is often referred to as 'systems with collocated actuators and sensors', discussed in, e.g., [1], [13], and [14].

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