Well-posed linear systems - a survey with emphasis on conservative systems

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Abstract. We survey the literature on well-posed linear systems, which has been an area of rapid development in recent years. We examine the particular subclass of conservative systems, and its connections to scattering theory. We study some transformations of well-posed systems, namely, duality and time-flow inversion, and their effect on the transfer function and the generating operators. We describe a simple way to generate conservative systems via a second order differential equation in a Hilbert space. We give results about the stability, controllability and observability of such conservative systems and illustrate these with a system modeling a controlled beam.

Key words. Well-posed linear system, regular linear system, operator semigroup, conservative system, scattering theory, time-flow-inversion, differential equations in Hilbert space, beam equation.

1. Introduction

By a *well-posed linear system* we mean a linear time-invariant system Σ such that on any finite time interval $[0,\tau]$, the operator Σ_τ from the initial state $x(0)$ and the input function u to the final state $x(\tau)$ and the output function y is bounded. The input space U, the state space X and the output space Y are Hilbert spaces, and the input and output functions are of class L^2_{loc}. For any $u \in L^2_{\text{loc}}$ and any $\tau \geq 0$, we denote by $\mathbf{P}_\tau u$ its truncation to the interval $[0, \tau]$. Then the well-posed system Σ consists of the family of bounded operators $\Sigma = (\Sigma_\tau)_{\tau \geq 0}$ such that

$$\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}. \tag{1.1}$$

The detailed definition and references will be given in Section 2. The well-posed linear system Σ is called *conservative* if for every $\tau \geq 0$, Σ_τ is a unitary operator from $X \times L^2([0, \tau]; U)$ to $X \times L^2([0, \tau]; Y)$. The fact that Σ is conservative means that the following two statements hold:

(i) Σ_τ is an isometry, i.e., the following balance equation holds:

$$\|x(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 dt = \|x(0)\|^2 + \int_0^\tau \|u(t)\|^2 dt,$$

(ii) Σ_τ is onto, which means that for every $x(\tau) \in X$ and every $\mathbf{P}_\tau y \in L^2([0, \tau], Y)$, we can find $x(0) \in X$ and $\mathbf{P}_\tau u \in L^2([0, \tau], U)$ such that (1.1) holds.

The modern control theory inspired version of the concept of a well-posed linear system was introduced in the paper [17] of D. Salamon in 1987 (significant parts of this theory are found already in the paper [9] by Helton from 1976). Conservative systems have a much older history. These systems appear in a scattering theory context in the book [12] by Lax and Phillips and the paper [1] by Adamjan and Arov (and in papers by the same authors from the 60's). They also play a central role in model theory for non-selfadjoint operators, which originated with the work of Livšic and his associates in the Soviet Union starting in the 50's (see [5]), with the work of Sz.-Nagy and Foias in the Eastern Europe in the 60's (see [27]), and with the work of de Branges and Rovnyak in the United States in the 60's (see [6]). For historical reasons, several competing sets of terminology and notation appear in the literature, which is making it difficult to translate results from one group of authors to another. In addition, the main part of the available literature about conservative systems is written in discrete time (though it can be converted to continuous time through the use of the Cayley transform).

This paper is a survey of available results about well-posed systems, with a special emphasis on results that are relevant to conservative systems (even if the result itself does not refer specifically to conservative systems). The authors are from the group studying well-posed systems with a control theoretic motivation, and of course their point of view is subjective. For many results we do not give proofs but, even so, we can only mention a small subset of what is known in this area and some readers may feel that our omissions are unfair. Important areas that we will (almost) not mention include: the differential representation of non-regular systems (which is prominent both in [17] and in most of the present Russian literature), functional models for contraction semigroups, admissibility of unbounded control and observation operators, exact and approximate controllability and observability, coprime and spectral factorizations, quadratic optimal control, H^∞ control. Our survey is somewhat unconventional in that it contains also results that have not yet been published. Only a few short proofs are included.

Section 2 is an overview of well-posed systems. We recall the concepts of control operator, observation operator and transfer function and we consider the behavior of the system on the whole real time axis.

In Section 3 we recall the concepts of regular and weakly regular linear system, the Λ-extension of an observation operator and we restate the main representation theorems for the transfer function and for the output function.

In Section 4 we investigate the connection between well-posed systems and scattering theory, in particular the semigroup of Lax and Phillips.

In Section 5 we discuss two transformations which lead from one well-posed system to another: duality and time-flow-inversion (these two transformations coincide in the case of a conservative system).

Section 6 is about conservative systems, in particular, about a surprizing simple way to generate conservative systems from certain differential equations.

Section 7 is a beam equation example, which illustrates several theoretical points of the paper. After adding a damping term, it becomes a conservative system of the type discussed in Section 6.

2. Well-posed linear systems

In this section we review the concept of a well-posed linear system, its control operator and observation operator, and some facts about transfer functions.

Notation 2.1. Let W be a Hilbert space. We regard $L^2_{\rm loc}((-\infty,\infty);W)$ as a Fréchet space, with the metric generated by the seminorms

$$\|u_n\| = \left(\int_{-n}^{n} \|u(t)\|^2 dt\right)^{1/2}, \qquad n\in\mathbb{N}.$$

For any interval J, we regard $L^2_{\rm loc}(J;W)$ as a subspace of $L^2_{\rm loc}((-\infty,\infty);W)$ (identifying $L^2_{\rm loc}(J;W)$ with the set of functions in $L^2_{\rm loc}((-\infty,\infty);W)$ which vanish outside of J), and similarly we regard $L^2(J;W)$ as a subspace of $L^2((-\infty,\infty);W)$. Let $\mathbf{P}_J$ be the projection of $L^2_{\rm loc}((-\infty,\infty);W)$ onto $L^2_{\rm loc}(J;W)$ (by truncation). We abbreviate $\mathbf{P}_\tau = \mathbf{P}_{[0,\tau]}$ (where $\tau\geq 0$), $\mathbf{P}_- = \mathbf{P}_{(-\infty,0]}$ and $\mathbf{P}_+ = \mathbf{P}_{[0,\infty)}$. The operator $\mathbf{S}_\tau$ is the (unilateral) right shift by τ on $L^2_{\rm loc}([0,\infty);W)$, and $\mathbf{S}^*_\tau$ is the left shift by τ on the same space. (If we restrict $\mathbf{S}_\tau$ and $\mathbf{S}^*_\tau$ from $L^2_{\rm loc}$ to L^2, then they are adjoint to each other.) For any $u,v\in L^2_{\rm loc}([0,\infty);W)$ and any $\tau\geq 0$, the *τ-concatenation* of u and v, denoted $u \underset{\tau}{\diamond} v$, is the function defined by

$$u \underset{\tau}{\diamond} v = \mathbf{P}_\tau u + \mathbf{S}_\tau v.$$

Thus, $(u \underset{\tau}{\diamond} v)(t) = u(t)$ for $t\in[0,\tau)$, while $(u \underset{\tau}{\diamond} v)(t) = v(t-\tau)$ for $t\geq\tau$.

Definition 2.2. Let U, X and Y be Hilbert spaces and denote $\Omega = L^2([0,\infty);U)$, $\Gamma = L^2([0,\infty);Y)$. A *well-posed linear system* on Ω, X and Γ is a quadruple $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$, where

(i) $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ is a strongly continuous semigroup of linear operators on X,

(ii) $\Phi = (\Phi_t)_{t\geq 0}$ is a family of bounded linear operators from Ω to X such that

$$\Phi_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v, \tag{2.1}$$

for every $u, v \in \Omega$ and all $\tau, t \geq 0$,

(iii) $\Psi = (\Psi_t)_{t\geq 0}$ is a family of bounded linear operators from X to Γ such that

$$\Psi_{\tau+t} x_0 = \Psi_\tau x_0 \underset{\tau}{\diamond} \Psi_t \mathbb{T}_\tau x_0, \tag{2.2}$$

for every $x_0 \in X$ and all $\tau, t \geq 0$, and $\Psi_0 = 0$,

(iv) $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ is a family of bounded linear operators from Ω to Γ such that

$$\mathbb{F}_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathbb{F}_\tau u \underset{\tau}{\diamond} (\Psi_t \Phi_\tau u + \mathbb{F}_t v), \tag{2.3}$$

for every $u, v \in \Omega$ and all $\tau, t \geq 0$, and $\mathbb{F}_0 = 0$.

We call U the *input space*, X the *state space* and Y the *output space* of Σ. The operators Φ_τ are called *input maps*, the operators Ψ_τ are called *output maps*, and the operators $\mathbb{F}_\tau$ are called *input-output maps*.

The above definition follows [31] and [32], but the first equivalent definitions were formulated by D. Salamon in [17] and [18]. Other equivalent definitions appeared in [19], [20], and related definitions can be found in [3], [9], [15], [16] and [36].

The intuitive interpretation of the operator families introduced in this definition is in terms of a state trajectory x and the output function y corresponding to an initial state $x(0)$ and an input function u: these are related by (1.1), where

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix} \qquad \forall\, \tau \geq 0. \tag{2.4}$$

It follows from (2.1) with $t = 0$ and $v = 0$ that Φ is *causal*, the state does not depend on the future input: $\Phi_\tau \mathbf{P}_\tau = \Phi_\tau$ for all $\tau \geq 0$, in particular $\Phi_0 = 0$. It follows from this and the definitions that for all $\tau, t \geq 0$,

$$\Phi_{\tau+t}\mathbf{P}_\tau = \mathbb{T}_t \Phi_\tau, \quad \mathbf{P}_\tau \Psi_{\tau+t} = \Psi_\tau, \quad \mathbf{P}_\tau \mathbb{F}_{\tau+t} \mathbf{P}_\tau = \mathbf{P}_\tau \mathbb{F}_{\tau+t} = \mathbb{F}_\tau,$$

and hence $\mathbf{P}_\tau \mathbb{F}_{\tau+t} \mathbf{P}_{[\tau,\tau+t]} = 0$. The last identity says $\mathbb{F}$ is *causal* (the past output does not depend on the future input).

We now recall some less immediate consequences of Definition 2.2, following [30] and [32]. For the remainder of this section, we use the assumptions of Definition 2.2. We denote the generator of $\mathbb{T}$ by A. The space X_1 is defined as $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$, and X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. The choice of β is not important, since different choices lead to equivalent norms on X_1 and on X_{-1}. The semigroup $\mathbb{T}$ can be extended to X_{-1}, and then its generator is an extension of A, defined on X. We use the same notation for all these extensions as for the original operators.

It follows from assumptions (i) and (ii) in the definition that there exists a unique $B \in \mathcal{L}(U; X_{-1})$, called the *control operator* of Σ, such that for all $t \geq 0$,

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma. \tag{2.5}$$

The function $\Phi_t u$ depends continuously on t. The fact that $\Phi_t u \in X$ means that B is an *admissible control operator* for $\mathbb{T}$. Admissible control operators are a subspace of $\mathcal{L}(U; X_{-1})$, we refer to Weiss [29, 35] for investigations of these operators.

Using the identity $\mathbf{P}_\tau \Psi_{\tau+t} = \Psi_\tau$, we define the operator $\Psi_\infty \colon X \to L^2_{\text{loc}}([0,\infty); Y)$ by $\Psi_\infty x_0 = \lim_{t\to\infty} \Psi_t x_0$. Then Ψ_∞ satisfies $\mathbf{P}_\tau \Psi_\infty = \Psi_\tau$ for all $\tau \geq 0$. Ψ_∞ is called the *extended output map* of Σ. By letting $t \to \infty$ in (2.2), we get

$$\Psi_\infty x_0 = \Psi_\infty x_0 \underset{\tau}{\diamond} \Psi_\infty \mathbb{T}_\tau x_0, \tag{2.6}$$

for every $x_0 \in X$ and all $\tau \geq 0$. More generally, any continuous linear operator $\Psi_\infty : X \to L^2_{\text{loc}}([0,\infty); Y)$ which satisfies (2.6) for every $x_0 \in X$ and all $\tau \geq 0$ is called an *extended output map* for $\mathbb{T}$. For every such Ψ_∞ there exists a unique $C \in \mathcal{L}(X_1; Y)$, called the *observation operator* of Ψ_∞ (or of Σ), such that

$$(\Psi_\infty x_0)(t) = C \mathbb{T}_t x_0, \tag{2.7}$$

for every $x_0 \in X_1$ and all $t \geq 0$. This determines Ψ_∞, since X_1 is dense in X.

An operator $C \in \mathcal{L}(X_1; Y)$ is called an *admissible observation operator* for $\mathbb{T}$ if the estimate

$$\int_0^\tau \|C\mathbb{T}_t x_0\|^2 dt \leq k\|x_0\|^2$$

holds for some $\tau > 0$ and for every $x_0 \in \mathcal{D}(A)$. For further details about such operators we refer to [30, 35]. It is clear that if C is the observation operator of a well-posed linear system, then C is admissible.

Using the identity $\mathbf{P}_\tau \mathbb{F}_{\tau+t} = \mathbb{F}_\tau$, we define the operator $\mathbb{F}_\infty \colon L^2_{\text{loc}}([0,\infty); U) \to L^2_{\text{loc}}([0,\infty); Y)$ by $\mathbb{F}_\infty u = \lim_{t\to\infty} \mathbb{F}_t u$. Then $\mathbf{P}_\tau \mathbb{F}_\infty = \mathbb{F}_\tau$ for all $\tau \geq 0$. $\mathbb{F}_\infty$ is called the *extended input-output map* of Σ. By letting $t \to \infty$ in (2.3), we can get

$$\mathbb{F}_\infty (u \underset{\tau}{\diamond} v) = \mathbb{F}_\infty u \underset{\tau}{\diamond} (\Psi_\infty \Phi_\tau u + \mathbb{F}_\infty v), \tag{2.8}$$

for every $u, v \in \Omega$ and all $\tau \geq 0$. Taking $u = 0$ in (2.8) we get that

$$\mathbb{F}_\infty \mathbf{S}_\tau = \mathbf{S}_\tau \mathbb{F}_\infty, \tag{2.9}$$

for every $\tau \geq 0$. Any continuous operator $\mathbb{F}_\infty : L^2_{\text{loc}}([0,\infty); U) \to L^2_{\text{loc}}([0,\infty); Y)$ which satisfies (2.9) is called *shift-invariant* or *time-invariant*.

Definition 2.3. For any $x_0 \in X$ and any $u \in L^2_{\text{loc}}([0,\infty); U)$, the *state trajectory* $x\colon [0,\infty) \to X$ and the *output function* $y \in L^2_{\text{loc}}([0,\infty); Y)$ of Σ corresponding to the initial state x_0 and the input function u are defined by

$$\begin{aligned} x(t) &= \mathbb{T}_t x_0 + \Phi_t u, \qquad t \geq 0, \\ y &= \Psi_\infty x_0 + \mathbb{F}_\infty u. \end{aligned} \tag{2.10}$$

From here we can recover (1.1) with Σ_τ as in (2.4), by taking $t = \tau$ and applying $\mathbf{P}_\tau$ to the second equation.

Notation 2.4. For any Hilbert space W, any interval J and any $\omega \in \mathbb{R}$ we put

$$L^2_\omega(J; W) = e_\omega L^2(J; W),$$

where $(e_\omega v)(t) = e^{\omega t} v(t)$, with the norm $\|e_\omega v\|_{L^2_\omega} = \|v\|_{L^2}$. We denote by $\mathbb{C}_\omega$ the half-plane of all $s \in \mathbb{C}$ with $\operatorname{Re} s > \omega$. The *growth bound* of the operator semigroup $\mathbb{T}$ with generator A is denoted by $\omega_{\mathbb{T}}$. Thus,

$$\omega_{\mathbb{T}} = \lim_{t\to\infty} \frac{1}{t} \log \|\mathbb{T}_t\| = \inf_{t>0} \frac{1}{t} \log \|\mathbb{T}_t\|$$

and $(sI - A)^{-1}$ is uniformly bounded on $\mathbb{C}_\omega$ if and only if $\omega > \omega_{\mathbb{T}}$.

As shown in [30, Proposition 2.3] and [32, Proposition 4.1], for every $\omega > \omega_{\mathbb{T}}$, Ψ_∞ is bounded from X to $L^2_\omega([0,\infty); Y)$ and $\mathbb{F}_\infty$ is bounded from $L^2_\omega([0,\infty); U)$ to $L^2_\omega([0,\infty); Y)$. For each $x_0 \in X$, the Laplace integral of $\Psi_\infty x_0$ converges absolutely for $\operatorname{Re} s > \omega_{\mathbb{T}}$, and the Laplace transform is given by

$$(\widehat{\Psi_\infty x_0})(s) = C(sI - A)^{-1} x_0, \qquad \operatorname{Re} s > \omega_{\mathbb{T}}, \tag{2.11}$$

see [30, formula (3.6)]. We can represent $\mathbb{F}_\infty$ via the *transfer function* $\mathbf{G}$ of Σ, which is a bounded analytic $\mathcal{L}(U; Y)$-valued function on $\mathbb{C}_\omega$ for every $\omega > \omega_{\mathbb{T}}$ (possibly also for some $\omega \leq \omega_{\mathbb{T}}$). If $x_0 \in X$ and $u \in L^2_\omega([0,\infty); U)$ with $\omega > \omega_{\mathbb{T}}$, then the corresponding output function $y = \Psi_\infty x_0 + \mathbb{F}_\infty u$ of Σ is in $L^2_\omega([0,\infty); Y)$ and its Laplace transform is given, according to (2.11) and Theorem 3.6 in [32], by

$$\hat{y}(s) = C(sI - A)^{-1} x_0 + \mathbf{G}(s)\hat{u}(s), \qquad \operatorname{Re} s > \omega. \tag{2.12}$$

Moreover, $\mathbf{G}$ satisfies

$$\begin{aligned} \mathbf{G}(s) - \mathbf{G}(\beta) &= (\beta - s) C(\beta I - A)^{-1}(sI - A)^{-1} B \\ &= C\left[(sI - A)^{-1} - (\beta I - A)^{-1}\right] B, \end{aligned} \tag{2.13}$$

for all $s, \beta \in \mathbb{C}_{\omega_{\mathbb{T}}}$ (equivalently, $\mathbf{G}'(s) = -C(sI - A)^{-2}B$). This shows that $\mathbf{G}$ is determined by A, B and C up to an additive constant operator.

We denote by $\gamma_{\mathbb{F}}$ the infimum of those $\omega \in \mathbb{R}$ for which $\mathbb{F}_\infty$ is bounded from $L^2_\omega([0,\infty);U)$ to $L^2_\omega([0,\infty);Y)$. Equivalently, $\gamma_{\mathbb{F}}$ is the infimum of all those $\omega \in \mathbb{R}$ for which $\mathbf{G}$ has a bounded analytic extension to $\mathbb{C}_\omega$. This number $\gamma_{\mathbb{F}} \in [-\infty, \infty)$ is called the *growth bound* of $\mathbb{F}_\infty$. It follows from what we have already said that $\gamma_{\mathbb{F}} \leq \omega_{\mathbb{T}}$. Moreover, if $\omega > \gamma_{\mathbb{F}}$, $u \in L^2_\omega([0,\infty);U)$ and $y = \mathbb{F}_\infty u$, then

$$\hat{y}(s) = \mathbf{G}(s)\hat{u}(s), \qquad \operatorname{Re} s > \omega. \tag{2.14}$$

It follows that for such ω, the norm of $\mathbb{F}_\infty$ from L^2_ω to L^2_ω is the supremum of $\|\mathbf{G}(s)\|$ over all $s \in \mathbb{C}_\omega$. By the maximum modulus theorem, denoting $\|\mathbb{F}_\infty\|_\omega = \|\mathbb{F}_\infty\|_{\mathcal{L}(L^2_\omega)}$,

$$\|\mathbb{F}_\infty\|_\omega = \sup_{\operatorname{Re} s = \omega} \|\mathbf{G}(s)\|. \tag{2.15}$$

Until now we have considered the time to be positive. It is sometimes important to think of a well-posed linear system Σ functioning on the time intervals $(-\infty, 0]$ or $(-\infty,\infty)$. To treat these cases, we introduce some further notation and we extend Φ_t and $\mathbb{F}_\infty$ so that they depend also on the values of the input for negative times.

Notation 2.5. Let W be a Hilbert space. The operator $\mathcal{S}_\tau$ (with $\tau \in \mathbb{R}$) is the (bilateral) right shift by τ on $L^2_{\text{loc}}((-\infty,\infty);W)$, so that $\mathcal{S}_{-\tau}$ denotes the (bilateral) left shift by τ on the same space. Recall the projections $\mathbf{P}_-$, $\mathbf{P}_+$ and the spaces $L^2_\omega(J;W)$ introduced at the beginning of this section. The space $L^2_{\omega,\text{loc}}((-\infty,\infty);W)$ consists of all the functions $u \in L^2_{\text{loc}}((-\infty,\infty);W)$ for which $\mathbf{P}_- u \in L^2_\omega((-\infty,0];W)$. We regard $L^2_{\omega,\text{loc}}((-\infty,\infty);W)$ as a Fréchet space, with the metric generated by the seminorms

$$\|u\|_n = \left(\int_{-\infty}^n e^{-2\omega t}\|u(t)\|^2 dt\right)^{1/2}, \qquad n \in \mathbb{N}.$$

The unilateral right shift $\mathbf{S}_\tau$ (with $\tau \geq 0$) was originally defined on $L^2_{\text{loc}}([0,\infty);W)$, but we extend it to $L^2_{\text{loc}}((-\infty,\infty);W)$ by $\mathbf{S}_\tau = \mathcal{S}_\tau \mathbf{P}_+$. Note that $\mathbf{S}_\tau = \mathbf{P}_{[\tau,\infty)}\mathcal{S}_\tau$.

Proposition 2.6. *Assume that $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system with input space U, state space X, output space Y, transfer function $\mathbf{G}$ and growth bounds $\omega_{\mathbb{T}}$ and $\gamma_{\mathbb{F}}$. Note that Φ_t was originally defined on $L^2([0,\infty);U)$, but Φ_t has an obvious extension to $L^2_{\text{loc}}((-\infty,\infty);U)$, still given by (2.5).*

For all $u \in L^2_{\omega,\text{loc}}((-\infty,\infty);U)$ with $\omega > \omega_{\mathbb{T}}$ and for all $t \in \mathbb{R}$, the following limit exists in X:

$$\widetilde{\Phi}_t u = \lim_{\tau \to \infty} \Phi_{\tau + t}\mathcal{S}_\tau u. \tag{2.16}$$

We have

$$\widetilde{\Phi}_t u = \int_{-\infty}^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma \tag{2.17}$$

and there exists a $k_\omega \geq 0$ (independent of t and u) such that

$$\|\widetilde{\Phi}_t u\| \leq k_\omega e^{\omega t} \|\mathbf{P}_{(-\infty,t]} u\|_{L^2_\omega}. \tag{2.18}$$

For all $u \in L^2_{\omega,\mathrm{loc}}((-\infty,\infty);U)$ with $\omega > \gamma_{\mathbb{F}}$, the following limit exists in the Fréchet space $L^2_{\omega,\mathrm{loc}}((-\infty,\infty);Y)$:

$$\mathcal{F}u = \lim_{\tau\to\infty} \mathcal{S}_{-\tau}\mathbb{F}_\infty \mathcal{S}_\tau u. \tag{2.19}$$

The operator $\mathcal{F}$ defined in this way is a bilaterally shift-invarant *and* causal *extension of $\mathbb{F}_\infty$, which means that*

$$\mathcal{F}\mathbf{P}_+ = \mathbb{F}_\infty, \qquad \mathcal{F}\mathcal{S}_t = \mathcal{S}_t\mathcal{F}, \qquad \mathbf{P}_{(-\infty,t]}\mathcal{F}\mathbf{P}_{[t,\infty)} = 0 \tag{2.20}$$

for all $t \in \mathbb{R}$. For each $\omega > \gamma_{\mathbb{F}}$, $\mathcal{F}$ maps $L^2_\omega((-\infty,\infty);U)$ into $L^2_\omega((-\infty,\infty);Y)$ and we denote by $\|\mathcal{F}\|_\omega$ the corresponding operator norm. Using also notation from (2.15)*, we have*

$$\|\mathcal{F}\|_\omega = \|\mathbb{F}_\infty\|_\omega = \sup_{s\in\mathbb{C}_\omega} \|\mathbf{G}(s)\|. \tag{2.21}$$

For the proof of this proposition we refer to [25].

We call the operators $\widetilde{\Phi}_t$ from (2.16) the *extended input maps* of Σ. Using (2.1) to express $\Phi_{\tau+t}$ in (2.16), we obtain that for all $t \geq 0$,

$$\widetilde{\Phi}_t = \mathbb{T}_t\widetilde{\Phi}_0 + \Phi_t. \tag{2.22}$$

By replacing τ by T, t by $\tau + t$ and u by $\mathcal{S}_\tau u$ in (2.16), we find that for all $t, \tau \in \mathbb{R}$, $\widetilde{\Phi}_{\tau+t}\mathcal{S}_\tau = \widetilde{\Phi}_t$. Multiplying this by $\mathcal{S}_{-\tau}$ to the right and using (2.22), we get the following extension of (2.1): for all $\tau \in \mathbb{R}$ and all $t \geq 0$,

$$\widetilde{\Phi}_{\tau+t} = \mathbb{T}_t\widetilde{\Phi}_\tau + \Phi_t\mathcal{S}_{-\tau}. \tag{2.23}$$

By replacing τ in (2.8) by $\tau + T$, multiplying by $\mathcal{S}_T u$ to the right, by $\mathcal{S}_{-T}$ to the left, and letting $T \to \infty$ we get the following extension of (2.8): for all $\tau \in \mathbb{R}$,

$$\mathcal{F} = \mathbf{P}_{(-\infty,\tau]}\mathcal{F} + \mathcal{S}_\tau\Psi_\infty\widetilde{\Phi}_\tau + \mathcal{S}_\tau\mathbb{F}_\infty\mathcal{S}_{-\tau}. \tag{2.24}$$

Remark 2.7. In the formulation of Staffans [19, 20], a well-posed linear system is defined in terms the semigroup $\mathbb{T}_t$ (denoted by $\mathcal{A}(t)$) and the extended operators $\widetilde{\Phi}_0$ (denoted by $\mathcal{B}$), Ψ_∞ (denoted by $\mathcal{C}$), and $\mathcal{F}$ (denoted by $\mathcal{D}$). The original operator families of input maps Φ_τ, output maps Ψ_τ, and and input-output maps $\mathbb{F}_\tau$ can be recovered from $\widetilde{\Phi}_0$, Ψ_∞, and $\mathcal{F}$ by means of

$$\Phi_\tau = \widetilde{\Phi}_0\mathcal{S}_{-\tau}\mathbf{P}_+, \qquad \Psi_\tau = \mathbf{P}_\tau\Psi_\infty, \qquad \mathbb{F}_\tau = \mathbf{P}_\tau\mathcal{F}\mathbf{P}_\tau.$$

Moreover, Staffans writes the algebraic conditions (2.1)–(2.3) as (in our notation)

$$\begin{aligned}
\mathbb{T}_t\widetilde{\Phi}_0 &= \widetilde{\Phi}_0\mathcal{S}_{-t}\mathbf{P}_-, \qquad t \geq 0,\\
\Psi_\infty\mathbb{T}_t &= \mathbf{S}_t^*\Psi_\infty, \qquad t \geq 0,\\
\mathbf{P}_-\mathcal{F}\mathbf{P}_+ = 0, \quad \mathbf{P}_+\mathcal{F}\mathbf{P}_- &= \Psi_\infty\widetilde{\Phi}_0, \quad \mathcal{S}_t\mathcal{F} = \mathcal{F}\mathcal{S}_t, \quad t \in \mathbb{R}.
\end{aligned}$$

3. Regular linear systems

In this section we review the main facts about regular and weakly regular systems, but without the feedback theory from [33]. The notation is as in Section 2.

Definition 3.1. Let X and Y be Hilbert spaces, let $\mathbb{T}$ be a strongly continuous semigroup on X and let $C \in \mathcal{L}(X_1, Y)$. The Λ-*extension* of C is the operator

$$C_\Lambda x_0 = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} x_0,$$

with its domain $\mathcal{D}(C_\Lambda)$ consisting of those $x_0 \in X$ for which the limits exist.

It is easy to see that C_Λ is indeed an extension of C. This extension has various interesting properties, for which we refer to [30], [33]. In the sequel, we assume that $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system, with input space U, state space X, output space Y, semigroup generator A, control operator B, observation operator C, transfer function $\mathbf{G}$ and semigroup growth bound $\omega_{\mathbb{T}}$. We denote by χ the characteristic function of $[0, \infty)$ (so that $\chi(t) = 1$ for all $t \geq 0$).

Definition 3.2. For any $\mathrm{v} \in U$, the function $y_{\mathrm{v}} = \mathbb{F}_\infty(\chi \cdot \mathrm{v})$ is the *step response* of Σ corresponding to v. The system Σ is called *regular* if the following limit exists in Y, for every $\mathrm{v} \in U$:

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau y_{\mathrm{v}}(\sigma)\, d\sigma = D\mathrm{v}. \tag{3.1}$$

The operator $D \in \mathcal{L}(U; Y)$ defined by (3.1) is called the *feedthrough operator* of Σ.

Equivalent characterizations of regularity will be given in Theorem 3.5. The following theorem gives the "local" representation of regular linear systems. The first part of the theorem holds for any well-posed linear system.

Theorem 3.3. (i) *For any initial state $x_0 \in X$ and any input $u \in L^2_{\mathrm{loc}}([0, \infty); U)$, the state trajectory x defined in* (2.10) *is the unique strong solution of*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad t \geq 0, \\ x(0) &= x_0. \end{aligned} \tag{3.2}$$

More precisely, x is continuous, and x is unique with the property

$$x(t) = x_0 + \int_0^t [Ax(\sigma) + Bu(\sigma)] d\sigma$$

for all $t \geq 0$, the integral being computed in X_{-1}.

(ii) *If Σ is regular, and if we denote the feedthrough operator of Σ by D, then the output y of Σ defined in* (2.10) *is given by*

$$y(t) = C_\Lambda x(t) + Du(t), \tag{3.3}$$

for almost every $t \geq 0$ (in particular, $x(t) \in \mathcal{D}(C_\Lambda)$ for almost every $t \geq 0$). If $t \geq 0$ is such that both u and y are continuous from the right at t, then (using those right limits) (3.3) *holds at t (in particular, $x(t) \in \mathcal{D}(C_\Lambda)$).*

The proof is in [31], [32] (these papers use another extension of C, denoted C_L, but C_Λ is an extension of C_L, so that Theorem 3.3 follows). Part (ii) of Theorem 3.3 implies the following formula for $\mathbb{F}_\infty$ for regular systems:

$$(\mathbb{F}_\infty u)(t) = C_\Lambda \int_0^t \mathbb{T}_{t-\sigma} Bu(\sigma)d\sigma + Du(t), \tag{3.4}$$

valid for every $u \in L^2_{\mathrm{loc}}([0,\infty);U)$ and almost every $t \geq 0$ (in particular, the integral above is in $\mathcal{D}(C_\Lambda)$ for almost every $t \geq 0$).

The operators A, B, C and D are called the *generating operators* of Σ, because Σ is completely determined by them via (3.2) and (3.3).

Theorem 3.4. *Assume that Σ is regular. Then $\mathbf{G}$ is given by*

$$\mathbf{G}(s) = C_\Lambda(sI-A)^{-1}B + D, \qquad \operatorname{Re} s > \omega_{\mathbb{T}}$$

(in particular, $(sI-A)^{-1}BU \subset \mathcal{D}(C_\Lambda)$).

The proof of this theorem, as well as of the following one, is given in [32].

We introduce a notation for angular domains in $\mathbb{C}$: for any $\psi \in (0,\pi)$,

$$\mathcal{W}(\psi) = \left\{ re^{i\phi} \;\middle|\; r \in (0,\infty),\ \phi \in (-\psi,\psi) \right\}.$$

Theorem 3.5. *The following statements are equivalent:*

(1) *Σ is regular, i.e., for every $\mathrm{v} \in U$ the limit in* (3.1) *exists.*

(2) *For every $s \in \rho(A)$ we have that $(sI-A)^{-1}BU \subset \mathcal{D}(C_\Lambda)$ and $C_\Lambda(sI-A)^{-1}B$ is an analytic $\mathcal{L}(U;Y)$-valued function of s on $\rho(A)$, uniformly bounded on any half-plane $\mathbb{C}_\omega$ with $\omega > \omega_{\mathbb{T}}$.*

(3) *There exists $s \in \rho(A)$ such that $(sI-A)^{-1}BU \subset \mathcal{D}(C_\Lambda)$.*

(4) *Any state trajectory of Σ is almost always in $\mathcal{D}(C_\Lambda)$.*

(5) *For every $\mathrm{v} \in U$ and every $\psi \in \left(0,\frac{\pi}{2}\right)$, $\mathbf{G}(s)\mathrm{v}$ has a limit as $|s| \to \infty$ and $s \in \mathcal{W}(\psi)$.*

(6) *For every $\mathrm{v} \in U$, $\mathbf{G}(\lambda)\mathrm{v}$ has a limit as $\lambda \to +\infty$ in $\mathbb{R}$.*

Moreover, if the limits mentioned in statements (1), (5) *and* (6) *above exist, then they are equal to $D\mathrm{v}$, where D is the feedthrough operator of Σ.*

The weak Λ-extension of C, denoted $C_{\Lambda w}$, is defined similarly as C_Λ, but with a weak limit, so that its domain is larger. *Weak regularity* is defined similarly as regularity, but with a weak limit, see [34] and [25]. Everything we said about regularity and C_Λ remains valid for the more general concept of weak regularity and for $C_{\Lambda w}$. The main reason why we need the concept of regularity (instead of using just weak regularity) is the feedback theory from [33] and its applications. This theory has substantial parts that we cannot extend in full generality to weakly regular systems (such as explicit formulas for the generating operators of a closed-loop system, in terms of the generating operators of the original system).

4. The connection with scattering theory

Starting from an arbitrary well-posed linear system Σ, it is possible to define a strongly continuous semigroup which resembles those encountered in the scattering theory of Lax and Phillips [12, 13], and which contains all the information about Σ. We explore this connection in this section. We give proofs, because they do not seem to be readily available in the published literature, in the context that we need.

Like in the previous section, we assume that $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system with input space U, state space X, output space Y, transfer function $\mathbf{G}$ and the two growth bounds $\omega_{\mathbb{T}}$ and $\gamma_{\mathbb{F}}$. We continue to use the notation $\mathbf{P}_-$, $\mathbf{P}_+$, $\mathbf{S}_t$, $\mathbf{S}_t^*$, $\mathbb{C}_\omega$, L^2_ω, $\mathcal{S}_t$, $\widetilde{\Phi}_t$ and $\mathcal{F}$ introduced in Section 2.

Proposition 4.1. *Let $\omega \in \mathbb{R}$, $\mathcal{Y} = L^2_\omega((-\infty, 0]; Y)$ and $\mathcal{U} = L^2_\omega([0, \infty); U)$. For all $t \geq 0$ we define on $\mathcal{Y} \times X \times \mathcal{U}$ the operator $\mathfrak{T}_t$ by*

$$\mathfrak{T}_t = \begin{bmatrix} \mathcal{S}_{-t} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathbf{S}_t^* \end{bmatrix} \begin{bmatrix} I & \Psi_t & \mathbb{F}_t \\ 0 & \mathbb{T}_t & \Phi_t \\ 0 & 0 & I \end{bmatrix}.$$

Then $\mathfrak{T} = (\mathfrak{T}_t)_{t \geq 0}$ is a strongly continuous semigroup. Take $y_0 \in \mathcal{Y}$, $x_0 \in X$ and $u_0 \in \mathcal{U}$. We denote by x the state trajectory $x(t) = \mathbb{T}_t x_0 + \Phi_t u_0$ and by y the "bilateral" output function, equal to y_0 for $t \leq 0$, and equal to $\Psi_\infty x_0 + \mathbb{F}_\infty u_0$ for $t \geq 0$. Then for all $t \geq 0$,

$$\begin{bmatrix} \mathbf{P}_{(-\infty,t]} y \\ x(t) \\ \mathbf{P}_{[t,\infty)} u_0 \end{bmatrix} = \begin{bmatrix} \mathcal{S}_t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \mathbf{S}_t \end{bmatrix} \mathfrak{T}_t \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix}. \tag{4.1}$$

The formula (4.1) shows that at any time $t \geq 0$, the first component of $\mathfrak{T}_t \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix}$ represents the past output, the second component represents the present state and the third component represents the future input.

Proof. The semigroup property $\mathfrak{T}_{\tau+t} = \mathfrak{T}_t \mathfrak{T}_\tau$ follows (via elementary algebra) from the formulas in Definition 2.2 and the fact that the left shifts $\mathcal{S}_{-t}$ and $\mathbf{S}_t^*$ are semigroups on $\mathcal{Y}$ and $\mathcal{U}$, respectively. The initial condition $\mathfrak{T}_0 = I$ is clearly satisfied. The formula (4.1) is a direct consequence of Definition 2.3.

To prove the strong continuity, we split $\begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix} \in \mathcal{Y} \times X \times \mathcal{U}$ into $\begin{bmatrix} y_0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_0 \\ u_0 \end{bmatrix}$. The continuity of $\mathfrak{T}_t$ applied to the first vector follows from the strong continuity of $\mathcal{S}_{-t}$. The continuity of $\mathfrak{T}_t$ applied to the second vector follows from the strong continuity of $\mathbf{S}_t^*$ and $\mathcal{S}_{-t}$ and from (4.1) (using the continuity of state trajectories). □

In the case where $\omega = 0$ and $\mathfrak{T}$ is contractive (or unitary), $\mathfrak{T}$ is isomorphic to a semigroup of the type studied by Lax and Phillips (the unitary case is treated in [12] and the contractive case in [13]; an extension to the general case is given by Helton

[9]). For this reason, we call $\mathfrak{T}$ the *Lax-Phillips semigroup* corresponding to the system Σ, see also [25]. Assuming $U = Y$ and $\omega = 0$, we identify the unperturbed unitary group in [12, 13] with the left shift group $\mathcal{S}_{-t}$ on $L^2((-\infty,\infty);U)$. The spaces $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal incoming and outgoing subspaces of $\mathcal{S}_{-t}$, respectively, and $\mathcal{F}$ is the scattering operator. Much useful information on how to translate scattering theory into the language of systems theory is found in [9]. We mention that in [12] and [13], in addition to the contractivity assumption on $\mathfrak{T}$, some further controllability and observability type assumptions are made.

In [9, 12, 13], the operator

$$W_- = \begin{bmatrix} \mathbf{P}_-\mathcal{F} \\ \widetilde{\Phi}_0 \\ \mathbf{P}_+ \end{bmatrix}$$

(denoted by very different symbols) is called the backward wave operator, and its action on exponential inputs (restricted to $(-\infty,0]$) is investigated. Translated into our language and our somewhat different framework, the result is as follows:

Proposition 4.2. *Denote the generator of* $\mathbb{T}$ *by* A *and the control operator of* Σ *by* B. *Then for every* $\mathrm{v} \in U$, *for all* $\lambda \in \mathbb{C}_{\omega_{\mathbb{T}}}$ *and for all* $t \in \mathbb{R}$,

$$\widetilde{\Phi}_t(e_\lambda \mathrm{v}) = e^{\lambda t}(\lambda I - A)^{-1}B\mathrm{v}\,, \tag{4.2}$$

$$\mathcal{F}(e_\lambda \mathrm{v}) = e_\lambda \mathbf{G}(\lambda)\mathrm{v}, \tag{4.3}$$

where e_λ *is the function* $e_\lambda(t) = e^{\lambda t}$, *for all* $t \in \mathbb{R}$.

Proof. To prove (4.2), we substitute $u = e_\lambda \mathrm{v}$ in (2.17) to get

$$\widetilde{\Phi}_t(e_\lambda \mathrm{v}) = \int_{-\infty}^{0} e^{\lambda(\sigma+t)}\mathbb{T}_{-\sigma}B\mathrm{v}d\sigma = e^{\lambda t}\int_0^\infty e^{-\lambda\sigma}\mathbb{T}_\sigma B\mathrm{v}d\sigma = e^{\lambda t}(\lambda I - A)^{-1}B\mathrm{v}\,.$$

To prove (4.3), denote $y = \mathcal{F}(e_\lambda \mathrm{v})$. Since $\mathcal{F}$ is shift-invariant, we have for all $\tau \in \mathbb{R}$,

$$\mathcal{S}_\tau y = \mathcal{F}(\mathcal{S}_\tau e_\lambda \mathrm{v}) = e^{-\lambda\tau}\mathcal{F}(e_\lambda \mathrm{v}) = e^{-\lambda\tau}y\,.$$

Thus, y is an eigenvector of $\mathcal{S}_\tau$ for every $\tau \in \mathbb{R}$, which implies that it is in the domain of the generator of the operator group $\mathcal{S}_\tau$, and hence y is continuous. Denoting $y_0 = y(0)$, this implies that $y = e_\lambda y_0$. To complete the proof we have to show that $y_0 = \mathbf{G}(\lambda)\mathrm{v}$. By (2.24) with $\tau = 0$ and by (4.2),

$$\begin{aligned} \mathbf{P}_+(e_\lambda y_0) = \mathbf{P}_+\mathcal{F}(e_\lambda \mathrm{v}) &= \Psi_\infty\widetilde{\Phi}_0(e_\lambda \mathrm{v}) + \mathbb{F}_\infty(e_\lambda \mathrm{v}) \\ &= \Psi_\infty(\lambda I - A)^{-1}B\mathrm{v} + \mathbb{F}_\infty(e_\lambda \mathrm{v})\,. \end{aligned}$$

We take the Laplace transform of both sides above and we use (2.12) and (2.13) to get that for all $s \in \mathbb{C}$ with $\operatorname{Re} s > \operatorname{Re}\lambda$,

$$(s-\lambda)^{-1}y_0 = C(sI - A)^{-1}(\lambda I - A)^{-1}B\mathrm{v} + \mathbf{G}(s)(s-\lambda)^{-1}\mathrm{v} = (s-\lambda)^{-1}\mathbf{G}(\lambda)\mathrm{v}\,.$$

From here we see that $y_0 = \mathbf{G}(\lambda)\mathrm{v}$, as claimed. $\square$

The last proposition is not stated in the most general form. Indeed, if $\gamma_{\mathbb{F}} < \omega_{\mathbb{T}}$, where $\gamma_{\mathbb{F}}$ is the growth bound of $\mathbb{F}_\infty$, then formula (4.3) remains valid on the larger half-plane $\lambda \in \mathbb{C}_{\gamma_{\mathbb{F}}}$. The most concise argument for this is to regard both sides as analytic functions defined on $\mathbb{C}_{\omega_{\mathbb{T}}}$ with values in the Fréchet space $L^2_{\omega,\mathrm{loc}}((-\infty,\infty);Y)$, where $\omega \in (\gamma_{\mathbb{F}}, \omega_{\mathbb{T}}]$. Both sides have analytic extensions to $\mathbb{C}_\omega$, and hence these extensions must be equal on $\mathbb{C}_\omega$. Since $\omega \in (\gamma_{\mathbb{F}}, \omega_{\mathbb{T}}]$ was arbitrary, we get equal analytic extensions on $\mathbb{C}_{\gamma_{\mathbb{F}}}$, meaning that (4.3) holds on $\mathbb{C}_{\gamma_{\mathbb{F}}}$.

In the scattering theory of Lax and Phillips [12, 13] (and also in [24]) the identity (4.3) is taken as the definition of $\mathbf{G}(\lambda)$, which is called the scattering matrix in that context. We refer to the survey paper of Arov [4], to Staffans [23] or [24] and to our paper [25] for further discussions of the connection between scattering theory and the theory of well-posed linear systems.

5. Duality and time-flow inversion

There are various transformations which lead from one well-posed system to another: static output feedback, duality, time-inversion, flow-inversion and time-flow inversion. We shall discuss here only duality and time-flow inversion, which in the conservative case are equivalent to each other.

As in Sections 2 and 3, we assume that $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ is a well-posed linear system with input space U, state space X, output space Y, transfer function $\mathbf{G}$ and the two growth bounds $\omega_{\mathbb{T}}$ and $\gamma_{\mathbb{F}}$. For all the proofs we refer to [26].

Notation 5.1. Let W be a Hilbert space. For every $u \in L^2_{\mathrm{loc}}((-\infty,\infty);W)$ and all $\tau \geq 0$, we define

$$(\mathbf{Я}u)(t) = u(-t), \qquad t \in \mathbb{R},$$

$$(\mathbf{Я}_\tau u)(t) = \begin{cases} u(\tau - t) & \text{for } t \in [0,\tau], \\ 0 & \text{for } t \notin [0,\tau]. \end{cases}$$

Using the time-reflection operators $\mathbf{Я}_\tau$, we introduce the dual system:

Theorem 5.2. *Let $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system with input space U, state space X and output space Y. Define Σ^d_τ (for all $\tau \geq 0$) by*

$$\Sigma^d_\tau = \begin{bmatrix} \mathbb{T}^d_\tau & \Phi^d_\tau \\ \Psi^d_\tau & \mathbb{F}^d_\tau \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}^*_\tau & \Psi^*_\tau \\ \Phi^*_\tau & \mathbb{F}^*_\tau \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbf{Я}_\tau \end{bmatrix}. \tag{5.1}$$

Then $\Sigma^d = (\mathbb{T}^d, \Phi^d, \Psi^d, \mathbb{F}^d)$ is a well-posed linear system with input space Y, state space X and output space U. Let $x_0 \in X$, $x^d_0 \in X$, $u \in L^2_{\mathrm{loc}}([0,\infty);U)$ and $y^d \in L^2_{\mathrm{loc}}([0,\infty);Y)$. Let x and y be the state trajectory and the output function of Σ corresponding to the initial state x_0 and the input function u, and let x^d and u^d be the state trajectory and the output function of Σ^d corresponding to the intial state

x_0^d *and the input function* y^d. *Then, for every* $\tau \geq 0$,

$$\begin{aligned} \langle x_0, x^d(\tau)\rangle &+ \int_0^\tau \langle u(\sigma), u^d(\tau-\sigma)\rangle \, d\sigma \\ &= \langle x(\tau), x_0^d\rangle + \int_0^\tau \langle y(\sigma), y^d(\tau-\sigma)\rangle \, d\sigma. \end{aligned} \tag{5.2}$$

The system Σ^d introduced above is called the *dual system* corresponding to Σ. It is easy to verify (from (5.1)) that applying the duality transformation twice, we get back the original system: $(\Sigma^d)^d = \Sigma$. Cleary $\omega_{\mathbb{T}} = \omega_{\mathbb{T}^d}$ (since $\mathbb{T}_\tau^d = \mathbb{T}_\tau^*$).

Proposition 5.3. *If* A, B *and* C *are the semigroup generator, control operator and observation operator of the well-posed linear system* Σ *with semigroup growth bound* $\omega_{\mathbb{T}}$, *then the corresponding operators for* Σ^d *are* A^*, C^* *and* B^*. *The transfer functions are related by*

$$\mathbf{G}^d(s) = \mathbf{G}^*(\overline{s}), \qquad \operatorname{Re} s > \omega_{\mathbb{T}}.$$

In particular, the input-output growth bounds are equal: $\gamma_{\mathbb{F}} = \gamma_{\mathbb{F}^d}$.

Some clarifications may be needed. Let us denote, as usual, by U, X and Y the input, state and output space of Σ. The spaces X_1 and X_{-1} are as in Section 2. We denote the corresponding spaces that we get by replacing A by A^* by X_1^d and X_{-1}^d, i.e., X_1^d is $\mathcal{D}(A^*)$ with the norm $\|z\|_1^d = \|(\beta I - A^*)z\|$, where $\beta \in \rho(A^*)$, and X_{-1}^d is the completion of X with respect to the norm $\|z\|_{-1}^d = \|(\beta I - A^*)^{-1}z\|$. Thus, we have the continuous and dense embeddings $X_1^d \subset X \subset X_{-1}^d$, similarly as for the spaces X_1 and X_{-1} introduced in Section 2. The scalar product of X has continuous extensions to $X_1 \times X_{-1}^d$ and to $X_1^d \times X_{-1}$, and X_{-1}^d (respectively X_{-1}) may be regarded as the dual of X_1 (respectively of X_1^d).

Proposition 5.4. *If the system* Σ *is weakly regular then its dual system* Σ^d *is weakly regular as well, and their feedthrough operators, denoted by* D *and* D^d, *are related by*

$$D^d = D^*.$$

We mention that if Σ is weakly regular and its input space U is finite-dimensional, then Σ^d is regular. There are regular systems whose dual is not regular.

Now we introduce the time-flow-inverted system corresponding to a well-posed linear system and state some of its properties. In the time-flow-inverted system we still let the relationship between $x(0)$, $\mathbf{P}_\tau u$, $x(\tau)$, and $\mathbf{P}_\tau y$ be the same as in (2.4), but this time we interpret $\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix}$ as the initial data and $\begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}$ as the final data. Clearly, a necessary and sufficient condition for $\begin{bmatrix} x(0) \\ \mathbf{P}_\tau u \end{bmatrix}$ to depend (uniquely and) continuously on $\begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix}$ is that for all $\tau > 0$, the operator Σ_τ is invertible.

Theorem 5.5. *Suppose that Σ_τ is invertible as an operator from $X \times L^2([0,\tau];U)$ to $X \times L^2([0,\tau];Y)$ for some $\tau > 0$. Then Σ_τ is invertible between these spaces for all $\tau \geq 0$ (note that Σ_0 is the identity on $X \times \{0\}$). Define $\Sigma_\tau^{\leftarrow}$ (for all $\tau \geq 0$) by*

$$\Sigma_\tau^{\leftarrow} = \begin{bmatrix} \mathbb{T}_\tau^{\leftarrow} & \Phi_\tau^{\leftarrow} \\ \Psi_\tau^{\leftarrow} & \mathbb{F}_\tau^{\leftarrow} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \boldsymbol{Я}_\tau \end{bmatrix} \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & \boldsymbol{Я}_\tau \end{bmatrix}. \tag{5.3}$$

Then $\Sigma^{\leftarrow} = (\mathbb{T}^{\leftarrow}, \Phi^{\leftarrow}, \Psi^{\leftarrow}, \mathbb{F}^{\leftarrow})$ is a well-posed linear system. If x and y are the state trajectory and the output function of Σ corresponding to the initial state $x_0 \in X$ and the input function $u \in L^2_{\mathrm{loc}}([0,\infty);U)$ (so that $x(0) = x_0$), then for all $\tau \geq 0$,

$$\begin{bmatrix} x(0) \\ \boldsymbol{Я}_\tau u \end{bmatrix} = \begin{bmatrix} \mathbb{T}_\tau^{\leftarrow} & \Phi_\tau^{\leftarrow} \\ \Psi_\tau^{\leftarrow} & \mathbb{F}_\tau^{\leftarrow} \end{bmatrix} \begin{bmatrix} x(\tau) \\ \boldsymbol{Я}_\tau y \end{bmatrix}.$$

The system $\Sigma^{\leftarrow}$ defined above is called the *time-flow-inverted system* corresponding to Σ. It is easy to verify that applying time-flow inversion twice, we get back the original system: $(\Sigma^{\leftarrow})^{\leftarrow} = \Sigma$. Intuitively, time-flow inversion can be imagined as a combination of time-inversion (reversing the direction of time) and flow-inversion (changing the roles of inputs and outputs). Rigorously speaking, such an interpretation is not always correct, because the two individual inversions may not be well defined for a system, even if its time-flow-inversion is well defined.

Regularity or weak regularity are not preserved under time-flow-inversion in general (of course, weak regularity is preserved in the conservative case, since time-flow-inversion is equivalent to the duality transformation is this case). Even if both systems are regular, we do not know how to express the generating operators of $\Sigma^{\leftarrow}$ in terms of the generating operators of Σ, without additional assumptions.

6. Conservative linear systems

The definition of a conservative well-posed linear system has been given in the Introduction. The differential form of the balance equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2,$$

and the "global" form of the balance equation (see Section 1) is equivalent to the fact that this holds for almost every $t \geq 0$ (all terms are in L^1_{loc}). The system Σ is conservative if and only if the balance equation (in either global or differential form) holds for all state trajectories of Σ as well as for all state trajectories of the dual system Σ^d. This concept is equivalent to what Arov and Nudelman [3] call a conservative scattering system and it goes back to the work of Lax and Phillips [12]. Related material can be found in Livšic [14] (see also the survey Arov [4]).

We mention that if the generating operators A, B, C, and D of Σ are bounded (for example, if Σ is finite-dimensional), so that Σ is described by

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases},$$

then Σ is conservative if and only if

$$A + A^* = -C^*C, \quad B = -C^*D, \quad D^*D = I, \quad DD^* = I; \tag{6.1}$$

see [3, p. 16] (this implies $A + A^* = -BB^*$ and $C = -DB^*$). Moreover, the corresponding transfer function $\mathbf{G}(s) = C(sI - A)^{-1}B + D$ is both inner and co-inner: it is bounded and analytic on the half-plane $\mathbb{C}_0$ and, for almost all $\omega \in \mathbb{R}$,

$$\mathbf{G}^*(i\omega)\mathbf{G}(i\omega) = \mathbf{G}(i\omega)\mathbf{G}(i\omega)^* = I. \tag{6.2}$$

The general well-posed version of these results is more involved. According to [3, Proposition 4.5] (and a little extra reasoning), a well-posed system is conservative if and only if for some (hence, for every) pair of numbers $s, z \in \mathbb{C}_0$,

$$\begin{aligned} A + A^* &= -C^*C, \\ B^*(zI - A^*)^{-1}(zI + A) &= -\mathbf{G}^d(z)C, \\ (z + s)B^*(zI - A^*)^{-1}(sI - A)^{-1}B &= I - \mathbf{G}^d(z)\mathbf{G}(s), \end{aligned} \tag{6.3}$$

and the same conditions are true when we replace Σ by the dual system Σ^d. Here, A, B and C have their usual meaning as in Section 2, while $\mathbf{G}$ and $\mathbf{G}^d$ are the transfer functions of Σ and Σ^d, as in Section 5. The dual version of (6.3) implies that for any $x \in \mathcal{D}(A^*)$ and for all $s \in \mathbb{C}_0$,

$$\mathbf{G}(s)B^*x = C\left[(sI - A)^{-1}BB^*x - x\right].$$

It is proved in [3, pp. 32–33] that the transfer function $\mathbf{G}$ could be any analytic function on the right half-plane $\mathbb{C}_0$ whose values are contractions in $\mathcal{L}(U, Y)$ (an operator-valued Schur function). Thus, the nice property (6.2) is lost in general (but (6.2) still holds if $\mathbb{T}$ and $\mathbb{T}^*$ are strongly stable). An "extreme" example for the loss of (6.2) is as follows: take the state space to be $X = L^2[0, \infty)$, let $\mathbb{T}_t = \mathbf{S}_t^*$ (left shift by t), let $U = \{0\}$ (the system has no input), take $Y = \mathbb{C}$ and $Cx = x(0)$. It is easy to verify that this is a conservative system with transfer function $\mathbf{G} = 0$. We can construct a new system by taking this system together with its dual (which has no output). This leads to a conservative system with $U = Y = \mathbb{C}$, with non-trivial input and output signals, but with transfer function zero.

Proposition 6.1. *Suppose that Σ is a conservative linear system with finite-dimensional and equal input and output spaces, i.e., $U = Y = \mathbb{C}^n$. We denote by $\mathbb{T}$ the semigroup of Σ. Then the following four assertions are equivalent:*

(1) $\mathbb{T}$ *is strongly stable.*

(2) Σ *is exactly observable in infinite time.*

(3) $\mathbb{T}^*$ *is strongly stable.*

(4) Σ *is exactly controllable in infinite time.*

If one (hence, all) of the above assertions holds, then also (6.2) *is true.*

Condition (2) above means that Ψ_∞ from (2.7) is bounded from below, when considered with the range space $L^2([0,\infty);Y)$. Condition (4) above means that the operator $\widetilde{\Phi}_0$ from (2.17) (with $t=0$) is onto X, when considered with the domain $L^2(-\infty,0];U)$. The proof is a combination of well known and simple facts about conservative systems, and its outline is (1)$\Longleftrightarrow$(2), (1)$\Longrightarrow$(6.2), (2)&(6.2)$\Longrightarrow$(3), (3)$\Longleftrightarrow$(4), (3)$\Longrightarrow$(6.2), (4)&(6.2)$\Longrightarrow$(1). The details will be in a paper on conservative linear systems that we plan to write. The fact that $U=Y$ is needed in Proposition 6.1 in order to obtain the equivalence of $\mathbf{G}$ being inner and $\mathbf{G}$ being co-inner. Note that the "extreme" example described before the proposition satisfies (1), but it does not satisfy $U=Y$, and so the proposition does not apply. Indeed, assertions (3) and (4) are false for this example. The modified "extreme" example (also described before the proposition) has $U=Y=\mathbb{C}$, so that now the four assertions must be equivalent, and they are false.

If we restrict our attention to weakly regular conservative systems, so that the generating operators A, B, C and D are all defined, then D^*D must be a contraction in $\mathcal{L}(U)$, but (unlike in the bounded case shown in (6.1)) it need not be the identity. This is clear from the "extreme" example described above, but even if the transfer function is assumed to be inner and co-inner, nothing special about D^*D can be concluded. This can be seen from the following fundamental example of a conservative system: a delay line of length τ. Such a delay line has a simple realization as a regular linear system with state space $X=L^2[-\tau,0]$, with $\mathbb{T}_t$ being the left shift by t on X, see [32]. This is a conservative system with $U=Y=\mathbb{C}$ and $\mathbf{G}(s)=e^{-\tau s}$, so that $D=0$. Note that $\mathbf{G}$ is inner and co-inner.

On the positive side, for a weakly regular conservative system, by letting first $z\to+\infty$ and then $s\to+\infty$ (along the real axis) in (6.3), we get a generalized version of the first three equations in (6.1), namely

$$A+A^* = -C^*C, \quad B^*_\Lambda = -D^*C, \quad \lim_{s\to+\infty} B^*_\Lambda(sI-A)^{-1}B = I-D^*D \tag{6.4}$$

(the first two equations above hold on $\mathcal{D}(A)$). Note that the limit is taken along the real axis. The dual versions of these equations are

$$A+A^* = -BB^*, \quad C_\Lambda = -DB^*, \quad \lim_{z\to+\infty} C_\Lambda(zI-A^*)^{-1}C^* = I-DD^* \tag{6.5}$$

(the first two equations above hold on $\mathcal{D}(A^*)$). At this time, it is not clear to us if (6.4) and (6.5) are sufficient for Σ to be conservative.

The following result shows how to construct a conservative linear system from very simple ingredients. It turns out that our construction appears naturally in mathematical models of vibrating systems with damping. We outline the construction and state the main results. The proofs and further details can be found in [28].

Let H be a Hilbert space, and let $A_0:\mathcal{D}(A_0)\to H$ be a self-adjoint, positive and boundedly invertible operator. We introduce the scale of Hilbert spaces H_α, $\alpha\in\mathbb{R}$, as follows: for every $\alpha\geq 0$, $H_\alpha=\mathcal{D}(A_0^\alpha)$, with the norm $\|z\|_\alpha=\|A_0^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows:

$H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. Equivalently, $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{-\alpha} = \left\|A_0^{-\alpha} z\right\|_H$. The operator A_0 can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A_0 : H_\alpha \to H_{\alpha-1} \qquad \forall\, \alpha \in \mathbb{R}.$$

The second ingredient needed for our construction is a bounded linear operator $C_0 : H_{\frac{1}{2}} \to U$, where U is another Hilbert space. We identify U with its dual, so that $U = U^*$. We denote $B_0 = C_0^*$, so that $B_0 : U \to H_{-\frac{1}{2}}$. We consider the system described by

$$\ddot{z}(t) + A_0 z(t) + \frac{1}{2} B_0 \frac{d}{dt} C_0 z(t) \;=\; B_0 u(t), \tag{6.6}$$

$$y(t) \;=\; \frac{d}{dt} C_0 z(t) - u(t), \tag{6.7}$$

where $t \in [0, \infty)$ is the time. A dot over a variable denotes its derivative with respect to time, possibly in the sense of distributions. The equation (6.6) is understood as an equation in $H_{-\frac{1}{2}}$, i.e., all the terms are in $H_{-\frac{1}{2}}$. Most of the linear equations modelling the damped vibrations of elastic structures can be written in the form (6.6), where z stands for the displacement field and the term $B_0 \frac{d}{dt} C_0 z(t)$, informally written as $B_0 C_0 \dot{z}(t)$, represents a viscous feedback damping. The signal $u(t)$ is an external input with values in U (often a displacement, a force or a moment acting on the boundary) and the signal $y(t)$ is the output (measurement) with values in U as well. The state $x(t)$ of this system and its state space X are defined by

$$x(t) \;=\; \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}, \qquad X \;=\; H_{\frac{1}{2}} \times H\,.$$

This means that in order to solve (6.6), initial values for $z(t)$ and $\dot{z}(t)$ at $t = 0$ have to be specified, and we take $z(0) \in H_{\frac{1}{2}}$ and $\dot{z}(0) \in H$. As we shall see, if $u \in L^2([0, \infty); U)$ then also $y \in L^2([0, \infty); U)$. Moreover, if $x(0) = 0$ then $\|u\| = \|y\|$ (in the norm of $L^2([0, \infty); U)$). We need some notation: for any Hilbert space W, the Sobolev spaces $\mathcal{H}^p(0, \infty; W)$ of W-valued functions (with $p > 0$) are defined in the usual way. The notation $C^n(0, \infty; W)$ (with $n \in \{0, 1, 2, \dots\}$) for n times continuously differentiable W-valued functions on $[0, \infty)$ is also quite standard. We denote by $BC^n(0, \infty; W)$ the space of those $f \in C^n(0, \infty; W)$ for which $f, f', \dots f^{(n)}$ are all bounded. Our main result is the following:

Theorem 6.2. *With the above assumptions, the equations* (6.6) *and* (6.7) *determine a conservative linear system* Σ*, in the following sense:*

There exists a conservative linear system Σ *whose input and output spaces are both* U *and whose state space is* X*. If* $u \in L^2([0, \infty); U)$ *is the input function,* $x = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ *is the state trajectory and* y *is the output function corresponding to* u *and some initial state in* X*, then*

(1) $$z \in BC(0,\infty;H_{\frac{1}{2}}) \cap BC^1(0,\infty;H) \cap \mathcal{H}^2(0,\infty;H_{-\frac{1}{2}}).$$

(2) *The two components of x are related by $w = \dot{z}$.*

(3) $C_0 z \in \mathcal{H}^1(0,\infty;U)$ *and the equations* (6.6) *(in $H_{-\frac{1}{2}}$) and* (6.7) *(in U) hold for almost every $t \geq 0$.*

If $\dot{z}$ is a continuous function of the time t, with values in $H_{\frac{1}{2}}$ (we shall derive conditions for this to be true), then (6.6) and (6.7) can be rewritten in the form

$$\ddot{z}(t) + A_0 z(t) + \frac{1}{2} B_0 C_0 \dot{z}(t) = B_0 u(t), \tag{6.8}$$

$$y(t) = C_0 \dot{z}(t) - u(t). \tag{6.9}$$

We can rewrite the equations (6.8), (6.9) as a first order system as follows:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= C^e x(t) - u(t), \end{cases} \tag{6.10}$$

where

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -\frac{1}{2}B_0C_0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \;\middle|\; A_0 z + \frac{1}{2} B_0 C_0 w \in H \right\},$$

$$C^e : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to U, \qquad C^e = [0 \;\; C_0].$$

We denote by C the restriction of C^e to $\mathcal{D}(A)$.

Proposition 6.3. *With the above notation, the semigroup generator of Σ is A, its control operator is B and its observation operator is C. The transfer function of Σ is given for all $s \in \mathbb{C}_0$ by*

$$\mathbf{G}(s) = C^e(sI - A)^{-1}B - I = C_0 s \left(s^2 I + A_0 + \frac{s}{2} B_0 C_0 \right)^{-1} B_0 - I.$$

The system Σ is isomorphic to its dual, and this (together with the fact that Σ is conservative) implies the following:

Proposition 6.4. *With the above notation, the following assertions are equivalent:*

(1) *The pair (A, B) is exactly controllable.*

(2) *The pair (A, C) is exactly observable.*

(3) *The semigroup generated by A is exponentially stable.*

A similar result holds for strong stability, with an additional assumption:

Proposition 6.5. *With the above notation, assume that $(\beta I - A)^{-1}$ is compact for some $\beta \in \rho(A)$ or that the intersection $\sigma(A) \cap i\mathbb{R}$ is countable. Then the following assertions are equivalent:*

(1) *The pair (A, B) is approximately controllable in infinite time.*

(2) *The pair (A, C) is approximately observable in infinite time.*

(3) *The semigroup generated by A is strongly stable.*

It is often interesting to examine the well-posedness of the undamped system corresponding to (6.6):

$$\ddot{z}(t) + A_0 z(t) = B_0 u(t), \tag{6.11}$$

with the same assumptions on A_0 and B_0, and with the output signal given again by (6.7). It is interesting that in this special context, only the well-posedness of the transfer function has to be checked, the admissibilities of the control and observation operators follow.

Proposition 6.6. *The following statements are equivalent:*

(1) *The function $\mathbf{G}^o : \mathbb{C}_0 \to \mathcal{L}(U)$ defined by*

$$\mathbf{G}^o(s) = C_0 s \left(s^2 I + A_0\right)^{-1} B_0 - I$$

is bounded on a vertical line contained in $\mathbb{C}_0$.

(2) *The function $\mathbf{G}^o$ defined above is bounded on every right half-plane $\mathbb{C}_\omega$ with $\omega > 0$.*

(3) *The equations (6.11) and (6.7) determine a well-posed linear system with the state space $X = H_{\frac{1}{2}} \times H$.*

If the above statements are true, then the transfer function of the system from point (3) is $\mathbf{G}^o$ from point (1).

7. A Rayleigh beam with piezoelectric actuator

In this section we provide an example of well-posed linear system, described by a second order differential equation in a Hilbert space, without damping and with unbounded control and observation operators. After adding a damping term, we obtain a conservative system that fits into the framework discussed in Section 5. This will enable us to prove that the damped system is exactly controllable and exponentially stable. Since the results in this section are new, we include the proofs.

The physical system that we have in mind consists of an elastic beam with a piezoelectric actuator. We suppose that both ends of the beam are hinged and that the actuator is excited in a manner so as to produce pure bending moments. The input is the voltage acting on the actuator and the measurement is the rate of the mean curvature of the piezoelectric actuator (or, equivalently, the difference of the angular velocities of the extremities of the actuator). We model this system by an initial and boundary value problem representing a homogenous Rayleigh beam, situated along the interval $[0, \pi]$, with the actuator occupying the interval $(\xi, \eta) \subset (0, \pi)$. The equations are (see, for instance, [7] or [8]):

$$\ddot{w}(x,t) - \alpha \frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^4 w}{\partial x^4}(x,t) = u(t) \frac{d}{dx}[\delta_\eta(x) - \delta_\xi(x)], \tag{7.1}$$

$$w(0,t) = w(\pi,t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\pi,t) = 0, \quad t \geq 0, \tag{7.2}$$

$$w(x,0) = w^0(x), \quad \dot{w}(x,0) = w^1(x), \quad 0 < x < \pi, \tag{7.3}$$

$$y(t) = \frac{\partial \dot{w}}{\partial x}(\eta, t) - \frac{\partial \dot{w}}{\partial x}(\xi, t). \tag{7.4}$$

In these equations $w(x,t)$ represents the transverse displacement of the beam at position $x \in [0, \pi]$, $\xi, \eta \in (0, \pi)$ denote the endpoints of the actuator, δ_a is the Dirac mass at the point a and $\alpha > 0$ is a constant, proportional to the moment of inertia of the cross section of the beam (α is proportional to the square of the thickness of the beam and is often neglected). The input is $u : [0,T] \to \mathbb{R}$ representing the voltage applied to the actuator.

Let us now denote $H = \mathcal{H}_0^1(0,\pi)$ and consider the dual $[\mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)]^*$ of $\mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)$ with respect to the pivot space $L^2(0,\pi)$. We consider also the bounded linear isomorhism $\mathcal{R} : [\mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)]^* \to L^2(0,\pi)$ defined by

$$\mathcal{R} = \left(I - \alpha \frac{d^2}{dx^2}\right)^{-1}.$$

We also define the linear operator $A_0 : \mathcal{D}(A_0) \to H$ by

$$\mathcal{D}(A_0) = \left\{ \phi \in \mathcal{H}^3(0,\pi) \,\middle|\, \phi(0) = \phi(\pi) = 0, \ \frac{d^2\phi}{dx^2}(0) = \frac{d^2\phi}{dx^2}(\pi) = 0 \right\},$$

$$A_0 \phi = \mathcal{R} \frac{d^4 \phi}{dx^4}, \qquad \forall\, \phi \in \mathcal{D}(A_0). \tag{7.5}$$

We can easily show that A_0 is self-adjoint and strictly positive. Denoting $H_1 = \mathcal{D}(A_0)$ and then defining H_μ for $\mu \in \mathbb{R}$ by fractional powers of A_0 and duality, as in Section 5, we have

$$H_{\frac{1}{2}} = \mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi), \quad H_{-\frac{1}{2}} = L^2(0,\pi).$$

Let us consider the operator

$$B_0 : \mathbb{C} \to H_{-\frac{1}{2}},$$

defined by

$$B_0 \mathrm{v} = \mathrm{v}\frac{d}{dx}[\delta_\eta - \delta_\xi].$$

With this notation, the system (7.1)-(7.4) can be written as

$$\ddot{w}(t) + A_0 w(t) = B_0 u(t),$$

$$w(0) = w^0, \quad \dot{w}(0) = w^1,$$

$$y(t) = B_0^* \dot{w}(t).$$

Theorem 7.1. *Equations* (7.1)-(7.4) *define a well-posed linear system with input space $U = \mathbb{C}$, state space $X = [\mathcal{H}^2(0,\pi) \cap \mathcal{H}_0^1(0,\pi)]$ and output space $\mathbb{C}$.*

Since the problem (7.1)-(7.4) is linear, in order to establish the above well-posedness result, it suffices to consider controls supported at only one point $a \in (0,\pi)$. More precisely, we consider the following initial and boundary value problem:

$$\ddot{\psi}(x,t) - \alpha\frac{\partial^2 \ddot{\psi}}{\partial x^2}(x,t) + \frac{\partial^4 \psi}{\partial x^4}(x,t) = v(t)\frac{d\delta_a}{dx}, \quad 0 < x < \pi, \ t > 0, \tag{7.6}$$

$$\psi(0,t) = \psi(\pi,t) = 0, \quad \frac{\partial^2 \psi}{\partial x^2}(0,t) = \frac{\partial^2 \psi}{\partial x^2}(\pi,t) = 0, \ t > 0, \tag{7.7}$$

$$\psi(x,0) = \psi^0(x), \quad \frac{\partial \psi}{\partial t}(x,0) = \psi^1(x), \quad 0 < x < \pi. \tag{7.8}$$

Theorem 7.1 clearly follows from the following proposition:

Proposition 7.2. *For any $v \in L^2(0,T)$ and for any $a \in (0,\pi)$, the problem* (7.6)-(7.8) *has a unique solution*

$$\psi \in C(0,T;\mathcal{H}^2(0,\pi)) \cap C^1(0,T;\mathcal{H}_0^1(0,\pi)). \tag{7.9}$$

Moreover, for any $b \in (0,\pi)$, we have that $\frac{\partial \psi}{\partial x}(b,\cdot) \in \mathcal{H}^1(0,T)$ and there exists $C > 0$ such that

$$\|\psi(\cdot,T)\|^2_{\mathcal{H}^2(0,\pi)} + \|\dot{\psi}(\cdot,T)\|^2_{\mathcal{H}^1(0,\pi)} + \left\|\frac{\partial \psi}{\partial x}(b,\cdot)\right\|^2_{\mathcal{H}^1(0,T)}$$

$$\leq C\left(\|\psi^0\|^2_{H_{\frac{1}{2}}} + \|\psi^1\|^2_H + \|v\|^2_{L^2(0,T)}\right). \tag{7.10}$$

Before proving Proposition 7.2, we consider the following homogeneous problem:

$$\ddot{\phi}(x,t) - \alpha\frac{\partial^2 \ddot{\phi}}{\partial x^2}(x,t) + \frac{\partial^4 \phi}{\partial x^4}(x,t) = 0,\ \ 0 < x < \pi,\ \ t > 0, \tag{7.11}$$

$$\phi(0,t) = \phi(\pi,t) = \frac{\partial^2 \phi}{\partial x^2}(0,t) = \frac{\partial^2 \phi}{\partial x^2}(\pi,t) = 0,\ \ t > 0, \tag{7.12}$$

$$\phi(x,0) = \phi^0(x),\ \frac{\partial \phi}{\partial t}(x,0) = \phi^1(x),\ \ 0 < x < \pi. \tag{7.13}$$

Lemma 7.3. *For any initial data* $(\phi^0,\phi^1) \in \mathcal{H}^2(0,1) \times \mathcal{H}^1_0(0,\pi)$*, there exists a unique weak solution of* (7.11)-(7.13) *in the class* $\phi \in C(0,T;\mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)) \cap C^1(0,T;\mathcal{H}^1_0(0,\pi))$*. Moreover, for all* $b \in (0,\pi)$ *we have that* $\frac{\partial \phi}{\partial x}(b,\cdot) \in \mathcal{H}^1(0,T)$ *and there exists* $C > 0$ *such that*

$$\left\| \frac{\partial \phi}{\partial x}(b,\cdot) \right\|^2_{\mathcal{H}^1(0,T)} \le C \left(\|\phi^0\|^2_{\mathcal{H}^2(0,\pi)} + \|\phi^1\|^2_{\mathcal{H}^1(0,\pi)} \right). \tag{7.14}$$

Proof. It is easy to see, by the semigroup method, that the problem (7.11)-(7.13) is well-posed in the state space $X = [\mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)] \times \mathcal{H}^1_0(0,\pi)$.

In order to prove (7.14), we put

$$\phi^0(x) = \sum_{k \ge 1} a_k \sin(kx), \quad \phi^1(x) = \sum_{k \ge 1} b_k \sin(kx),$$

with $(k^2 a_k), (k b_k) \in l^2(\mathbb{R})$. Obviously, we have

$$\phi(x,t) = \sum_{k \ge 1} \left\{ a_k \cos\left[\frac{k^2}{\sqrt{1+k^2}} t\right] + \frac{b_k\sqrt{1+k^2}}{k^2} \sin\left[\frac{k^2}{\sqrt{1+k^2}} t\right] \right\} \sin(kx), \tag{7.15}$$

which implies that, for all $T > 0$, we have $\frac{\partial \phi}{\partial x}(b,\cdot) \in H^1(0,T)$ and

$$\int_0^T \left| \frac{\partial \dot{\phi}}{\partial x}(b,t) \right|^2 dt \le C \sum_{k \ge 1} k^2 \left[a_k^2 \frac{k^4}{1+k^2} + b_k^2 \right] \le C \sum_{k \ge 1} k^2 \left[a_k^2 k^2 + b_k^2 \right],$$

which clearly yields (7.14). □

The extension of A_0 from (7.5) to an operator from $H_{\frac{1}{2}} = \mathcal{H}^2(0,\pi) \cap \mathcal{H}^1_0(0,\pi)$ to $H_{-\frac{1}{2}} = L^2(0,\pi)$ is still denoted A_0. Moreover, let us put $\mathcal{D}(A_1) = H_{\frac{1}{2}}$ and define

$$A_1 : H_{\frac{1}{2}} \to L^2(0,\pi),\ A_1\phi = -\frac{1}{\alpha}\frac{d^2\phi}{dx^2} \qquad \forall\, \phi \in H_{\frac{1}{2}}.$$

The operators A_0 and A_1 are related as follows:

Lemma 7.4. *The linear operator $L = A_0 - A_1$ is bounded from $H_{\frac{1}{2}}$ to $H_{\frac{1}{2}}$.*

Proof. If $\phi \in H_{\frac{1}{2}}$, then ϕ can be written as

$$\phi(x) = \sum_{n\geq 1} a_n \sin(nx),$$

with $\sum_{n\geq 1} n^4 a_n^2 < \infty$. Then

$$L\phi = -\sum_{n\geq 1} \frac{n^2}{1+\alpha n^2} a_n \sin(nx),$$

which clearly implies the conclusion of the lemma. □

Proof of Proposition 7.2. The existence and the uniqueness of a solution ψ satisfying (7.9) follows from Lemma 7.3 by duality (see Jaffard and Tucsnak [11] for details). In order to prove the regularity of the trace of ψ at $x = b$ we notice that the equation

$$\ddot{\psi} + A_0\psi = v(t)\mathcal{R}\left(\frac{d\delta_a}{dx}\right), \tag{7.16}$$

holds in $L^2(0,T;L^2(0,\pi))$. We consider the initial value problem

$$\ddot{\psi}_1 + A_1\psi_1 = v(t)\mathcal{R}\left(\frac{d\delta_a}{dx}\right), \tag{7.17}$$

$$\psi_1(0) = 0,\ \dot{\psi}_1(0) = 0. \tag{7.18}$$

Relation above imply that $\psi_2 = \psi - \psi^1$ satisfies

$$\ddot{\psi}_2 + A_1\psi_2 = L\psi,$$

$$\psi_2(0) = \psi^0,\ \dot{\psi}_2(0) = \psi^1,$$

which, by (7.9) and Lemma 7.4, implies that $\frac{\partial\psi_2}{\partial\partial x}(b,\cdot) \in \mathcal{H}^1(0,T)$. The fact that $\frac{\partial\psi_1}{\partial x}(b,\cdot) \in \mathcal{H}^1(0,T)$ follows by applying the Laplace transform to (7.17) and then by direct calculations (see [2] for details). □

Remark 7.5. In [11] the authors considered the two-dimensional version of (7.1)-(7.3). More precisely for $T > 0$, $u \in L^2(0,T)$, $\Omega \subset \mathbb{R}^2$ an open bounded set, $\Gamma = \partial\Omega$, $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and γ a curve included in Ω, they considered the problem

$$\ddot{w}(x,y,t) - \alpha\Delta\ddot{w}(x,y,t) + \Delta^2 w(x,y,t) = u(t)\frac{\partial\delta_\gamma}{\partial\nu},\ \text{in } Q, \tag{7.19}$$

$$w(x,y,t) = \Delta w(x,y,t) = 0,\ \text{on } \Sigma, \tag{7.20}$$

$$w(x, y, 0) = 0, \; w'(x, y, 0) = 0, \text{ in } \Omega, \tag{7.21}$$

where $\frac{\partial \delta_\gamma}{\partial \nu}$ stands for the derivative of the Dirac mass concentrated on γ with respect to the normal to γ, and $\alpha > 0$ is a constant. The main result in [11] asserts that, for any $u \in L^2(0, T)$, the problem (7.19)-(7.21) admits a unique solution $w \in C(0, T; \mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)) \cap C^1(0, T; \mathcal{H}_0^1(\Omega)$. In other words, if we consider the input space $U = \mathbb{C}$ and the state space $X = [\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)] \times \mathcal{H}_0^1(\Omega)$, then the application from the input to the state is bounded. We conjecture that, if we define the output

$$y(t) = \int_\gamma \frac{\partial \dot{w}}{\partial \nu}, \tag{7.22}$$

then (7.19)-(7.22) defines a well-posed linear system, with input space $U = \mathbb{C}$, state space $X = [\mathcal{H}^2(\Omega) \cap \mathcal{H}_0^1(\Omega)] \times \mathcal{H}_0^1(\Omega)$ and output space U.

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