Passive Behaviors and their Passive State/Signal Realizations in Continuous Time

Damir Z. Arov and Mikael Kurula and Olof J. Staffans

Abstract— **In this talk we discuss passive and conservative state/signal systems in continuous time. Such systems can be used to model, e.g., passive linear electrical circuits containing lumped and/or distributed resistances, capacitors, inductors, and wave guides, etc. Most of the standard partial differential equations appearing in physics on can be written in state/signal form.**

A passive state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ consists of three components: 1) an internal Hilbert *state space* $\mathcal{X}$, 2) a Kreĭn *signal space* $\mathcal{W}$ through which the system interacts with the external world, and 3) a *generating subspace* V of the product space $\mathcal{X} \times \mathcal{X} \times \mathcal{W}$. The generating subspace is required to be maximal nonnegative with respect to a certain power inner product and to satisfy an extra non-degeneracy condition.

A *classical future trajectory* $(x(\cdot), w(\cdot))$ of Σ consists of a par of functions $x(\cdot) \in C^1([0,\infty); \mathcal{X})$ and $w(\cdot) \in C([0,\infty); \mathcal{W})$ satisfying

$$(\dot{x}(t), x(t), w(t)) \in V, \qquad t \in [0,\infty).$$

The set of all generalized future trajectories of Σ is obtained from the family of all classical trajectories by a standard approximation procedure.

By the *future behavior* of Σ we mean the set of all signal parts $w(\cdot)$ of all stable future trajectories $(x(\cdot), w(\cdot))$ of Σ satisfying the extra condition $x(0) = 0$. The future behavior of a passive state/signal system is a right-shift invariant subspace of $L^2([0,\infty); \mathcal{W})$ and it is maximal nonnegative with respect to the Kreĭn space inner product in $L^2([0,\infty); \mathcal{W})$ inherited from $\mathcal{W}$. Such a subspace is called a *passive future behavior*.

Each passive future behavior can be realized as the future behavior of a passive state/signal system Σ, and it is possible to require Σ to have, for example, one of the following five additional sets of properties: Σ is a) observable and co-energy preserving, b) controllable and energy preserving, c) simple and conservative, d) minimal and optimal, e) minimal and $*$-optimal. Realizations satisfying one of the sets of conditions a)–e) are determined by the given future behavior up to unitary similarity. It is even possible to construct *canonical* realizations, i.e., realizations with satisfy a)–e), and which are *uniquely determined by the given data*.

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I. Passive Future Behaviors

Let $\mathcal{W}$ be a Kreĭn (signal) space. Let $K^2_+(\mathcal{W})$ be the Kreĭn space of $\mathcal{W}$-valued functions $w(\cdot)$ in $L^2_+(\mathcal{W}) := L^2([0,\infty); \mathcal{W})$ with indefinite inner product inhered from the inner product $[\cdot,\cdot]_{\mathcal{W}}$ in $\mathcal{W}$, i.e., $[w(\cdot), w(\cdot)]_{K^2_+(\mathcal{W})} := \int_0^\infty [w(t), w(t)]\,\mathrm{d}t$. Let $s \mapsto T^s$ be the semigroup of right shifts in $K^2_+(\mathcal{W})$. A maximal nonnegative T^s-invariant subspace $\mathfrak{W}_+$ of $K^2_+(\mathcal{W})$ is called as *passive future behavior* on $\mathcal{W}$.

The notion of a passive future behavior plays a role in the passive s/s (state/signal) systems theory that is analogical to the role of a scattering operator S in the the passive i/o (input/output) systems theory. These two notions correspond to each other in the following way: if $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$ then $K^2_+(\mathcal{W}) = -L^2_+(\mathcal{Y})\,[\dot{+}]\,L^2_+(\mathcal{U})$ is a fundamental decomposition of $K^2_+(\mathcal{W})$, and $\mathfrak{W}_+$ is the graph of a linear contractive *scattering operator* operator S from $L^2_+(\mathcal{U})$ into $L^2_+(\mathcal{Y})$ that intertwines the semigroups of right shifts in these spaces. The converse is also true. Thus, to a passive future behavior corresponds a family of scattering operators, that are defined by this behavior and by a fundamental decomposition of $\mathcal{W}$.

The Laplace transformation is a unitary map from L^2_+-spaces onto the Hardy H^2 spaces of holomorphic functions on the right half plane $\mathbb{C}^+$, and also from the Kreĭn space $K^2_+(\mathcal{W})$ onto the Kreĭn space $\widetilde{K}^2_+(\mathcal{W}) = -H^2(\mathcal{Y})\,[\dot{+}]\,H^2(\mathcal{U})$. The Laplace transformation maps $\mathfrak{W}_+$ onto a maximal nonnegative $\widetilde{T}^s$-invariant subspace $\widetilde{\mathfrak{W}}^+$ of $\widetilde{K}^2_+(\mathcal{W})$. Such a subspace is called a *passive future frequency domain behavior* on $\mathcal{W}$. Here $s \mapsto \widetilde{T}^s$ is the semigroup of multiplication of a function $\widetilde{w}(\cdot)$

in $\widetilde{K}^2_+(\mathcal{W}))$ by the scalar function $z \mapsto \exp(-sz)$. The subspace $\widetilde{\mathfrak{W}}_+$ is the graph of the operator $H^2(\mathcal{U}) \to H^2(\mathcal{Y})$ which multiplies a function $\widetilde{u}(\cdot)$ in $H^2(\mathcal{U})$ by a $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued holomorphic contractive function $\widetilde{S}(\cdot)$. The function $\widetilde{S}(\cdot)$ is the symbol of the scattering operator S and it is called a scattering matrix.

II. Passive S/S Systems and their Future Behaviors

A passive linear continuous time invariant s/s system consists of three components: 1) an internal Hilbert *state space* $\mathcal{X}$, 2) a Kreĭn *signal space* $\mathcal{W}$, and 3) a *generating subspace* V of the *node space* $\mathfrak{K} := \mathcal{X} \times \mathcal{X} \times \mathcal{W}$. We equip $\mathfrak{K}$ with the Kreĭn space inner product $[\cdot,\cdot]_{\mathfrak{K}}$, defined by

$$[k_1,k_2]_{\mathfrak{K}} = -(z_1,x_2)_X - (x_1,z_2)_X + [w_1,w_2]_W$$

for $k_i = (z_i,x,w_i) \in \mathfrak{K}$, $i = 1,2$, and require the generating subspace to be a *maximal nonnegative* subspace of $\mathfrak{K}$. In addition we assume that V satisfies the condition that $(z,0,0) \in V$ only if $z = 0$. We denote this system by $\Sigma = (V;\mathcal{X},\mathcal{W})$.

A *classical future trajectory* $(x(\cdot),w(\cdot))$ of Σ consists of a par of functions $x(\cdot) \in C^1([0,\infty);\mathcal{X})$ and $w(\cdot) \in C([0,\infty);\mathcal{W})$ satisfying

$$(\dot{x}(t),x(t),w(t)) \in V, \qquad t \in [0,\infty).$$

The set of all *generalized future trajectories* of Σ is the closure in $C([0,\infty);\mathcal{X}) \times L^2_{\mathrm{loc}}([0,\infty);\mathcal{W})$ of the family of all classical future trajectories of Σ. A generalized future trajectory is *stable* if x is bounded on $[0,\infty)$ and $w \in K^2_+(\mathcal{W})$, and it is *externally generated* if $x(0) = 0$.

The set $\mathfrak{W}^{\Sigma}_+$ of all signal components $w(\cdot)$ of all externally generated stable generalized trajectories of Σ is called the *future behavior* of Σ. It is a passive future behavior on $\mathcal{W}$. The converse is also true: any passive future behavior $\mathfrak{W}_+$ on a Kreĭn space $\mathcal{W}$ may be realized as the future behavior $\mathfrak{W}^{\Sigma}_+$ of a passive s/s system Σ. Moreover, Σ may be chosen to belong to any one of the following classes of passive s/s systems: a) energy preserving and controllable, b) co-energy preserving and observable, c) conservative and simple, d) optimal and minimal, e) $*$-optimal and minimal. Each such system is defined by $\mathfrak{W}_+$ up to unitary similarity.

III. Definitions of Classes a)-e)

A system $\Sigma = (V,\mathcal{X},\mathcal{W})$ is called *energy preserving*, or *co-energy preserving*, or *conservative* if $V \subset V^{[\perp]}$, or $V^{[\perp]} \subset V$, or $V = V^{[\perp]}$, respectively, where $V^{[\perp]}$ is the orthogonal companion of V.

A system Σ is called *controllable* if the closure $\mathfrak{R}_\Sigma$ of the set of all states $x(t)$ taken from all externally generated trajectories of Σ is the full state space $\mathcal{X}$. A system Σ is called *observable* if the set $\mathfrak{N}_\Sigma$ of all initial states $x(0)$ of all unobservable trajectories of Σ, i.e., trajectories $(x(\cdot),w(\cdot))$ on $[0,\infty)$ with $w \equiv 0$, is the zero subspace of $\mathcal{X}$. A system Σ is called *simple* if $(\mathfrak{R}_\Sigma)^\perp \cap \mathfrak{N}_\Sigma = \{0\}$. A system Σ is *minimal* if it is controllable and observable. A minimal system Σ is *optimal ($*$-optimal)* if its stable externally generated trajectories $(x(\cdot),w(\cdot))$ have the property that at any time t the norm $\|x(t)\|$ is the smallest (largest) possible one among all externally generated trajectories with the same signal part $w(\cdot)$ of any minimal passive s/s systems with the same future behavior as Σ.

IV. Scattering Representations of Passive S/S Systems

A passive linear continuous time i/s/o (input/state/output) scattering system $\Sigma_{\mathrm{sc}} = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ consists of three Hilbert spaces $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Y}$ and a closed linear operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \mathcal{X} \times \mathcal{U} \to \mathcal{X} \times \mathcal{Y}$ with properties defined in [AN96] and [Sta05]. The definitions of trajectories $(x(\cdot),u(\cdot),y(\cdot))$ of such a system and of its scattering matrix $\widetilde{S}(\cdot)$ can also be found there, together with the definitions of the analogues of properties a)–e) for s/s systems listed above.

To any passive scattering system Σ_{sc} corresponds a unique passive s/s system $\Sigma = (V,\mathcal{X},\mathcal{W})$ with the same state space $\mathcal{X}$ and signal space $\mathcal{W} = -\mathcal{Y}\,[+]\,\mathcal{U}$, such that there is a one-to-one correspondence between the trajectories $(x(\cdot),u(\cdot),y(\cdot))$ of Σ_{sc} and the trajectories $(x(\cdot),w(\cdot))$ of Σ with $w(\cdot) = y(\cdot) + u(\cdot)$, $y(\cdot) \in L^2_+(\mathcal{Y})$, and $u(\cdot) \in L^2_+(\mathcal{U})$. The future behavior of Σ is the graph of the corresponding i/o scattering operator S.

Conversely, if $\Sigma = (V,\mathcal{X},\mathcal{W})$ is a passive linear continuous time invariant s/s system and $\mathcal{W} = -\mathcal{Y}[+]\mathcal{U}$ is a fundamental decomposition of $\mathcal{W}$, then there is a unique passive scattering system Σ_{sc} such that Σ may be recovered from Σ_{sc} as described above. We call Σ_{sc} a scattering representation of Σ. This correspondence between passive scattering systems and passive s/s systems is invariant with respect to the above classifications a)–e).

V. Canonical models

Canonical models of the passive s/s realizations in the classes a)–e) have been obtained that are continuous time analogues of the discrete time models obtained earlier by D. Arov and O. Staffans, and presented in [AS09], [AS10]. Some of the results presented above have been obtained earlier in [AN96], [KS09], and [Kur10]. More details on the continuous time canonical models will be presented in [AKS10].

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