

## State/signal linear time-invariant systems theory: passive discrete time systems

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Dedicated to Vladimir A. Yakubovich, on the occasion of his 80th birthday

### SUMMARY

This is a continuation of previous work where we developed a discrete time-invariant linear state/signal systems theory in a general setting. In this article, the state space is required to be a Hilbert space, as earlier, but the signal space is taken to be a Kreĭn space. The notion of the adjoint of a given state/signal system is introduced and exploited throughout the paper, and in particular, in the definition and the study of passive and conservative state/signal systems, which is the main subject of this paper. It is shown that each fundamental decomposition of the Kreĭn signal space is admissible for a passive state/signal system, meaning that there is a corresponding input/state/output representation of the system, a so-called scattering representation. The connection between different scattering representations and their scattering matrices (i.e. transfer functions) is explained. We show that every passive state/signal system has a minimal conservative orthogonal dilation and minimal passive orthogonal compressions. Passive signal behaviours are defined, and their passive, conservative, and  $H$ -passive realizations are studied. It is shown that the set of all positive self-adjoint operators  $H$  (which need not be bounded or have a bounded inverse) for which a state/signal system  $\Sigma$  is  $H$ -passive coincides with the set of generalized positive solutions  $H$  of the Kalman–Yakubovich–Popov inequality for an arbitrary scattering representation of  $\Sigma$ , and consequently, this set does not depend on the particular representation. Under an extra minimality assumption this set contains a minimal solution which defines the available storage, and a maximal solution which defines the required supply. Copyright © 2006 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

It is a great pleasure for the authors to dedicate the present work to Vladimir Andreevich Yakubovich, the founder of the absolute stability theory through the conception of the theory of passive systems.<sup>§</sup> During the last 40 years the absolute stability theory has developed intensively within pure and applied control theory. The classical results by Kalman, Yakubovich, and Popov in this area are now so well known that they are typically regarded as ‘folklore’, and consequently, exact references to the original publications are often not included.

Our present contribution to the passive systems theory, i.e. the introduction of the class of discrete time passive linear state/signal systems, extends the classical theory in the respect that we do not distinguish between inputs and outputs of a system; they are both considered as parts of the *signal component* of the system. The same feature is found in the behavioural theory developed by Willems, which like the absolute stability theory has had a great impact on modern control theory. However, passivity considerations force us to always include an *explicit state component* in the system which is usually either missing or only implicit in the behavioural theory. It is the inclusion of this state component that makes it possible to obtain a natural input/output-free mathematical model of a passive linear infinite-dimensional system that interchanges energy with the surroundings, thereby making it possible to extend the absolute stability theory into a behavioural state/signal framework.

Our findings can be roughly summarized as follows (exact definitions and details will be given later in this article). Let  $\Sigma$  be a minimal state/signal system with a passive behaviour. Then the signal space  $\mathcal{W}$  of  $\Sigma$  can always be split into an input space  $\mathcal{U}$  and an output space  $\mathcal{Y}$  in such a way that we get an input/state/output representation  $\Sigma_{i/s/o}$  of  $\Sigma$  of the classical *scattering type*. Here the word ‘scattering’ means that the *supply rate* which describes the exchange of energy between the system and the surroundings is given by  $j(u, y) = \|u\|_{\mathcal{U}}^2 - \|y\|_{\mathcal{Y}}^2$ , where  $u$  is the input and  $y$  is the output. In other words, the amount of energy flowing into the system is proportional to  $\|u\|_{\mathcal{U}}^2$ , and the amount of energy flowing out of the system is proportional to  $\|y\|_{\mathcal{Y}}^2$ . The passivity of the behaviour of the system guarantees that the map from the input to the output is contractive in the  $\ell^2$ -norm, and hence the generalized Kalman–Yakubovich–Popov (KYP) inequality (of scattering type) for  $\Sigma_{i/s/o}$  has a non-empty set of solutions. Each solution is a positive self-adjoint operator in the state space, but it may be unbounded and have an unbounded inverse. Indeed, in the infinite-dimensional setting the boundedness or unboundedness of these solutions and their inverses depend in a crucial way on the original choice of state space.<sup>¶</sup> However, the scattering representation mentioned above is not unique, and there also exist other input/state/output representations of  $\Sigma$  which are *not of scattering type*. In the state/signal setting the supply rate corresponds to an indefinite inner product in the signal space  $\mathcal{W}$ . Depending on how the signal space is split into an input space  $\mathcal{U}$  and an output space  $\mathcal{Y}$  we get representations of the supply rate of the state/signal system  $\Sigma$  that look different from the scattering rate  $j(u, y) = \|u\|_{\mathcal{U}}^2 - \|y\|_{\mathcal{Y}}^2$ . The two most commonly studied cases, in addition to the scattering rate mentioned above, are the *impedance* and *transmission* supply rates. The

<sup>§</sup>In particular, the first author remembers with great affection the moral support by V. A. Yakubovich and the research initiated by him, which resulted in the joint publication [1] at the difficult time of the first author’s unsuccessful attempt to defend a doctoral thesis in the partially anti-semitic atmosphere of that time.

<sup>¶</sup>In Reference [2] an example is given based on the heat equation where *all* solutions of the continuous time version of the generalized KYP inequality are unbounded and have an unbounded inverse.

coefficients of the different input/state/output representations  $\Sigma_{i/s/o}$  of  $\Sigma$  (i.e. the main operator, the control operator, the observation operator, and the feed-through operator) vary with the decomposition, and so do the coefficients defining the supply rate of  $\Sigma$  as a quadratic function of the input and output (so that in one representation it may be of impedance type and in another of transmission type), but *the set of solutions of the generalized KYP inequality always stays the same*. This provides us with a general tool to *convert known results for scattering systems (for example, those from the Yakubovich school) into analogous results for impedance and transmission systems, and the other way around*. See Remark 9.14 for details.

Infinite-dimensional systems theory tends to be technically rather complicated, especially in the case of a continuous time variable. A natural starting point is therefore to begin with the discrete time theory, as we have done here, although the ultimate goal is to develop an analogous theory for continuous time system that can be applied to boundary control systems of hyperbolic or parabolic type. By using the internal Cayley transform one can transform many of the results presented here to a continuous time setting. We plan to return to this elsewhere.

The general ‘topological’ part of the linear time-invariant state/signal systems theory in discrete time was introduced and studied in Reference [3], which we in the sequel refer to as ‘Part I’. There we throughout took both the state space  $\mathcal{X}$  and the signal space  $\mathcal{W}$  to be Hilbert spaces. Here we still take the state space  $\mathcal{X}$  to be Hilbert space (i.e. at the moment we only consider systems whose ‘internal energy’ is non-negative), but in order for our passive state/signal systems to be extensions of classical passive input/state/output systems we are forced to use an indefinite inner product in the signal space  $\mathcal{W}$ , corresponding to the desired supply rate. Thus, in this article  $\mathcal{W}$  will be a Kreĩn space instead of a Hilbert space. As we mentioned above, in Part I we took both  $\mathcal{X}$  and  $\mathcal{W}$  to be Hilbert spaces. However, we did not make any explicit use of the inner products in  $\mathcal{X}$  and  $\mathcal{W}$ ; the only Hilbert space property that we used was that in a Hilbert space every closed subspace is complemented. The same statement is true in a Kreĩn space, so the theory in Part I applies directly to the present situation where  $\mathcal{X}$  is a Hilbert space and  $\mathcal{W}$  is a Kreĩn spaces (as well as to the even more general case where both  $\mathcal{X}$  and  $\mathcal{W}$  are allowed to be Kreĩn spaces).

After this general discussion, let us now turn to details. The trajectory  $(x(\cdot), w(\cdot))$  of a state/signal system consists of a state sequence  $x(n) \in \mathcal{X}$  and a signal sequence  $w(n) \in \mathcal{W}$ ,  $n \in \mathbb{Z}^+$  that satisfy the system of equations

$$x(n+1) = F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \quad n \in \mathbb{Z}^+ \tag{1}$$

$$x(0) = x_0$$

where  $F$  is a bounded linear operator with closed domain  $\mathcal{D}(F)$  in the product space  $\begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$  and range  $\mathcal{R}(F) \subset \mathcal{X}$ . The domain of  $F$  has the property that for every  $x \in \mathcal{X}$  there is at least one  $w \in \mathcal{W}$  such that  $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{D}(F)$ . This property guarantees that for every  $x_0 \in \mathcal{X}$  there exists at least one trajectory  $(x(\cdot), w(\cdot))$  of the system with initial state  $x(0) = x_0$ . The above properties of  $F$  and  $\mathcal{D}(F)$  are equivalent to properties (i)–(iv) of the graph  $V$  of  $F$  in the product space

$$\mathfrak{R} = \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$$

listed at the beginning of Section 3. An equivalent way of writing (1) is

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+, \quad x(0) = x_0 \quad (2)$$

By a *state/signal node* we mean a colligation  $\Sigma = (V; \mathcal{X}; \mathcal{W})$  satisfying properties (i)–(iv) (so that  $V$  is the graph of an operator  $F$  of the type described above). By a *linear discrete time-invariant state/signal system* we understand a state/signal node together with the set of all trajectories  $(x(\cdot), w(\cdot))$  on  $\mathbb{Z}^+$ , and we use the same notation  $\Sigma = (V; \mathcal{X}; \mathcal{W})$  for both the node and the system.

A state/signal system  $\Sigma := (V; \mathcal{X}; \mathcal{W})$  with a Hilbert state space  $\mathcal{X}$  and a Kreĭn signal space  $\mathcal{W}$  is called *forward passive* (or *forward conservative*) if all trajectories  $(x(\cdot), w(\cdot))$  of  $\Sigma$  satisfy the inequality<sup>||</sup>

$$\|x(n+1)\|_{\mathcal{X}}^2 - \|x(n)\|_{\mathcal{X}}^2 \leq [w(n), w(n)]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+ \quad (3)$$

$$\left( \text{or } \|x(n+1)\|_{\mathcal{X}}^2 - \|x(n)\|_{\mathcal{X}}^2 = [w(n), w(n)]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+, \text{ respectively} \right) \quad (4)$$

It is easy to give an energy interpretation of (3) and (4): at each time  $n$  the final energy  $\|x(n+1)\|_{\mathcal{X}}^2$  is no bigger than (or equal to, respectively) the initial energy  $\|x(n)\|_{\mathcal{X}}^2$  plus the energy which has been absorbed from the surrounding signal space. It is also easy to check that forward passivity (or forward conservativity) is equivalent to the following properties of  $V$ :

$$-\|z\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + [w, w]_{\mathcal{W}} \geq 0, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \quad (5)$$

$$\left( \text{or } -\|z\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + [w, w]_{\mathcal{W}} = 0, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V, \text{ respectively} \right) \quad (6)$$

This makes it natural to introduce an indefinite inner product  $\langle \cdot, \cdot \rangle_{\mathfrak{K}}$  in  $\mathfrak{K}$  by the formula

$$\left[ \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z' \\ x' \\ w' \end{bmatrix} \right]_{\mathfrak{K}} = -(z, z')_{\mathcal{X}} + (x, x')_{\mathcal{X}} + [w, w']_{\mathcal{W}}, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z' \\ x' \\ w' \end{bmatrix} \in \mathfrak{K} \quad (7)$$

With this inner product  $\mathfrak{K} = -\mathcal{X}[+] \mathcal{X}[+] \mathcal{W} = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$  becomes a Kreĭn space. The forward passivity property (5) (or forward conservativity property (6)) mean that  $V$  is a non-negative (or neutral, respectively) subspace of  $\mathfrak{K}$ .

<sup>||</sup>The more general setting of Yakubovich where the internal energy is given by a positive quadratic form instead of the square of the norm in the state space is discussed in Section 9.

Above we have defined what we mean by *forward* passivity or conservativity. The corresponding *backward* notions are defined by means of the *adjoint state/signal system*  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  of  $\Sigma$ . Here  $\mathcal{W}_* = -\mathcal{W}$  (i.e. the same space, but with the inner product  $[\cdot, \cdot]_{\mathcal{W}_*} = -[\cdot, \cdot]_{\mathcal{W}}$ ), and  $V_*$  is a subspace of

$$\mathfrak{R}_* = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W}_* \end{bmatrix}$$

which in a certain sense is the annihilator of  $V$ . This construction is explained in detail in Section 4.

A system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is *backward passive* or *backward conservative* if the adjoint system  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  is forward passive or forward conservative, respectively. Finally,  $\Sigma$  is *passive* or *conservative* if it is both forward and backward passive or conservative, respectively. Equivalently, a system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is *passive* if and only if  $V$  is *maximally non-negative*, and it is *conservative* if and only if  $V$  is *Lagrangian*.

As we can see from the discussion above, in this work we make extensive use of the geometry of a Kreĭn space. For the convenience of the reader we have gathered in Section 2 the basic results on Kreĭn spaces that we need. Section 3 is a short overview of the material in Part I, adapted to the case where the signal space is a Kreĭn space, followed by a more detailed discussion of pseudo-similarity than what is found in Part I. In particular, we recall the three basic types of representations of a state/signal system, namely *driving variable*, *output nulling*, and *input/state/output* representations.

As we mentioned earlier, Section 4 is devoted to duality theory. Here we also introduce the adjoint of a given behaviour.

The notion of passivity and conservativity of state/signal systems that we described briefly above is introduced in Section 5. We shall see that if  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is passive, then every fundamental decomposition  $\mathcal{W} = -\mathcal{W}_- [ \begin{smallmatrix} + \\ - \end{smallmatrix} ] \mathcal{W}_+$  is an admissible input/output decomposition of  $\mathcal{W}$  if we take the input and output space to be the Hilbert spaces  $\mathcal{U} = \mathcal{W}_+$  and  $\mathcal{Y} = \mathcal{W}_-$ , respectively. This means that  $\Sigma$  has a corresponding input/state/output representation  $\Sigma_{i/s/o} = ([ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} ]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-)$ , called a *scattering representation*. The trajectories  $(x(\cdot), u(\cdot), y(\cdot))$  of  $\Sigma_{i/s/o}$  are defined by the system of equations

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n) \\ y(n) &= Cx(n) + Du(n) \\ w(n) &= y(n) + u(n), \quad n \in \mathbb{Z}^+ \\ x(0) &= x_0 \end{aligned} \tag{8}$$

This representation is a linear discrete time-invariant passive scattering system, i.e. the trajectories satisfy

$$\|x(n+1)\|_{\mathcal{X}}^2 + \|y(n)\|_{\mathcal{Y}}^2 \leq \|x(n)\|_{\mathcal{X}}^2 + \|u(n)\|_{\mathcal{U}}^2, \quad n \in \mathbb{Z}^+ \tag{9}$$

where

$$\|u(n)\|_{\mathcal{U}}^2 = (u(n), u(n))_{\mathcal{U}} = [u(n), u(n)]_{\mathcal{U}}$$

$$\|y(n)\|_{\mathcal{Y}}^2 = (y(n), y(n))_{\mathcal{Y}} = -[y(n), y(n)]_{\mathcal{Y}}$$

If, in addition,  $\Sigma$  is forward conservative, then

$$\|x(n+1)\|_{\mathcal{X}}^2 + \|y(n)\|_{\mathcal{Y}}^2 = \|x(n)\|_{\mathcal{X}}^2 + \|u(n)\|_{\mathcal{U}}^2, \quad n \in \mathbb{Z}^+ \quad (10)$$

Clearly, these two conditions correspond to the forward passivity inequality (3) and forward conservativity equality (4). We remark that in the case of an input/state/output system already the forward inequality (9) is sufficient to imply also backward passivity.

As we have seen above, from each passive state/signal system  $\Sigma$  we get infinitely many passive scattering representations of  $\Sigma$ , one for each fundamental decomposition of  $\mathcal{W}$ . The connection between these representations and their transfer functions, or *scattering matrices*, is studied in Section 6. In particular, we prove that two scattering matrices  $\mathfrak{D}$  and  $\mathfrak{D}_1$  which are obtained in this way are connected by a linear fractional transformation of the type

$$\mathfrak{D}_1 = [\Phi_{11} \mathfrak{D} + \Phi_{12}][\Phi_{21} \mathfrak{D} + \Phi_{22}]^{-1} \quad (11)$$

where  $\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}$  is the decomposition of the identity operator on  $\mathcal{W}$  with respect to the two given fundamental decompositions of  $\mathcal{W}$ . The restrictions of these scattering matrices to the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$  belong to the Schur class  $S(\mathbb{D}; \mathcal{U}, \mathcal{Y})$  of holomorphic  $\mathcal{B}(\mathcal{U}; \mathcal{Y})$ -valued contractive functions on  $\mathbb{D}$ .

In Section 7 we prove that every passive state/signal system  $\Sigma$  has an orthogonal conservative dilation which is unique up to unitary similarity under a natural minimality assumption. This dilation need not be simple. If it is, then  $\Sigma$  is said to have *minimal losses*. We also prove that every passive state/signal system has an orthogonal compression which is *minimal* (i.e. it cannot be compressed any further, or equivalently, it is controllable and observable).

In Section 8 we take a look at *passive behaviours* and their realizations by means of a simple conservative, or controllable passive and forward conservative, or observable passive and backward conservative state/signal systems. All of these are unique up to unitary similarity. It is also possible to construct minimal passive realizations, which are unique only up to pseudo-similarity.

Up to now we have only treated the case where the ‘internal energy’ of the system is described by the square of the norm of the state. V. A. Yakubovich and his successors typically allow the internal energy to be a more general quadratic function of the state. We study this case in Section 9 by introducing the class of  $H$ -passive state/signal systems. Here  $H$  is a positive self-adjoint operator in the state space which may be unbounded and may have an unbounded inverse. This is done in such a way that a state/signal system  $\Sigma$  is  $H$ -passive if and only if the adjoint system  $\Sigma_*$  is  $H^{-1}$ -passive. We show that if a state/signal system  $\Sigma$  is  $H$ -passive, then any fundamental decomposition  $\mathcal{W} = -\mathcal{W}_- \begin{bmatrix} + \\ - \end{bmatrix} \mathcal{W}_+$  of its signal space  $\mathcal{W}$  is admissible, i.e. there exists a scattering representation  $\Sigma_{i/s/o} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y}$  of  $\Sigma$  with  $\mathcal{U} = \mathcal{W}_+$  and  $\mathcal{Y} = \mathcal{W}_-$ . Let  $M_{\Sigma}$  be the set of all  $H$  for which  $\Sigma$  is  $H$ -passive. Then, for each scattering representation  $\Sigma_{i/s/o}$ ,  $M_{\Sigma}$  coincides with the set  $M_{\Sigma_{i/s/o}}$  of all generalized positive self-adjoint solution of the

discrete time scattering KYP (*inequality*) for  $\Sigma_{i/s/o}$

$$\begin{bmatrix} A^*HA - H + C^*C & A^*HB + C^*D \\ B^*HA + D^*C & B^*HB + D^*D - I_{\mathcal{U}} \end{bmatrix} \preceq 0 \quad (12)$$

in a sense that will be explained in Section 9.\*\* In particular,  $M_{\Sigma_{i/s/o}}$  is determined uniquely by the state/signal system  $\Sigma$ , and it does not depend on the particular scattering representation. We also prove a similar statement for admissible orthogonal decompositions of the signal space, i.e. for *admissible transmission* representations of  $\Sigma$ . In our next paper the same statement will be proved for admissible input/state/output (impedance) representations of  $\Sigma$  which correspond to decompositions  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  of  $\mathcal{W}$  into two Lagrangean subspaces  $\mathcal{Y}$  and  $\mathcal{U}$ .

We prove that  $H \in M_{\Sigma}$  if and only if  $H^{1/2}$  is a pseudo-similarity between  $\Sigma$  and a passive state/signal system  $\Sigma_H$ . Let  $M_{\Sigma}^{\min}$  be the subset of  $M_{\Sigma}$  for which  $\Sigma_H$  is minimal (i.e. controllable and observable). If  $\Sigma$  is minimal and  $M_{\Sigma}$  is non-empty, then  $M_{\Sigma}^{\min}$  is non-empty and  $M_{\Sigma}^{\min}$  contains a minimal element  $H_{\circ}$  and a maximal element  $H_{\bullet}$  with respect to the standard partial ordering of (possibly unbounded) self-adjoint operators on  $H$ . The operators  $H_{\circ}$  and  $H_{\bullet}$  correspond to Willems' [15, 16] *available storage* and *required supply*, respectively.

The results presented here have natural applications to several subclasses of passive discrete time state/signals systems, such as optimal, balanced, strongly stable, and lossless systems. These applications will be presented elsewhere, together with related results on Darlington representations of passive lossy behaviours. In this connection we shall also discuss stability of the system in the case where  $\Sigma_H$  is minimal. Here the stability of  $x(\cdot)$  is not with respect to the original norm  $\|\cdot\|_{\mathcal{X}}$  in the state space, but with respect to the 'energy' norm defined by the *storage (or Lyapunov) function*

$$E_H(x) = (x, Hx)_{\mathcal{X}} := \|H^{1/2}x\|_{\mathcal{X}}^2, \quad H \in M_{\Sigma}^{\min}$$

The difference is significant since both  $H$  and  $H^{-1}$  may be unbounded (an example where *all* solutions of the continuous time version of the generalized KYP inequality (12) must be unbounded and have an unbounded inverse is given in Reference [2]).

In the next two papers in this series we shall present additional results related to the transmission case where the signal space  $\mathcal{W}$  is decomposed into an orthogonal sum  $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$  which is not fundamental. Even if the state/signal system  $\Sigma$  is passive it need not be true that every such orthogonal decomposition is admissible. To study this case we introduce affine generalizations of the notion of an input/state/output representation and a transfer function. Similar considerations apply to the impedance case, too, where  $\mathcal{W}$  is decomposed into a sum  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ , where both  $\mathcal{Y}$  and  $\mathcal{U}$  are Lagrangean subspaces of  $\mathcal{W}$  (in particular, they are not orthogonal to each other in  $\mathcal{W}$ ).

In the sequel we shall often need to refer to results taken from Reference [3]. As we mentioned earlier, we shall refer to this publication as 'Part I'. When we cite a particular result in

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\*\*There is a rich literature on the finite-dimensional version of this inequality and the corresponding equality with scattering supply rate; see, e.g. References [4–6], and the references mentioned there. This inequality is named after Kalman [7], Popov [8], and Yakubovich [9]. In the seventies the classical results on the KYP inequalities were extended to systems with  $\dim \mathcal{X} = \infty$  by V. A. Yakubovich and his students and collaborators (see References [10–12] and the references listed there). There is now also a rich literature on this subject; see, e.g. the discussion in Reference [13] and the references cited there. The notion of a generalized solution of (12) that we use was introduced and studied in Reference [14].

Reference [3] we shall do this by adding a roman number ‘I’ to the corresponding number appearing in Reference [3]. Thus, for example, Definition I.2.1 stands for Definition 2.1 in Part I, and (I.3.9) stands for formula (3.9) in Part I.

*Notation*

The space of bounded linear operators from one Kreĭn space  $\mathcal{X}$  to another Kreĭn space  $\mathcal{Y}$  is denoted by  $\mathcal{B}(\mathcal{X}; \mathcal{Y})$ , and we abbreviate  $\mathcal{B}(\mathcal{X}; \mathcal{X})$  to  $\mathcal{B}(\mathcal{X})$ . The domain, range, and kernel of a linear operator  $A$  is denoted by  $\mathcal{D}(A)$ ,  $\mathcal{R}(A)$ , and  $\mathcal{N}(A)$ , respectively. The restriction of  $A$  to some subspace  $\mathcal{Z} \subset \mathcal{D}(A)$  is denoted by  $A|_{\mathcal{Z}}$ . The identity operator on  $\mathcal{X}$  is denoted by  $1_{\mathcal{X}}$ . For each  $A \in \mathcal{B}(\mathcal{X})$  we let  $\Lambda_A$  be the set of points  $z \in \mathbb{C}$  for which  $(1_{\mathcal{X}} - zA)$  has a bounded inverse, plus the point at infinity if  $A$  is boundedly invertible. We denote the projection onto a closed subspace  $\mathcal{Y}$  of a space  $\mathcal{X}$  along some complementary subspace  $\mathcal{U}$  by  $P_{\mathcal{Y}}^{\mathcal{U}}$ , and by  $P_{\mathcal{Y}}$  if  $\mathcal{Y}$  is orthogonal to  $\mathcal{U}$ .

$\mathbb{C}$  is the complex plane,  $\mathbb{D}$  is the open unit disk in  $\mathbb{C}$ ,  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$  and  $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ . The sequence space  $\ell^2(\mathbb{Z}^+; \mathcal{U})$  contain those  $\mathcal{U}$ -valued sequences  $u(\cdot)$  on  $\mathbb{Z}^+$  which satisfy  $\sum_{n \in \mathbb{Z}^+} \|u(n)\|^2 < \infty$ .

We denote the ordered product of the two locally convex topological vector spaces  $\mathcal{X}$  and  $\mathcal{Y}$  by  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ . In particular, although  $\mathcal{X}$  and  $\mathcal{Y}$  may be Hilbert spaces (in which case the product topology on  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$  is induced by an inner product), we shall not require that  $\begin{bmatrix} \mathcal{X} \\ 0 \end{bmatrix} \perp \begin{bmatrix} 0 \\ \mathcal{Y} \end{bmatrix}$  in  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ . We identify a vector  $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ 0 \end{bmatrix}$  with  $x \in \mathcal{X}$  and a vector  $\begin{bmatrix} 0 \\ y \end{bmatrix} \in \begin{bmatrix} 0 \\ \mathcal{Y} \end{bmatrix}$  with  $y \in \mathcal{Y}$ . (We also denote the ordered direct sum  $\mathcal{X} \dot{+} \mathcal{Y}$  by  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ .) We denote the inner product in the Hilbert space  $\mathcal{X}$  by  $(\cdot, \cdot)_{\mathcal{X}}$ , the inner product in the Kreĭn space  $\mathcal{W}$  by  $[\cdot, \cdot]_{\mathcal{W}}$ . The set of all vectors that are orthogonal to a set  $\mathcal{G}$  is denoted by  $G^{[\perp]}$  in the case of a Kreĭn space and by  $\mathcal{G}^{\perp}$  in the case of a Hilbert space.

In the sequel the acronym ‘s/s’ stands for ‘state/signal’, and the acronym ‘i/s/o’ for ‘input/state/output’.

## 2. KREĬN SPACES

For the reader’s convenience we collect here various results concerning the geometry of Kreĭn spaces which we shall use in the sequel. For more thorough treatments of Kreĭn spaces we refer to References [17–19].

By a Kreĭn space we mean a linear space  $\mathcal{W}$  endowed with an indefinite inner product  $[\cdot, \cdot]_{\mathcal{W}}$  which is *complete* in the following sense: there are two subspaces  $-\mathcal{W}_-$  and  $\mathcal{W}_+$  of  $\mathcal{W}$  such that the restriction of  $[\cdot, \cdot]_{\mathcal{W}}$  to  $\mathcal{W}_+ \times \mathcal{W}_+$  makes  $\mathcal{W}_+$  a Hilbert space while the restriction of  $-\![\cdot, \cdot]_{\mathcal{W}}$  to  $\mathcal{W}_- \times \mathcal{W}_-$  makes  $\mathcal{W}_-$  a Hilbert space, and  $\mathcal{W} = -\mathcal{W}_- \dot{+} \mathcal{W}_+$  is a  $[\cdot, \cdot]_{\mathcal{W}}$ -orthogonal direct sum decomposition of  $\mathcal{W}$ . In this case the decomposition  $\mathcal{W} = -\mathcal{W}_- \dot{+} \mathcal{W}_+$  is said to form a *fundamental decomposition* for the Kreĭn space  $\mathcal{W}$ . A fundamental decomposition is never unique, except in the trivial situation where  $\mathcal{W}_-$  or  $\mathcal{W}_+$  is the zero space. It is true that  $\text{ind}_- \mathcal{W} := \dim \mathcal{W}_-$  and  $\text{ind}_+ \mathcal{W} := \dim \mathcal{W}_+$  are uniquely determined; in case either one of  $\text{ind}_- \mathcal{W}$  or  $\text{ind}_+ \mathcal{W}$  is finite, then  $\mathcal{W}$  is said to be a *Pontryagin space*. A choice of fundamental decomposition  $\mathcal{W} = -\mathcal{W}_- \dot{+} \mathcal{W}_+$  determines a Hilbert space norm on  $\mathcal{W}$  by

$$\|w_- + w_+\|_{\mathcal{W}_- \oplus \mathcal{W}_+}^2 = -[w_-, w_-]_{\mathcal{W}} + [w_+, w_+]_{\mathcal{W}}, \quad w_- \in \mathcal{W}_-, \quad w_+ \in \mathcal{W}_+ \tag{13}$$



While the norm  $\|\cdot\|_{\mathcal{W}_- \oplus \mathcal{W}_+}$  itself depends on the choice of fundamental decomposition  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  for  $\mathcal{W}$ , all these norms are equivalent and the resulting strong and weak topologies are each independent of the choice of the fundamental decomposition. In particular, the weak topology is the weakest topology with respect to which each of the linear functionals  $w \mapsto [w, w']_{\mathcal{W}}$  is continuous with respect to the (uniquely determined) norm topology on  $\mathcal{W}$ , and every continuous linear functional on  $\mathcal{W}$  is of this type. Any norm on  $\mathcal{W}$  arising in this way from some choice of fundamental decomposition  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  for  $\mathcal{W}$  we shall call an *admissible norm* on  $\mathcal{W}$ , and we shall refer to the corresponding positive inner product on  $\mathcal{W}_- \oplus \mathcal{W}_+$  as an *admissible Hilbert space inner product* on  $\mathcal{W}$ .

For each Kreĭn space  $\mathcal{W}$  we define its *anti-space*  $-\mathcal{W}$  to be algebraically and topologically the same space as  $\mathcal{W}$  but with the new inner product  $[\cdot, \cdot]_{-\mathcal{W}} = -[\cdot, \cdot]_{\mathcal{W}}$ . If  $\mathcal{W}$  is a Hilbert space, then we call  $-\mathcal{W}$  an *anti-Hilbert space*. Observe that a Kreĭn space and its anti-space have the same admissible norms and admissible Hilbert space inner products.

A subspace  $\mathcal{G}$  of a Kreĭn space is said to be *non-negative*, *neutral* or *non-positive* if  $[g, g]_{\mathcal{W}} \geq 0$  for all  $g \in \mathcal{G}$ ,  $[g, g]_{\mathcal{W}} = 0$  for all  $g \in \mathcal{G}$ , or  $[g, g]_{\mathcal{W}} \leq 0$  for all  $g \in \mathcal{G}$ , respectively. Subspaces of these types are called *semi-definite*. In each semi-definite subspace  $\mathcal{G}$  the Cauchy inequality  $|[g, g']_{\mathcal{W}}|^2 \leq [g, g][g', g']_{\mathcal{W}}$  holds for all  $g, g' \in \mathcal{G}$ . In particular, in each neutral subspace we have  $[g, g']_{\mathcal{W}} = 0$  for all  $g, g' \in \mathcal{G}$ . A subspace is *maximal non-negative* (respectively, *maximal non-positive*) if it is non-negative (non-positive) and if it is not properly contained in any other non-negative (non-positive) subspace. If  $[g, g]_{\mathcal{W}} > 0$  for all  $g \in \mathcal{G}$  with  $g \neq 0$ , we say that  $\mathcal{G}$  is *positive*; similarly,  $\mathcal{G}$  is *negative* if  $[g, g]_{\mathcal{W}} < 0$  for all  $g \in \mathcal{G}$  with  $g \neq 0$ . In case that there is a  $\delta > 0$  so that  $[g, g]_{\mathcal{W}} \geq \delta \|g\|^2$  (respectively,  $[g, g]_{\mathcal{W}} \leq -\delta \|g\|^2$ ) for some admissible choice of norm  $\|\cdot\|$  on  $\mathcal{W}$  and all  $g \in \mathcal{G}$ , we shall say that  $\mathcal{G}$  is *uniformly positive* (respectively, *uniformly negative*).

A bounded linear operator  $A$  on a Kreĭn space  $\mathcal{W}$  is called *non-negative* (and we write  $A \geq 0$ ) or *non-positive* ( $A \leq 0$ ) if  $[w, Aw]_{\mathcal{W}} \geq 0$  or  $[w, Aw]_{\mathcal{W}} \leq 0$ , respectively, for all  $w \in \mathcal{W}$ . It is *positive* ( $A > 0$ ) or *negative* ( $A < 0$ ) if  $[w, Aw]_{\mathcal{W}} > 0$  or  $[w, Aw]_{\mathcal{W}} < 0$ , respectively, for all non-zero  $w \in \mathcal{W}$ . By  $A \leq B$ , where both  $A$  and  $B$  are bounded linear operators, we mean that  $A - B \leq 0$ , etc.

The orthogonal companion  $\mathcal{G}^{\perp}$  of an arbitrary subset  $\mathcal{G} \subset \mathcal{W}$  in the Kreĭn space inner product  $[\cdot, \cdot]_{\mathcal{W}}$  is defined as

$$\mathcal{G}^{\perp} = \{w \in \mathcal{W} \mid [w, g]_{\mathcal{W}} = 0 \text{ for all } g \in \mathcal{G}\}$$

If  $\mathcal{W}$  is a Hilbert space, then we write  $\mathcal{G}^{\perp}$  instead of  $\mathcal{G}^{[\perp]}$ . This is always a closed subspace of  $\mathcal{W}$ . Note that, by definition, a subspace  $\mathcal{G}$  is neutral if and only if  $\mathcal{G} \subset \mathcal{G}^{[\perp]}$ . A stronger notion than a neutral subspace is that of a Lagrangean subspace: we say that a subspace  $\mathcal{G} \subset \mathcal{W}$  is *Lagrangean* if  $\mathcal{G} = \mathcal{G}^{[\perp]}$ .

If we fix a fundamental decomposition  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$ , we may view elements of  $\mathcal{W}$  as consisting of column vectors

$$w = \begin{bmatrix} w_- \\ w_+ \end{bmatrix} \in \begin{bmatrix} -\mathcal{W}_- \\ \mathcal{W}_+ \end{bmatrix}$$

where we view  $\mathcal{W}_-$  and  $\mathcal{W}_+$  as Hilbert spaces, and the Kreĩn space inner product on  $\mathcal{W}$  is given by

$$\begin{aligned} \left[ \begin{bmatrix} w_- \\ w_+ \end{bmatrix}, \begin{bmatrix} w'_- \\ w'_+ \end{bmatrix} \right]_{\mathcal{W}} &= \left( \begin{bmatrix} w_- \\ w_+ \end{bmatrix}, \begin{bmatrix} -1_{\mathcal{W}_-} & 0 \\ 0 & 1_{\mathcal{W}_+} \end{bmatrix} \begin{bmatrix} w'_- \\ w'_+ \end{bmatrix} \right)_{\mathcal{W}_- \oplus \mathcal{W}_+} \\ &= -(w_-, w'_-)_{\mathcal{W}_-} + (w_+, w'_+)_{\mathcal{W}_+} \end{aligned} \tag{14}$$

*Lemma 2.1*

Let  $\mathcal{W}$  be a Kreĩn space  $\mathcal{W}$  with the inner product  $[\cdot, \cdot]_{\mathcal{W}}$ , and let  $(\cdot, \cdot)_{\mathcal{H}}$  be an admissible Hilbert space inner product in  $\mathcal{W}$ . Then there exists a unique operator  $J \in \mathcal{B}(\mathcal{W})$  such that

$$[w, w']_{\mathcal{W}} = (w, Jw')_{\mathcal{H}}, \quad w, w' \in \mathcal{W} \tag{15}$$

The operator  $J$  is both unitary and self-adjoint with respect to both the inner products  $[\cdot, \cdot]_{\mathcal{W}}$  and  $(\cdot, \cdot)_{\mathcal{H}}$ .

*Proof*

Let  $J = \begin{bmatrix} -1_{\mathcal{W}_-} & 0 \\ 0 & 1_{\mathcal{W}_+} \end{bmatrix}$  be the operator in (14). Then  $J$  is self-adjoint and unitary both in the Kreĩn space  $\mathcal{W}$  and in the Hilbert space  $\mathcal{W}_- \oplus \mathcal{W}_+$ , and (15) holds. Clearly,  $J$  is determined uniquely by (15).  $\square$

An operator which is both self-adjoint and unitary is usually called a *signature operator*.

Non-negative, neutral, non-positive, and Lagrangean subspaces are characterized as follows by means of an arbitrary fundamental decomposition of  $\mathcal{W}$ .

*Proposition 2.2*

Let  $\mathcal{W}$  be a Kreĩn space represented in the form  $\mathcal{W} = \begin{bmatrix} \mathcal{W}_- \\ \mathcal{W}_+ \end{bmatrix}$  with the Kreĩn space inner product given by (14). Then the following claims are true:

- (1)  $\mathcal{G}$  is non-negative if and only if there is a linear Hilbert space contraction  $K_+ : \mathcal{D}_+ \mapsto \mathcal{W}_-$  from some domain  $\mathcal{D}_+ \subset \mathcal{W}_+$  into  $\mathcal{W}_-$  such that

$$\mathcal{G} = \left[ \begin{array}{c} K_+ \\ 1_{\mathcal{W}_+} \end{array} \right]_{\mathcal{D}_+} = \left\{ \left[ \begin{array}{c} K_+ d_+ \\ d_+ \end{array} \right] \middle| d_+ \in \mathcal{D}_+ \right\} \tag{16}$$

$\mathcal{G}$  is maximal non-negative if and only if, in addition,  $\mathcal{D}_+ = \mathcal{W}_+$ .

- (2)  $\mathcal{G}$  is non-positive if and only if there is a linear contraction  $K_- : \mathcal{D}_- \mapsto \mathcal{W}_+$  from some domain  $\mathcal{D}_- \subset \mathcal{W}_-$  into  $\mathcal{W}_+$  such that

$$\mathcal{G} = \left[ \begin{array}{c} 1_{\mathcal{W}_-} \\ K_- \end{array} \right]_{\mathcal{D}_-} = \left\{ \left[ \begin{array}{c} d_- \\ K_- d_- \end{array} \right] \middle| d_- \in \mathcal{D}_- \right\} \tag{17}$$

$\mathcal{G}$  is maximal non-positive if and only if, in addition,  $\mathcal{D}_- = \mathcal{W}_-$ .

- (3)  $\mathcal{G}$  is neutral if and only if there is an isometry  $U_+$  mapping a subspace  $\mathcal{D}_+$  of  $\mathcal{W}_+$  isometrically onto a subspace  $\mathcal{D}_-$  of  $\mathcal{W}_-$ , or equivalently, an isometry  $U_-$  mapping

$\mathcal{D}_- \subset \mathcal{W}_-$  isometrically onto  $\mathcal{D}_+ \subset \mathcal{W}_+$ , such that

$$\mathcal{G} = \begin{bmatrix} U_+ \\ 1_{\mathcal{W}_+} \end{bmatrix} \mathcal{D}_+ = \begin{bmatrix} 1_{\mathcal{W}_-} \\ U_- \end{bmatrix} \mathcal{D}_- \quad (18)$$

$\mathcal{G}$  is Lagrangean if and only if, in addition,  $\mathcal{D}_+ = \mathcal{W}_+$  and  $\mathcal{D}_- = \mathcal{W}_-$ .

- (4)  $\mathcal{G}$  is maximal non-negative if and only if  $\mathcal{G}$  is closed and  $\mathcal{G}^{[\perp]}$  is maximal non-positive. More precisely, if  $\mathcal{G}$  has representation (16) with  $\mathcal{D}_+ = \mathcal{W}_+$ , then  $\mathcal{G}^{[\perp]}$  has the representation

$$\mathcal{G}^{[\perp]} = \begin{bmatrix} 1_{\mathcal{W}_-} \\ K_+^* \end{bmatrix} \mathcal{W}_- \quad (19)$$

where  $K_+^*$  is computed with respect to the Hilbert space inner product in  $\mathcal{W}_-$  (instead of the anti-Hilbert space inner product in  $-\mathcal{W}_-$  inherited from  $\mathcal{W}$ ).

- (5)  $\mathcal{G}$  is maximal non-negative if and only if  $\mathcal{G}$  is closed and non-negative and  $\mathcal{G}^{[\perp]}$  is non-positive. In particular,  $\mathcal{G}$  is Lagrangean if and only if  $\mathcal{G}$  is both maximal non-negative and maximal non-positive.

#### *Proof*

See the following theorems in Reference [18]: Theorem 11.7 on p. 54, Theorems 4.2 and 4.4 on pp. 105–106, and Lemma 4.5 on p. 106.  $\square$

The fundamental decompositions that we have considered above are a special case of *orthogonal decompositions*  $\mathcal{W} = -\mathcal{Y}[\perp]\mathcal{U}$  of  $\mathcal{W}$ , where  $\mathcal{Y}$  and  $\mathcal{U}$  are orthogonal with respect to  $[\cdot, \cdot]_{\mathcal{W}}$ , and both  $\mathcal{Y}$  and  $\mathcal{U}$  are Krein spaces with the inner products inherited from  $-\mathcal{W}$  and  $\mathcal{W}$ , respectively. Thus, if  $w = y + u$  with  $y \in \mathcal{Y}$  and  $u \in \mathcal{U}$ , then

$$[w, w]_{\mathcal{W}} = [y, y]_{\mathcal{W}} + [u, u]_{\mathcal{W}} = -[y, y]_{\mathcal{Y}} + [u, u]_{\mathcal{U}} \quad (20)$$

This orthogonal decomposition is fundamental if and only if  $\mathcal{Y}$  and  $\mathcal{U}$  are Hilbert spaces, i.e. if they are both non-negative.

### 3. STATE/SIGNAL NODES AND SYSTEMS WITH KREIN SIGNAL SPACES

In this section we recall a number of definitions from Part I. There both the state space  $\mathcal{X}$  and the signal space  $\mathcal{W}$  were taken to be Hilbert spaces. Here we still require  $\mathcal{X}$  to be a Hilbert space, but take  $\mathcal{W}$  to be a Krein space. In Part I we defined the *node space*  $\mathfrak{R}$  to be a Hilbert space, namely the product of two copies of  $\mathcal{X}$  and one copy of  $\mathcal{W}$ . This time we interpret  $\mathfrak{R}$  as a Krein space with the inner product  $\langle \cdot, \cdot \rangle_{\mathfrak{R}}$  given by (7). In other words, we replace the first copy of  $\mathcal{X}$  by the anti-Hilbert space  $-\mathcal{X}$ , and use the indefinite inner product in  $\mathfrak{R}$  which it inherits from its components, so that  $\mathfrak{R} = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ . Note that  $\mathfrak{R}$  *cannot be a Pontryagin space* unless  $\mathcal{X}$  is finite-dimensional. Recall that we in Part I only made marginal use of the assumption that  $\mathcal{X}$ ,  $\mathcal{W}$ , and  $\mathfrak{R}$  were Hilbert spaces, i.e. the only Hilbert space property that we used there was that every closed subspace of a Hilbert space is complemented. Thus, the results of Part I are still valid in the new setting where  $\mathcal{W}$  and  $\mathfrak{R}$  are Krein spaces.

Thus, in the setting of this paper a *s/s* (i.e. state/signal) *node*  $\Sigma := (V; \mathcal{X}, \mathcal{Y})$  is a colligation where the state space  $\mathcal{X}$  is a Hilbert space, the signal space  $\mathcal{W}$  is a Krein space, and  $V$  is a subspace of the node space  $\mathfrak{R}$  with the following four properties:

- (i)  $V$  is closed in  $\mathfrak{R}$ ;
- (ii) For every  $x \in \mathcal{X}$  there is some  $\begin{bmatrix} z \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$  such that  $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$ ;
- (iii) If  $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$ , then  $z = 0$ ;
- (iv) The set  $\left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix} \mid \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \text{ for some } z \in \mathcal{X} \right\}$  is closed in  $\begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ .

By the *s/s system* generated by the *s/s node*  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  we mean this node itself together with the set of all its trajectories, i.e. all sequences of pairs  $(x(\cdot), w(\cdot))$  satisfying  $\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V$  for all  $n \in \mathbb{Z}^+$ . We use the same notation  $\Sigma$  for the system as for the original node.

In Sections 3–5 of Part I we developed the following three different kinds of representations of a *s/s system*.

*Proposition 3.1*

Let  $V$  be a subspace of the node space  $\mathfrak{R}$ . Then the following assertions are equivalent:

- (1)  $V$  has properties (i)–(iv), i.e.  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is a *s/s node*.
- (2)  $V$  has a *driving variable* representation

$$V = \mathcal{R} \left( \begin{bmatrix} A' & B' \\ 1_{\mathcal{X}} & 0 \\ C' & D' \end{bmatrix} \right) \tag{21}$$

for some bounded linear operators  $\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \in \mathcal{B}(\begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix})$  with the additional requirement that  $D'$  is injective and has closed range. Here  $\mathcal{L}$  is an auxiliary Hilbert space, called the *driving variable space*.

- (3)  $V$  has an *output nulling* representation

$$V = \mathcal{N} \left( \begin{bmatrix} -1_{\mathcal{X}} & A'' & B'' \\ 0 & C'' & D'' \end{bmatrix} \right) \tag{22}$$

for some bounded linear operators  $\begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix} \in \mathcal{B}(\begin{bmatrix} \mathcal{X} \\ \mathcal{W} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{K} \end{bmatrix})$  with the additional requirement that  $D''$  is surjective. Here  $\mathcal{K}$  is an auxiliary Hilbert space, called the *error space*.

- (4)  $V$  has an *i/s/o* (i.e. input/state/output) *representation*

$$V = \mathcal{R} \left( \begin{bmatrix} A & B \\ 1_{\mathcal{X}} & 0 \\ C & D \\ 0 & 1_{\mathcal{Y}} \end{bmatrix} \right) = \mathcal{N} \left( \begin{bmatrix} -1_{\mathcal{X}} & A & 0 & B \\ 0 & C & -1_{\mathcal{Y}} & D \end{bmatrix} \right) \tag{23}$$

for some bounded linear operators  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix})$ , where  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  is a direct sum decomposition of  $\mathcal{W}$ . We call  $\mathcal{Y}$  the *output space* and  $\mathcal{U}$  the *input space*.

This follows from Lemmas I.3.1 and I.4.1 and Theorem I.5.1.

A decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  of the signal space is called *admissible* for a s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  if  $\Sigma$  has an i/s/o representation  $\Sigma_{i/s/o}$  with respect to this decomposition. This representation  $\Sigma_{i/s/o}$  is uniquely determined by  $\Sigma$  and by the decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ .

The three different representations of  $V$  in Proposition 3.1 provide us with three different representations of the original s/s node  $\Sigma$  and the corresponding s/s system. In the driving variable representation of  $V$  given in (21) the trajectories of  $\Sigma$  are described by the system of equations

$$\begin{aligned} x(n+1) &= A'x(n) + B'\ell(n) \\ w(n) &= C'x(n) + D'\ell(n), \quad n \in \mathbb{Z}^+ \end{aligned} \quad (24)$$

$$x(0) = x_0$$

where each  $\ell(n) \in \mathcal{L}$ . If we instead use the output nulling representation of  $V$  given in (22), then the trajectories of  $\Sigma$  are described by the system of equations

$$\begin{aligned} x(n+1) &= A''x(n) + B''w(n) \\ 0 &= C''x(n) + D''w(n), \quad n \in \mathbb{Z}^+ \end{aligned} \quad (25)$$

$$x(0) = x_0$$

Finally, in the i/s/o representation of  $V$  given in (23) the trajectories of  $\Sigma$  are described by the system of equations (8).

In Part I we used the following notations: a driving variable representation of  $\Sigma$  was typically denoted by  $\Sigma_{dv/s/s} := \left( \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$ , an output nulling representation by  $\Sigma_{s/s/on} = \left( \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}; \mathcal{X}, \mathcal{W}, \mathcal{K} \right)$ , and an i/s/o representation was denoted by  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ . In the case of a driving variable representation and an output nulling representation these notations still contain a sufficient amount of information so that we can recover the original s/s system from the given information. In the case of an i/s/o representation our earlier notation does not explicitly tell us how to recreate the Krein space inner product in  $\mathcal{W}$  from the subspaces  $\mathcal{U}$  and  $\mathcal{Y}$  alone. In the present part we shall consider only i/s/o representations corresponding to *orthogonal input/output decompositions*  $\mathcal{W} = -\mathcal{Y} \dot{+} \mathcal{U}$  of the signal space, so once we know the Krein space inner products in  $\mathcal{U}$  and  $\mathcal{Y}$  we also know the Krein space inner product in  $\mathcal{W}$ . In the case where the decomposition is *fundamental*, i.e.  $\mathcal{Y}$  and  $\mathcal{U}$  are *Hilbert spaces*, we call the corresponding i/s/o representation  $\Sigma_{i/s/o}$  a *scattering representation* of  $\Sigma$ , and in the more general case where  $\mathcal{Y}$  and  $\mathcal{U}$  are *Krein spaces* we call  $\Sigma_{i/s/o}$  a *transmission representation* of  $\Sigma$ . In the former case, the *transfer function*

$$\mathfrak{D}(z) = D + zC(1_x - zA)^{-1}B, \quad z \in \Lambda_A \quad (26)$$

of  $\Sigma_{i/s/o}$  is called the *scattering matrix* of  $\Sigma_{i/s/o}$ , and in the latter case it is called the *transmission matrix* of  $\Sigma_{i/s/o}$ .

Let us end this section with a short review of the notion of a *causal signal behaviour* introduced in Section I.7. There we restricted ourselves to the case where the signal space is a Hilbert space, but the same construction applies to the case where the signal space is a Kreĭn space, too.

The notion of a signal behaviour is closely related to the notion of an *externally generated trajectory* of a s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$ , i.e. a trajectory  $(x(\cdot), w(\cdot))$  on  $\mathbb{Z}^+$  satisfying  $x(0) = 0$ . By the *(causal signal) behaviour of  $\Sigma$*  (or *induced by  $\Sigma$* , or *realized by  $\Sigma$* ) on the signal space  $\mathcal{W}$  we mean the set of all sequences in  $\mathcal{W}^{\mathbb{Z}^+}$  that are the signal components  $w(\cdot)$  of all externally generated trajectories  $(x(\cdot), w(\cdot))$  of  $\Sigma$ . This set is closed and right-shift invariant in the Fréchet space  $\mathcal{W}^{\mathbb{Z}^+}$ . More generally, by a *behaviour* on  $\mathcal{W}$  we mean an arbitrary closed right-shift invariant subspace of  $\mathcal{W}^{\mathbb{Z}^+}$ . If the behaviour  $\mathfrak{B}$  is induced by a s/s system  $\Sigma$ , then we say that  $\mathfrak{B}$  is *realizable*, and call  $\Sigma$  a *s/s realization of  $\mathfrak{B}$* . Two s/s systems with the same signal space  $\mathcal{W}$  are *externally equivalent* if they have the same signal behaviour. *Externally equivalent s/s systems have the same set of admissible decompositions of the signal space.*

Not every behaviour is realizable. A necessary and sufficient criterion for the realizability of a behaviour is given in Theorem I.7.5. An important role in this theorem is played by the *zero section*

$$\mathfrak{B}(0) = \{w(0) | w \in \mathfrak{B}\} \tag{27}$$

of the behaviour  $\mathfrak{B}$ . This is always a closed subspace of  $\mathcal{W}$ . If  $\mathfrak{B}$  is realizable and  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is a realization of  $\mathfrak{B}$ , then  $\mathfrak{B}(0)$  coincides with the *canonical input space*

$$\mathcal{U}_0 = \left\{ w \in \mathcal{W} \left| \begin{bmatrix} z \\ 0 \\ w \end{bmatrix} \in V \text{ for some } z \in \mathcal{X} \right. \right\} \tag{28}$$

of  $\Sigma$ . By Lemma I.5.7, every decomposition  $\mathcal{W} = \mathcal{Y}_0 \dot{+} \mathcal{U}_0$  (where  $\mathcal{Y}_0$  is an arbitrary complement to  $\mathcal{U}_0$ ) is admissible for  $\Sigma$ .

The behaviours that we shall consider in this part will be *passive*, and they will always be realizable. See Section 8 for details.

We finally review the notions of *minimality* and *pseudo-similarity*.

The *reachable subspace*  $\mathfrak{R}$  of a s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is the closure in  $\mathcal{X}$  of all possible values of the state components  $x(n)$ ,  $n \in \mathbb{Z}^+$ , of all *externally generated trajectories* of  $\Sigma$ , i.e. trajectories satisfying  $x(0) = 0$ . We call  $\Sigma$  *controllable* if  $\mathfrak{R} = \mathcal{X}$ . The *unobservable subspace*  $\mathfrak{U}$  of  $\Sigma$  consists of all initial values  $x(0)$  of all *unobservable trajectories*, i.e. trajectories  $(x(\cdot), w(\cdot))$  where  $w(n) = 0$ ,  $n \in \mathbb{Z}^+$ . We call  $\Sigma$  *observable* if  $\mathfrak{U} = \{0\}$ .

In Part I we also defined what we mean by the minimality of a s/s system  $\Sigma$ , and showed in Theorem I.8.26 that  $\Sigma$  is minimal if and only if  $\Sigma$  is both controllable and observable. (We shall say more about this in Section 7.)

The external equivalence of two minimal s/s systems is related to the notion of the *pseudo-similarity* of these two systems. We call a linear operator  $Q$  acting from the Hilbert space  $\mathcal{X}$  to the Hilbert space  $\mathcal{Y}$  a *pseudo-similarity* if it is closed and injective, its domain  $\mathcal{D}(Q)$  is dense in  $\mathcal{X}$ , and its range  $\mathcal{R}(Q)$  is dense in  $\mathcal{Y}$ .

*Definition 3.2*

Two s/s systems  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  and  $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$  are *pseudo-similar* if there exists a pseudo-similarity  $Q: \mathcal{X} \rightarrow \mathcal{X}_1$  such that the following conditions hold.

If  $(x(\cdot), w(\cdot))$  is a trajectory of  $\Sigma$  with  $x(0) \in \mathcal{D}(Q)$ , then  $x(n) \in \mathcal{D}(Q)$  for all  $n \in \mathbb{Z}^+$  and  $(Qx(\cdot), w(\cdot))$  is a trajectory of  $\Sigma_1$ , and conversely, if  $(x_1(\cdot), w(\cdot))$  is a trajectory of  $\Sigma_1$  with  $x_1(0) \in \mathcal{R}(Q)$ , then  $x_1(n) \in \mathcal{R}(Q)$  for all  $n \in \mathbb{Z}^+$  and  $(Q^{-1}x_1(\cdot), w(\cdot))$  is a trajectory of  $\Sigma$ .

An operator  $Q$  with the above properties is called a *pseudo-similarity between  $\Sigma$  and  $\Sigma_1$* . Clearly, if  $Q$  is a pseudo-similarity between  $\Sigma$  and  $\Sigma_1$ , then  $Q^{-1}$  is a pseudo-similarity between  $\Sigma_1$  and  $\Sigma$ .

*Proposition 3.3*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  and  $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$  be two s/s systems with the same signal space  $\mathcal{W}$ .

- (1) If  $\Sigma$  and  $\Sigma_1$  are pseudo-similar, then they are externally equivalent.
- (2) Conversely, if  $\Sigma$  and  $\Sigma_1$  are minimal and externally equivalent, then they are pseudo-similar.

*Proof*

Part (1) follows directly from Definition 3.2 (take  $x(0) = 0$  and  $x_1(0) = 0$ ). Part (2) is contained in Proposition I.7.11 and Theorem I.8.26.  $\square$

Two i/s/o systems  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  and  $\Sigma_{i/s/o}^1 = \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}; \mathcal{X}_1, \mathcal{U}, \mathcal{Y} \right)$  with the same input and output spaces are called pseudo-similar if there exists a pseudo-similarity  $Q: \mathcal{X} \rightarrow \mathcal{X}_1$  such that  $A\mathcal{D}(Q) \subset \mathcal{D}(Q)$ ,  $\mathcal{R}(B) \subset \mathcal{D}(Q)$ , and

$$\begin{bmatrix} A_1 Q & B_1 \\ C_1 Q & D_1 \end{bmatrix} = \begin{bmatrix} QA & QB \\ C & D \end{bmatrix} \Big|_{\substack{\mathcal{D}(Q) \\ \mathcal{W}}} \quad (29)$$

We shall apply the same similarity notion to driving variable and output nulling representations, too, interpreting them as i/s/o systems (as explained in Remark I.5.4).

*Proposition 3.4*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  and  $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$  be two s/s systems with the same signal space  $\mathcal{W}$ , and let  $Q: \mathcal{X} \rightarrow \mathcal{X}_1$  be a pseudo-similarity with graph

$$\mathcal{G}(Q) = \left\{ \begin{bmatrix} Qx \\ x \end{bmatrix} \Big| x \in \mathcal{D}(Q) \right\} \quad (30)$$

Then the following conditions are equivalent.

- (1)  $\Sigma$  and  $\Sigma_1$  are pseudo-similar with pseudo-similarity operator  $Q$ .
- (2) The following implication holds: If  $\begin{bmatrix} z \\ x \end{bmatrix} \in V$ ,  $\begin{bmatrix} z_1 \\ x_1 \end{bmatrix} \in V_1$ , and  $\begin{bmatrix} x_1 \\ x \end{bmatrix} \in \mathcal{G}(Q)$ , then  $\begin{bmatrix} z_1 \\ z \end{bmatrix} \in \mathcal{G}(Q)$ .
- (3) The systems  $\Sigma$  and  $\Sigma_1$  have the same set of admissible decompositions  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  of  $\mathcal{W}$ , and for every such decomposition the corresponding i/s/o representations  $\Sigma_{i/s/o}$  and  $\Sigma_{i/s/o}^1$  are pseudo-similar with pseudo-similarity operator  $Q$ .

- (4) There exists some decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  of  $\mathcal{W}$  which is admissible both for  $\Sigma$  and for  $\Sigma_1$ , and the corresponding i/s/o representations  $\Sigma_{i/s/o}$  and  $\Sigma_{i/s/o}^1$  are pseudo-similar with pseudo-similarity operator  $Q$ .
- (5)  $\Sigma$  and  $\Sigma_1$  have driving variable representations  $\Sigma_{dv/s/s}$  and  $\Sigma_{dv/s/s}^1$ , respectively, which are pseudo-similar with pseudo-similarity operator  $Q$ .
- (6)  $\Sigma$  and  $\Sigma_1$  have output nulling representations  $\Sigma_{s/s/on}$  and  $\Sigma_{s/s/on}^1$ , respectively, which are pseudo-similar with pseudo-similarity operator  $Q$ .

*Proof*

(1)  $\Leftrightarrow$  (2): It is easy to see that (2)  $\Rightarrow$  (1). Conversely, if (1) holds, then it follows from the fact that every trajectory of  $\Sigma$  or  $\Sigma_1$  on the interval  $[0, 0]$  can be extended to a full trajectory (see assertion 1) of Proposition I.2.2) that (2) holds.

(1)  $\Rightarrow$  (3): If (1) holds, then by Proposition 3.3,  $\Sigma$  and  $\Sigma_1$  are externally equivalent; hence they have the same admissible input/output decompositions of the signal space (see Theorem I.7.7). By using the standard i/s/o representation (8) of the trajectories it is easy to see that the pseudo-similarity of  $\Sigma$  and  $\Sigma_1$  implies that the corresponding i/s/o representations of  $\Sigma$  and  $\Sigma_1$  are pseudo-similar with the same pseudo-similarity operator.

(3)  $\Rightarrow$  (4): This is trivial.

(4)  $\Rightarrow$  (1): This follows easily from the i/s/o representation (8) of the trajectories of a s/s system.

(4)  $\Rightarrow$  (5) and (4)  $\Rightarrow$  (6): The i/s/o representations in (4) can be interpreted as driving variable representations or as output nulling representations as explained in Remark I.5.2, and it is easy to see that they are still pseudo-similar in the new (driving variable or output nulling) sense.

(5)  $\Rightarrow$  (1) and (6)  $\Rightarrow$  (1): The proofs of these implications are essentially the same as the proof of the implication (4)  $\Rightarrow$  (1), with the i/s/o representation replaced by the driving variable representation (24) or the output nulling representation (25). □

#### 4. ADJOINT STATE/SIGNAL NODES AND SYSTEMS

The present setting where the state space  $\mathcal{X}$  is a Hilbert space and the signal space  $\mathcal{W}$  is a Kreĭn space enables us to define the *adjoint* of a s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$ . This is another s/s system  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  which is needed, among others, in our definition of the *passivity* of a s/s system. We want our definition of the adjoint of a s/s system to be consistent with the standard definition of the adjoint of an i/s/o system with Hilbert input, state, and output spaces, and this makes it necessary to introduce a non-standard interpretation of the adjoints of the Kreĭn signal space  $\mathcal{W}$  and the Kreĭn node space  $\mathfrak{R}$ .

Instead of identifying the dual of the signal space  $\mathcal{W}$  with  $\mathcal{W}$  itself we shall identify the dual of  $\mathcal{W}$  with  $\mathcal{W}_* = -\mathcal{W}^{\dagger\dagger}$ . This we do in the following way. Let  $\mathcal{I}$  be the identity operator from  $\mathcal{W}_* = -\mathcal{W}$  to  $\mathcal{W}$ . Then every bounded linear functional on  $\mathcal{W}$  is of the form  $w \mapsto [w, \mathcal{I}w_*]_{\mathcal{W}}$  for some  $w_* \in \mathcal{W}_*$ . This defines a duality pairing

$$\langle w, w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = [w, \mathcal{I}w_*]_{\mathcal{W}} = [\mathcal{I}^*w, w_*]_{\mathcal{W}_*}, \quad w \in \mathcal{W}, \quad w_* \in \mathcal{W}_* \tag{31}$$

---

<sup>††</sup>The reason for this is that we want also the adjoint system to be causal rather than anti-causal.



between  $\mathcal{W}$  and  $\mathcal{W}_*$  so that with respect to this pairing  $\mathcal{W}$  and  $\mathcal{W}_*$  are adjoints of each other. Note that this pairing is anti-unitary in the sense that  $\mathcal{I}^* = -\mathcal{I}^{-1}$  since, for all  $v_*, w_* \in \mathcal{W}_*$ ,

$$[v_*, w_*]_{\mathcal{W}_*} = -[\mathcal{I}v_*, \mathcal{I}w_*]_{\mathcal{W}} = [v_*, -\mathcal{I}^* \mathcal{I}w_*]_{\mathcal{W}_*}$$

In particular,  $\mathcal{I}^{-1} = -\mathcal{I}^*$  is the identity operator from  $\mathcal{W}$  to  $-\mathcal{W}$ . In the sequel we shall keep the notation  $\mathcal{G}^{\perp}$  (introduced in Section 2) for the orthogonal companion of an arbitrary subset  $\mathcal{G} \subset \mathcal{W}$  with respect to  $[\cdot, \cdot]_{\mathcal{W}}$ . We denote the annihilator of  $\mathcal{G}$  in  $\mathcal{W}_*$  with respect to the duality pairing  $\langle \cdot, \cdot \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle}$  by  $\mathcal{G}^{\langle \perp \rangle}$ . Thus,

$$\mathcal{G}^{\langle \perp \rangle} = \{w_* \in \mathcal{W}_* \mid \langle g, w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0 \text{ for all } g \in \mathcal{G}\} = \mathcal{I}^{-1} \mathcal{G}^{\perp}$$

Since we now have two different adjoints of  $\mathcal{W}$ , namely  $\mathcal{W}$  itself and  $\mathcal{W}_*$ , it is possible to compute adjoints of operators defined on  $\mathcal{W}$  or mapping into  $\mathcal{W}$  in two different ways. In the sequel we denote adjoints with respect to the inner product  $[\cdot, \cdot]_{\mathcal{W}}$  by the superscript  $*$ , and we denote adjoints with respect to the duality pairing  $\langle \cdot, \cdot \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle}$  by the superscript  $\dagger$ . We restrict ourselves to the case where the operators in question map a Hilbert space  $\mathcal{X}$  into  $\mathcal{W}$ , or  $\mathcal{W}$  into a Hilbert space  $\mathcal{X}$ , or  $\mathcal{W}$  back into itself, and where the adjoint of  $\mathcal{X}$  is identified with itself. For example, if  $C \in \mathcal{B}(\mathcal{X}; \mathcal{W})$ , then  $C^\dagger \in \mathcal{B}(\mathcal{W}_*; \mathcal{X})$ ,  $C^* \in \mathcal{B}(\mathcal{W}; \mathcal{X})$ , and for all  $x \in \mathcal{X}$  and  $w_* \in \mathcal{W}_*$ ,

$$(x, C^\dagger w_*)_{\mathcal{X}} = \langle Cx, w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = [Cx, \mathcal{I}w_*]_{\mathcal{W}} = (x, C^* \mathcal{I}w_*)_{\mathcal{X}}$$

Thus,  $C^\dagger = C^* \mathcal{I}$ , or equivalently,  $C^* = -C^\dagger \mathcal{I}^*$ . For each operator  $B \in \mathcal{B}(\mathcal{W}; \mathcal{X})$  the analogous computation (valid for all  $x \in \mathcal{X}$  and all  $w_* \in \mathcal{W}_*$ )

$$\langle w, B^\dagger x \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = (Bw, x)_{\mathcal{X}} = [w, B^* x]_{\mathcal{W}} = \langle w, -\mathcal{I}^* B^* x \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle}$$

shows that  $B^\dagger = -\mathcal{I}^* B^*$  and  $B^* = \mathcal{I} B^\dagger$ . Finally, if  $D \in \mathcal{B}(\mathcal{W})$ , then for all  $w \in \mathcal{W}$  and  $w_* \in \mathcal{W}_*$ ,

$$\begin{aligned} \langle w, D^\dagger w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} &= \langle Dw, w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = [Dw, \mathcal{I}w_*]_{\mathcal{W}} = [w, D^* \mathcal{I}w_*]_{\mathcal{W}} \\ &= \langle w, -\mathcal{I}^* D^* \mathcal{I}w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} \end{aligned}$$

so that  $D^\dagger = \mathcal{I}^{-1} D^* \mathcal{I}$  and  $D^* = \mathcal{I} D^\dagger \mathcal{I}^{-1}$  with  $\mathcal{I}^{-1} = -\mathcal{I}^*$ . At this point it is important to observe that if we repeat the same construction twice, where we the second time interchange  $\mathcal{W}$  and  $\mathcal{W}_*$  but still use the same sign  $\dagger$  for all the adjoints, then  $(C^\dagger)^\dagger = -C$  (whereas  $(C^*)^* = C$ ) and  $(B^\dagger)^\dagger = -B$  (whereas  $(B^*)^* = B$ ). However,  $(D^\dagger)^\dagger = D$ .

After this discussion on the anti-dual of the signal space  $\mathcal{W}$  we now return to the full s/s node  $\Sigma = (V; \mathcal{X}, \mathcal{W})$ . We shall define the *adjoint* of  $\Sigma$  to be another s/s system  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  with the same state space  $\mathcal{X}$  and with the anti-space  $\mathcal{W}_*$  of  $\mathcal{W}$  as its signal space. Thus, the node space  $\mathfrak{R}_*$  of the adjoint s/s system will be  $\mathfrak{R}_* = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W}_* \end{bmatrix}$ . As in the case of the signal space  $\mathcal{W}$  we shall identify the dual of the node space  $\mathfrak{R}$  by the adjoint node space  $\mathfrak{R}_*$  as follows (compare this with the discussion above on how we identify the dual of  $\mathcal{W}$  with  $\mathcal{W}_*$ ). Each bounded linear functional on  $\mathfrak{R}$  has the (non-standard) representation

$$\begin{bmatrix} z \\ x \\ w \end{bmatrix} \mapsto \left[ \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} x_* \\ z_* \\ \mathcal{I}w_* \end{bmatrix} \right]_{\mathfrak{R}} = \left[ \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \right]_{\mathfrak{R}}$$

for some unique  $\begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in \mathfrak{R}_*$ , and this defines a duality pairing

$$\begin{aligned} \left\langle \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \right\rangle_{\langle \mathfrak{R}, \mathfrak{R}_* \rangle} &= \left[ \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \right]_{\mathfrak{R}} \\ &= -(z, x_*)_{\mathcal{X}} + (x, z_*)_{\mathcal{X}} + \langle w, w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} \end{aligned} \quad (32)$$

between  $\mathfrak{R}$  and  $\mathfrak{R}_*$ . Observe that like the duality pairing between  $\mathcal{W}$  and  $\mathcal{W}_*$  also this duality pairing is anti-unitary, since for all  $\begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix}, \begin{bmatrix} z'_* \\ x'_* \\ w'_* \end{bmatrix} \in \mathfrak{R}_*$  we have

$$\begin{aligned} &\left[ \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix}, \begin{bmatrix} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} z'_* \\ x'_* \\ w'_* \end{bmatrix} \right]_{\mathfrak{R}} \\ &= \left[ \begin{bmatrix} x_* \\ z_* \\ \mathcal{I}w_* \end{bmatrix}, \begin{bmatrix} x'_* \\ z'_* \\ \mathcal{I}w'_* \end{bmatrix} \right]_{\mathfrak{R}} \\ &= -(x_*, x'_*)_{\mathcal{X}} + (z_*, z'_*)_{\mathcal{X}} + [\mathcal{I}w_*, \mathcal{I}w'_*]_{\mathcal{W}} = - \left[ \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix}, \begin{bmatrix} z'_* \\ x'_* \\ w'_* \end{bmatrix} \right]_{\mathfrak{R}_*} \end{aligned}$$

The generating subspace  $V_*$  of the adjoint system is defined to be the annihilator of  $V$  in  $\mathfrak{R}_*$  with respect to the above duality pairing, i.e.

$$V_* = V^{\langle \perp \rangle} = \left[ \begin{array}{ccc} 0 & 1_{\mathcal{X}} & 0 \\ 1_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & -\mathcal{I}^* \end{array} \right] V^{\langle \perp \rangle} = \{k_* \in \mathfrak{R}_* \mid \langle k, k_* \rangle_{\langle \mathfrak{R}, \mathfrak{R}_* \rangle} = 0 \text{ for all } k \in V\} \quad (33)$$

where  $V^{\langle \perp \rangle}$  stands for the orthogonal companion of  $V$  in  $\mathfrak{R}$ .

*Proposition 4.1*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s node. Then the triple  $\Sigma_* = (V_*, \mathcal{X}, \mathcal{W}_*)$  defined above is also a s/s node.

*Proof*

Definition (33) may be rewritten as

$$-(z, x_*)_{\mathcal{X}} + (x, z_*)_{\mathcal{X}} + (w, w_*)_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0$$

for all  $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$  and all  $\begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in V_*$ . Let  $\Sigma_{\text{dv/s/s}} = \left( \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$  be a driving variable representation of the s/s node  $\Sigma = (V; \mathcal{X}, \mathcal{W})$ . By (21), this is equivalent to

$$-(A'x + B'\ell, x_*)_{\mathcal{X}} + (x, z_*)_{\mathcal{X}} + \langle C'x + D'\ell, w_* \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0$$

for all  $\begin{bmatrix} x \\ \ell \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{L} \end{bmatrix}$  and all  $\begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in V_*$ . Passing to adjoints we get the alternative equivalent relation (for the same set of data)

$$(x, z_* - (A')^*x_* + (C')^\dagger w_*)_{\mathcal{X}} + (\ell, -(B')^*x_* + (D')^\dagger w_*)_{\mathcal{L}} = 0$$

This says that  $\begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in V_*$  if and only if  $\begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in \mathcal{N} \left( \begin{bmatrix} -1_{\mathcal{X}} & (A')^* & -(C')^\dagger \\ 0 & -(B')^* & (D')^\dagger \end{bmatrix} \right)$ . The operator  $(D')^\dagger$  is surjective since  $D'$  is injective and has closed range. By Proposition 3.1,  $V_*$  has properties (i)–(iv), so  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  is a s/s node. At the same time we find that

$$(\Sigma_*)_{\text{s/s/on}} = \left( \begin{bmatrix} (A')^* & -(C')^\dagger \\ -(B')^* & (D')^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{W}_*, \mathcal{L} \right)$$

is an output nulling representation of  $\Sigma_*$ . □

#### Definition 4.2

Let  $\Sigma = (V, \mathcal{X}, \mathcal{W})$  be a s/s node, with a Hilbert state space  $\mathcal{X}$  and a Kreĭn signal space  $\mathcal{W}$ . The s/s node  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  defined above is called the *adjoint* of  $\Sigma$ . The corresponding s/s system is called the adjoint of the s/s system  $\Sigma$ .

#### Proposition 4.3

The adjoint of the adjoint  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  of a s/s node (or system)  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  coincides with the original node (or system)  $\Sigma$ , i.e.  $(\Sigma_*)_* = \Sigma$ .

We leave the straightforward proof to the reader ( $V_*$  is the annihilator of  $V$  in  $\mathfrak{R}_*$  if and only if  $V$  is the annihilator of  $V_*$  in  $\mathfrak{R}$ ).

#### Proposition 4.4

Let  $\mathcal{U}_0$  be the canonical input space of the s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  (defined in (28)), and let  $\mathcal{U}_{0*}$  be the canonical input space of the adjoint system  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$ . Then  $\mathcal{U}_{0*} = \mathcal{U}_0^{\langle \perp \rangle}$ . Consequently, also  $\mathfrak{B}_*(0) = \mathfrak{B}(0)^{\langle \perp \rangle}$ , where  $\mathfrak{B}(0)$  and  $\mathfrak{B}_*(0)$  are the zero sections (see (27)) of the behaviours of  $\Sigma$  and  $\Sigma_*$ , respectively.

*Proof*

Let  $\Sigma_{\text{dv/s/s}} = \left( \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$ . Then, by Proposition I.3.2,  $\mathcal{U}_0 = \mathcal{R}(D')$ . As we saw in the proof of Proposition 4.1)

$$(\Sigma_*)_{\text{s/s/on}} = \left( \begin{bmatrix} (A')^* & -(C')^\dagger \\ -(B')^* & (D')^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{W}_*, \mathcal{L} \right)$$

is an output nulling representation of  $\Sigma_*$ . Therefore, by Proposition I.4.2,  $\mathcal{U}_{0*} = N((D')^\dagger)$ . This implies that  $\mathcal{U}_{0*} = N((D')^\dagger) = \mathcal{R}(D')^{\langle \perp \rangle} = \mathcal{U}_0^{\langle \perp \rangle}$ . The statement about the zero sections follows immediately, since  $\mathfrak{B}(0) = \mathcal{U}_0$  and  $\mathfrak{B}_*(0) = \mathcal{U}_{0*}$ .  $\square$

It is possible to give an alternative characterization of  $\Sigma_*$  in terms of the following relationship between the trajectories of  $\Sigma$  and those of  $\Sigma_*$ .

*Proposition 4.5*

Let  $(x(\cdot), w(\cdot))$  be a trajectory of  $\Sigma$  on  $\mathbb{Z}^+$ , and let  $(x_*(\cdot), w_*(\cdot))$  be a trajectory of  $\Sigma_*$  on  $\mathbb{Z}^+$ . Then, for all  $n \in \mathbb{Z}^+$ ,

$$-(x(n+1), x_*(0))_{\mathcal{X}} + (x(0), x_*(n+1))_{\mathcal{X}} + \sum_{k=0}^n \langle w(k), w_*(n-k) \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0 \quad (34)$$

In particular, if both of these trajectories are externally generated (i.e.  $x(0) = 0$  and  $x_*(0) = 0$ ), then

$$\sum_{k=0}^n \langle w(k), w_*(n-k) \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0, \quad n \in \mathbb{Z}^+ \quad (35)$$

Conversely, if the set of trajectories  $(x(\cdot), w(\cdot))$  and  $(x_*(\cdot), w_*(\cdot))$  of two s/s systems  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  and  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  satisfy (34) for all  $n \in \mathbb{Z}^+$ , then  $\Sigma$  and  $\Sigma_*$  are adjoints of each other.

*Proof*

We begin with the direct statement. By the definition of trajectories of  $\Sigma$  and  $\Sigma_*$ , we have for all  $k, m \in \mathbb{Z}^+$ ,

$$-(x(k+1), x_*(m))_{\mathcal{X}} + (x(k), x_*(m+1))_{\mathcal{X}} + \langle w(k), w_*(m) \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0 \quad (36)$$

Taking  $m = n - k$  and summing over  $k \in [0, n]$  we get (34).

To prove the converse statement it suffices to take  $n = 0$  in (34), obtaining

$$-(x(1), x_*(0))_{\mathcal{X}} + (x(0), x_*(1))_{\mathcal{X}} + \langle w(0), w_*(0) \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0 \quad (37)$$

By assertion 1(c) of Proposition I.3.2, we can take  $\begin{bmatrix} x(1) \\ x(0) \\ w(0) \end{bmatrix}$  to be an arbitrary vector in  $V$  and  $\begin{bmatrix} x_*(1) \\ x_*(0) \\ w_*(0) \end{bmatrix}$  to be an arbitrary vector in  $V_*$  (every such vector is a trajectory along  $V$  or  $V_*$ , respectively, on the time interval  $[0, 0]$ ). This means that  $V_*$  is the annihilator of  $V$ , and hence  $\Sigma_*$  is the adjoint of  $\Sigma$ .  $\square$

*Proposition 4.6*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system with the adjoint  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$ .

- (1) The sequence  $(x(\cdot), w(\cdot))$  is a trajectory of  $\Sigma$  on  $\mathbb{Z}^+$  if and only if (34) holds for all trajectories of  $\Sigma_*$  on  $\mathbb{Z}^+$  and all  $n \in \mathbb{Z}^+$ .
- (2) The sequence  $(x_*(\cdot), w_*(\cdot))$  is a trajectory of  $\Sigma_*$  on  $\mathbb{Z}^+$  if and only if (34) holds for all trajectories of  $\Sigma$  on  $\mathbb{Z}^+$  and all  $n \in \mathbb{Z}^+$ .

*Proof*

We start by noticing that it suffices to prove part (1), since part (2) follows from (1) and Proposition 4.3.

The proof of part (1) is by induction over the length of the interval on which  $(x(\cdot), w(\cdot))$  is a solution. We begin with the one point interval  $[0, 0]$ . We take  $n = 0$  in (34) to get (34), and arguing as in the proof of Proposition 4.5 we find that  $\begin{bmatrix} x(1) \\ x(0) \\ w(0) \end{bmatrix} \in V$ . This shows that  $(x(\cdot), w(\cdot))$  is a

trajectory on the one point interval  $[0, 0]$ .

Next we move on to the general induction step. Assume that  $(x(\cdot), w(\cdot))$  is a trajectory of  $\Sigma$  on  $[0, p]$  for some  $p \in \mathbb{Z}^+$ . We claim that it is then also a trajectory on  $[0, p + 1]$ . It follows from the induction hypothesis that (36) holds for all  $k \in [0, p]$  and all  $m \in \mathbb{Z}^+$ . In particular, we can take  $m = p + 1 - k$  and sum over  $k \in [0, p]$  to get

$$-(x(p+1), x_*(1))_{\mathcal{X}} + (x(0), x_*(p+2))_{\mathcal{X}} + \sum_{k=0}^p \langle w(k), w_*(p+1-k) \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0$$

If we subtract this from (34) with  $n = p + 1$ , then we get

$$-(x(p+2), x_*(0))_{\mathcal{X}} + (x(p+1), x_*(1))_{\mathcal{X}} + \langle w(p+1), w_*(0) \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0$$

This being true for all  $\begin{bmatrix} x_*(1) \\ x_*(0) \\ w_*(0) \end{bmatrix} \in V_*$  we must have  $\begin{bmatrix} x(p+2) \\ x(p+1) \\ w(p+1) \end{bmatrix} \in V$ . This proves that  $(x(\cdot), w(\cdot))$  is a trajectory on the interval  $[0, p + 1]$ .  $\square$

We are now ready to prove the following duality relationship between the unobservable and reachable subspaces:

*Proposition 4.7*

Let  $\Sigma = (V, \mathcal{X}, \mathcal{W})$  be a s/s node, and let  $\Sigma_* = (V_*, \mathcal{X}, \mathcal{W}_*)$  be the adjoint s/s node. Let  $\mathfrak{R}$  and  $\mathfrak{U}$  be the reachable and unobservable subspaces of  $\Sigma$ , respectively, and let  $\mathfrak{R}_*$  and  $\mathfrak{U}_*$  be the reachable and unobservable subspaces of  $\Sigma_*$ . Then  $\mathfrak{R}_* = \mathfrak{U}^\perp$  and  $\mathfrak{U}_* = \mathfrak{R}^\perp$ .

*Proof*

Let  $(x_*(\cdot), 0)$  be a trajectory of  $\Sigma_*$ , and let  $(x(\cdot), w(\cdot))$  be an externally generated trajectory of  $\Sigma$  (i.e.  $x(0) = 0$ ). Then (34) becomes  $(x(n+1), x_*(0))_{\mathcal{X}} = 0$ ,  $n \in \mathbb{Z}^+$ , which implies that  $x_*(0) \in \mathfrak{R}^\perp$ . This shows that  $\mathfrak{U}_* \subset \mathfrak{R}^\perp$ . Conversely, suppose that  $x_* \in \mathfrak{R}^\perp$ . Decompose  $\mathcal{W}_*$  into a direct sum  $\mathcal{W}_* = \mathcal{Y}_* \dot{+} \mathcal{U}_*$  where  $\mathcal{U}_* = \mathfrak{B}_*(0)$  is the zero section of the behaviour of  $\Sigma_*$ , and  $\mathcal{Y}_*$  is an arbitrary direct complement. By Theorem I.7.5, this decomposition is admissible for  $\Sigma_*$ , hence

there exists a unique trajectory  $(x_*(\cdot), w_*(\cdot))$  of  $\Sigma_*$  satisfying  $x_*(0) = x_*$  and  $w_*(n) \in \mathcal{Y}_*$ ,  $n \in \mathbb{Z}^+$ . Once more, let  $(x(\cdot), w(\cdot))$  be an externally generated trajectory of  $\Sigma$ . The fact that  $x_*(0) \in \mathfrak{R}^\perp$  implies that (35) holds. Taking  $n = 0$  we find that  $w_*(0) \in \mathfrak{B}(0)^{\langle \perp \rangle}$  (since  $w(0)$  can be an arbitrary vector in  $\mathfrak{B}(0)$ ). But  $\mathfrak{B}(0)^{\langle \perp \rangle} = \mathfrak{B}_*(0) = \mathcal{U}_*$ , so  $w_*(0) \in \mathcal{U}_*$ . On the other hand,  $w_*(0) \in \mathcal{Y}_*$ , and  $\mathcal{Y}_* \cap \mathcal{U}_* = 0$ . Thus,  $w_*(0) = 0$ . Once we know that  $w_*(0) = 0$  we can repeat the same argument with  $n = 0$  replaced by  $n = 1$  to get  $w_*(1) = 0$ . The same process can be repeated, and by using induction we find that  $w_*(n) = 0$  for all  $n \in \mathbb{Z}^+$ . Thus  $(x_*(\cdot), 0)$  is an unobservable trajectory with  $x_*(0) = x_*$ , and so  $x_* \in \mathfrak{U}_*$ . This proves that  $\mathfrak{U} = \mathfrak{R}^\perp$ . Applying the same argument with  $\Sigma$  and  $\Sigma_*$  interchanged we find that also  $\mathfrak{R}_* = \mathfrak{U}^\perp$ .  $\square$

*Definition 4.8*

A s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  with a Hilbert state space  $\mathcal{X}$  is *simple* if  $\mathfrak{U} \cap \mathfrak{R}^\perp = 0$ , where  $\mathfrak{U}$  is the unobservable subspace and  $\mathfrak{R}$  is the reachable subspace.

Equivalently,  $\Sigma$  is simple if and only if the closed linear span of  $\mathfrak{R}$  and  $\mathfrak{U}^\perp$  is all of  $\mathcal{X}$ .

*Proposition 4.9*

A s/s node  $\Sigma = (V, \mathcal{X}, \mathcal{W})$  is controllable (or observable, or minimal, or simple) if the adjoint system  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  is observable (or controllable, or minimal, or simple, respectively).

*Proof*

This follows from the definitions of controllability and observability, the fact that a system is minimal if and only if it is controllable and observable, and Proposition 4.7.  $\square$

*Proposition 4.10*

Let  $\Sigma = (V, \mathcal{X}, \mathcal{W})$  be a s/s node, with the driving variable representation  $\Sigma_{\text{dv/s/s}} = \left( \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$  and the output nulling representation  $\Sigma_{\text{s/s/on}} = \left( \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}; \mathcal{X}, \mathcal{W}, \mathcal{K} \right)$ . Then  $(\Sigma_*)_{\text{s/s/on}} = \left( \begin{bmatrix} (A')^* & -(C')^\dagger \\ -(B')^* & (D')^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{W}_*, \mathcal{L} \right)$  is an output nulling representation of  $\Sigma_*$  and  $(\Sigma_*)_{\text{dv/s/s}} = \left( \begin{bmatrix} (A'')^* & (C'')^* \\ (B'')^\dagger & (D'')^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{K}, \mathcal{W}_* \right)$  is a driving variable representation of  $\Sigma_*$ .

*Proof*

That  $(\Sigma_*)_{\text{s/s/on}} = \left( \begin{bmatrix} (A')^* & -(C')^\dagger \\ -(B')^* & (D')^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{W}_*, \mathcal{L} \right)$  is an output nulling representation of  $\Sigma_*$  was proved at the end of Proposition 4.1.

To prove that  $(\Sigma_*)_{\text{dv/s/s}} = \left( \begin{bmatrix} (A'')^* & (C'')^* \\ (B'')^\dagger & (D'')^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{K}, \mathcal{W}_* \right)$  is a driving variable representation of  $\Sigma_*$

we argue as follows. We have  $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$  if and only if, for all  $x' \in \mathcal{X}$  and  $e \in \mathcal{K}$ ,

$$(-z + A''x + B''w, x')_{\mathcal{X}} = 0$$

$$(C''x + D''w, e)_{\mathcal{K}} = 0$$

Passing to adjoints we get for the same set of data

$$-(z, x')_{\mathcal{X}} + (x, (A'')^* x')_{\mathcal{X}} + \langle w, (B'')^\dagger x' \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0$$

$$(x, (C'')^* e)_{\mathcal{X}} + \langle w, (D'')^\dagger e \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0$$

which can be rewritten in the equivalent form

$$\left\langle \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} (A'')^* & (C'')^* \\ 1_{\mathcal{X}} & 0 \\ (B'')^\dagger & (D'')^\dagger \end{bmatrix} \begin{bmatrix} x' \\ e \end{bmatrix} \right\rangle_{\langle \mathcal{R}, \mathcal{R}_* \rangle} = 0$$

This characterizes  $V$  as the annihilator of the range of the operator  $\begin{bmatrix} (A'')^* & (C'')^* \\ 1_{\mathcal{X}} & 0 \\ (B'')^\dagger & (D'')^\dagger \end{bmatrix}$ . The range of this operator is closed since its adjoint is surjective. On the other hand,  $V$  is also the annihilator of  $V_*$ , so  $V_*$  must coincide with the range of this operator. The operator  $(D'')^\dagger$  is injective and has closed range since  $D''$  is surjective. Thus,  $\left( \begin{bmatrix} (A'')^* & (C'')^* \\ (B'')^\dagger & (D'')^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{K}, \mathcal{W}_* \right)$  is a driving variable representation of  $\Sigma_*$ .  $\square$

There is a similar relationship between the i/s/o representation of  $V$  given in part (4) of Proposition 3.1 and an analogous i/s/o representation for the adjoint system. The exact formulation is more complicated in the case where the input/output decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  of the signal space  $\mathcal{W}$  is not orthogonal. We postpone the treatment of this case to a later time, and here we present only the orthogonal case, i.e. the case where  $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$  (described at the end of Section 2). The input space  $\mathcal{U}$  and the output space  $\mathcal{Y}$  in this decomposition are Kreĭn spaces which are orthogonal to each other. We identify the duals of  $\mathcal{U}$  and  $\mathcal{Y}$  with themselves, and denote adjoints of operators defined on these subspaces or mapping into these subspaces by the superscript  $*$ .

*Proposition 4.11*

Let  $\Sigma = (V, \mathcal{X}, \mathcal{W})$  be a s/s node, and let  $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$  be a admissible orthogonal decomposition of  $\mathcal{W}$ , with the corresponding transmission representation  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  of  $\Sigma$ . Then  $\mathcal{W}_* = -\mathcal{U}[\dot{+}]\mathcal{Y}$  is an admissible orthogonal decomposition of  $\mathcal{W}_*$  for the adjoint s/s node  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  of  $\Sigma$ , and  $\Sigma_{i/s/o}^* = \left( \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}; \mathcal{X}, \mathcal{Y}, \mathcal{U} \right)$  is a transmission representation of  $\Sigma_*$ .

*Proof*

As described in Remark I.5.2, we interpret  $\Sigma_{i/s/o}$  as a driving variable representation  $\Sigma_{dv/s/s} =$

$\left( \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{W} \right)$  with

$$\begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \left[ \begin{array}{c|c} A & B \\ \hline C & D \\ 0 & 1_{\mathcal{U}} \end{array} \right]$$

According to Proposition 4.10,  $(\Sigma_*)_{s/s/on} = \left( \begin{bmatrix} (A')^* & -(C')^\dagger \\ -(B')^* & (D)^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{W}_*, \mathcal{U} \right)$  is an output nulling representation of  $\Sigma_*$ . Here  $(B')^* = B^*$ ,  $-(C')^\dagger = [C^* \ 0]$ , and  $(D)^\dagger = [-D^* \ 1_{\mathcal{U}}]$ . Thus, the resulting output nulling representation of  $V_*$  is equivalent to the following relationship between

the components of  $\begin{bmatrix} z_* \\ x_* \\ y_* \\ u_* \end{bmatrix} \in V_*$ :

$$z_* = A^*x_* + C^*y_*$$

$$0 = -B^*x_* - D^*y_* + u_*$$

This set of equations can alternatively be interpreted as an i/s/o representation of  $\Sigma_*$ , and it coincides with the representation  $\Sigma_{i/s/o}^*$  given in the statement of the theorem. Thus, in particular,  $\mathcal{W}_* = -\mathcal{U}[\cdot]_{\mathcal{Y}}$  is an admissible orthogonal input/output decomposition of  $\mathcal{W}$ .  $\square$

Our definition of the adjoint of a s/s system is based on a Kreĭn space inner product in the signal space  $\mathcal{W}$ . In the standard approach to duality of i/s/o systems one uses Hilbert space inner products in the input space  $\mathcal{U}$  and output space  $\mathcal{Y}$ , and computes the adjoints with respect to these inner products. In our s/s setting this amounts to using a Hilbert space inner product in  $\mathcal{W}$  (the cross product of the inner products in  $\mathcal{U}$  and  $\mathcal{Y}$ ). The question of how these two approaches are related to each other is answered in the following lemma:

*Lemma 4.12*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system with a Kreĭn signal space with inner product  $[\cdot, \cdot]_{\mathcal{W}}$ , let  $(\cdot, \cdot)_{\mathcal{W}}$  be an admissible Hilbert space inner product on  $\mathcal{W}$ , and let  $J$  be the signature operator defined in Lemma 2.1. Denote the adjoint of  $\Sigma$  with respect to the inner product  $[\cdot, \cdot]_{\mathcal{W}}$  by  $\Sigma_*$ , and denote the adjoint of  $\Sigma$  with respect to the inner product  $(\cdot, \cdot)_{\mathcal{W}}$  by  $\Sigma_*^1$ . Then  $(x_*(\cdot), w_*(\cdot))$  is a trajectory of  $\Sigma_*$  if and only if  $(x_*(\cdot), Jw_*(\cdot))$  is a trajectory of  $\Sigma_*^1$ .

*Proof*

This follows directly from Lemma 2.1 and the definition of the adjoint of a system.  $\square$

*Theorem 4.13*

If the two systems  $\Sigma$  and  $\Sigma^1$  with the same signal space  $\mathcal{W}$  are pseudo-similar with pseudo-similarity operator  $Q$ , then the adjoint systems  $\Sigma_*$  and  $\Sigma_*^1$  are pseudo-similar with pseudo-similarity operator  $(Q^*)^{-1}$ .

*Proof*

Choose some driving variable representations  $\Sigma_{dv/s/s}$  and  $\Sigma_{dv/s/s}^1$  of  $\Sigma$  and  $\Sigma^1$ , respectively, which are pseudo-similar with pseudo-similarity operator  $Q$  (according to Proposition 3.4 this is possible). We interpret these as i/s/o systems with input space equal to the common driving variable space  $\mathcal{L}$  and output space equal to the signal space  $\mathcal{W}$ . By replacing the Kreĭn space inner product  $[\cdot, \cdot]_{\mathcal{W}}$  by an admissible Hilbert space inner product  $(\cdot, \cdot)_{\mathcal{W}}$  we get two pseudo-similar i/s/o systems with Hilbert input and output spaces. For such systems it was proved in Reference [14, Proposition 3.1] that the adjoints of these systems are pseudo-similar with



pseudo-similarity operator  $(Q^*)^{-1}$ . By Lemma 4.12, the same statement is therefore true for the adjoints of  $\Sigma_{\text{dv/s/s}}$  and  $\Sigma_{\text{dv/s/s}}^1$  of  $\Sigma$  computed with respect to the original inner product  $[\cdot, \cdot]_{\mathcal{W}}$  in  $\mathcal{W}$ . By Proposition 4.10 these adjoints are output nulling representations of the adjoint s/s systems  $\Sigma_*$  and  $\Sigma_*^1$ , respectively. By Proposition 3.4,  $\Sigma_*$  and  $\Sigma_*^1$  are pseudo-similar with pseudo-similarity operator  $(Q^*)^{-1}$ .  $\square$

We end this section by introducing *the adjoint of a given behaviour*.

Let  $\mathfrak{B}$  be a behaviour on the Krein signal space  $\mathcal{W}$ . It is easy to see that the set  $\mathfrak{B}_*$  of all sequences  $w_*(\cdot)$  that satisfy condition (35) for all  $w(\cdot) \in \mathfrak{B}$  is a closed right-shift invariant subspace of  $\mathcal{W}_*^{\mathbb{Z}^+}$ , i.e.  $\mathfrak{B}_*$  is a behaviour on the adjoint signal space  $\mathcal{W}_*$ .

*Definition 4.14*

The *adjoint of the behaviour*  $\mathfrak{B}$  on the signal space  $\mathcal{W}$  is the behaviour  $\mathfrak{B}_*$  defined above.

*Theorem 4.15*

A behaviour  $\mathfrak{B}$  is realizable if and only if the adjoint behaviour  $\mathfrak{B}_*$  is realizable. Moreover, a s/s system  $\Sigma$  is a realization of  $\mathfrak{B}$  if and only if the adjoint s/s system  $\Sigma_*$  is a realization of  $\mathfrak{B}_*$ .

*Proof*

Let  $\Sigma$  be a realization of  $\mathfrak{B}$ . It is clear from Proposition 4.5 and Definition 4.14 that the behaviour of the adjoint s/s system  $\Sigma_*$  is contained in  $\mathfrak{B}_*$ . Thus, it suffices to prove the opposite inclusion, i.e. to show that each  $w_*(\cdot) \in \mathfrak{B}_*$  is the signal component of an externally generated trajectory of  $\Sigma_*$ .

Let  $\Sigma_{\text{dv/s/s}} = \left( \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$  be a driving variable representation of  $\Sigma$ . By Proposition 4.10,  $(\Sigma_*)_{\text{s/s/on}} = \left( \begin{bmatrix} (A')^* & -(C')^\dagger \\ -(B')^* & (D')^\dagger \end{bmatrix}; \mathcal{X}, \mathcal{W}_*, \mathcal{L} \right)$  is an output nulling representation of  $\Sigma_*$ , meaning that the externally generated trajectories  $(x_*(\cdot), w_*(\cdot))$  of  $\Sigma_*$  are the solutions of

$$\begin{aligned} x_*(n+1) &= (A')^* x_*(n) - (C')^\dagger w_*(n) \\ 0 &= -(B')^* x_*(n) + (D')^\dagger w_*(n), \quad n \in \mathbb{Z}^+ \end{aligned} \quad (38)$$

$$x_*(0) = 0$$

This set of equations can be iterated to produce the following equivalent set of equations (cf. formula (I.6.7))

$$x_*(0) = 0, \quad x_*(n) = - \sum_{k=0}^{n-1} ((A')^*)^k (C')^\dagger w_*(n-k-1), \quad n \geq 1 \quad (39)$$

$$(D')^\dagger w_*(n) + \sum_{k=0}^{n-1} (B')^* ((A')^*)^k (C')^\dagger w_*(n-k-1) = 0, \quad n \geq 1 \quad (40)$$

$$(D')^\dagger w_*(0) = 0 \tag{41}$$

This set of equations has a solution if and only if  $w_*(\cdot)$  satisfies (40)–(41), in which case (39) defines  $x_*(\cdot)$  as a unique function of  $w_*(\cdot)$ . Thus, it suffices to show that every  $w_*(\cdot) \in \mathfrak{W}_*$  satisfies (40)–(41).

The requirement that  $\Sigma_{\text{dv/s/s}}$  is a driving variable representation of  $\Sigma$  means that the trajectories  $(x(\cdot), w(\cdot))$  of  $\Sigma$  are parameterized by (24), and the requirement that  $w_*(\cdot) \in \mathfrak{W}_*$  means that (35) holds for all such trajectories. In particular, taking  $x(0) = 0$ ,  $\ell(0) = \ell_0$  arbitrary in  $\mathcal{L}$ , and  $\ell(n) = 0$  for  $n \geq 1$  we get the sequence  $w(\cdot) \in \mathfrak{W}$  defined by

$$w(0) = D'\ell_0, \quad w(k) = C'(A')^{k-1}B'\ell_0, \quad k \geq 1$$

Substituting this sequence into (35), and taking into account that  $\ell_0$  can be any vector in  $\mathcal{L}$  we get exactly (40)–(41). Thus, (39)–(41) have a solution  $(x_*(\cdot), w_*(\cdot))$ , meaning that  $w_*(\cdot)$  is an external trajectory of  $\Sigma_*$ . □

*Corollary 4.16*

Two s/s systems  $\Sigma$  and  $\Sigma^1$  with the same signal space  $\mathcal{W}$  are externally equivalent if and only if the adjoint s/s systems  $\Sigma_*$  and  $\Sigma_*^1$  are externally equivalent.

*Proof*

This follows directly from Theorem 4.15. □

### 5. PASSIVE STATE/SIGNAL SYSTEMS

We now arrive at the main theme of this paper, namely the notion of the passivity of a s/s system.

*Definition 5.1*

A s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$ , where  $\mathcal{X}$  is a Hilbert space and  $\mathcal{W}$  is a Kreĭn space, is *forward passive* (or *forward conservative*) if all its trajectories  $(x(\cdot), w(\cdot))$  satisfy inequalities (3) (or equalities (4), respectively).

It is easy to check that a s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is forward passive (or forward conservative) if and only if its generating subspace  $V$  is a non-negative (or neutral, respectively) subspace of the node space  $\mathfrak{R}$ .

*Theorem 5.2*

A s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is forward passive if and only if  $\mathcal{W}$  has an admissible orthogonal decomposition  $\mathcal{W} = -\mathcal{Y} \oplus \mathcal{U}$  where  $\mathcal{U}$  is uniformly positive, such that the operator  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  in the corresponding i/s/o representation  $\Sigma_{\text{i/s/o}} = (\begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$  is a linear contraction from  $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$  to  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ , where the inner products in  $\mathcal{U}$  and  $\mathcal{Y}$  are inherited from  $\mathcal{W}$  and  $-\mathcal{W}$ , respectively (in particular,  $\mathcal{U}$  is a Hilbert space and  $\mathcal{Y}$  is a Kreĭn space). It is forward conservative if and only if the operator  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is isometric.

*Proof*

The sufficiency of the given conditions for forward passivity or conservativity follows immediately from the i/s/o representation of  $V$  and the fact that forward passivity of  $\Sigma$  is equivalent to the non-negativity of  $V$ , whereas forward conservativity of  $\Sigma$  is equivalent to  $V$  being neutral.

Conversely, suppose that  $\Sigma$  is forward passive, i.e. that  $V$  is non-negative. This means that  $-\|z\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + [w, w]_{\mathcal{W}} \geq 0$  for all  $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$ . Taking  $x = 0$  we find from (28) that the canonical input space  $\mathcal{U}_0$  is non-negative in  $\mathcal{W}$ . By part (1) of Proposition 2.2,  $\mathcal{U}_0$  has the representation

$$\mathcal{U}_0 = \left[ \begin{array}{c} D \\ 1_{\mathcal{W}_+} \end{array} \right] \mathcal{U} = \left\{ \left[ \begin{array}{c} Du \\ u \end{array} \right] \middle| u \in \mathcal{U} \right\} \quad (42)$$

for some subspace  $\mathcal{U}$  of  $\mathcal{W}_+$ . The subspace  $\mathcal{U}$  is closed in  $\mathcal{W}_+$  since  $\mathcal{U}_0$  is closed in  $\mathcal{W}$ , and it is uniformly positive since  $\mathcal{W}_+$  is uniformly positive. Define  $\mathcal{Y} = -\mathcal{Y}_-[\dot{+}]\mathcal{Y}_+$ , where  $\mathcal{Y}_-$  is the orthogonal companion to  $\mathcal{U}$  in  $\mathcal{W}_+$  and  $\mathcal{Y}_+ = \mathcal{W}_-$ . This is a fundamental decomposition of  $\mathcal{Y}$ , and  $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$  is an orthogonal decomposition of  $\mathcal{W}$ . According to (42), the orthogonal projection of  $\mathcal{U}_0$  onto  $\mathcal{Y}_-$  is zero, so that with respect to the decomposition  $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$  the space  $\mathcal{U}_0$  has the graph representation

$$\mathcal{U}_0 = \left[ \begin{array}{c} D_+ \\ 0 \\ 1_{\mathcal{W}_+} \end{array} \right] \mathcal{U}$$

By Lemma I.5.7,  $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$  is an admissible input/output decomposition of  $\mathcal{W}$ . That the operator  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  in the corresponding i/s/o representation  $\Sigma_{i/s/o}$  is a contraction from the Hilbert space  $\begin{bmatrix} x \\ w \end{bmatrix}$  to the Kreĭn space  $\begin{bmatrix} x \\ y \end{bmatrix}$  follows directly from the fact that  $V$  is non-negative. Moreover,  $V$  is neutral if and only if  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is isometric.  $\square$

*Remark 5.3*

As the above proof shows, we can add the following conclusion to Theorem 5.2: The output space  $\mathcal{Y}$  has a fundamental decomposition  $\mathcal{Y} = -\mathcal{Y}_-[\dot{+}]\mathcal{Y}_+$  such that the decomposition of  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  with respect to this decomposition of  $\mathcal{Y}$  has the form

$$\left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] = \left[ \begin{array}{c|c} A & B \\ \hline C_- & 0 \\ \hline C_+ & D_+ \end{array} \right]$$

The property of forward passivity of a s/s system is not closed under duality: even if  $\Sigma$  is forward passive (or forward conservative), it need not be true that the adjoint system  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  is forward passive (or forward conservative). Indeed, forward passivity of  $\Sigma_*$  means that  $V_*$  is a non-negative subspace of  $\mathfrak{K}_*$ . This is true if and only if  $V^{[1]}$  is a non-positive subspace of  $\mathfrak{K}$ .

*Definition 5.4*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system.

- (1)  $\Sigma$  is *backward passive* (or *backward conservative*) if the adjoint system  $\Sigma_*$  is forward passive (or forward conservative, respectively).
- (2)  $\Sigma$  is *passive* (or *conservative*) if it is both forward and backward passive (or forward and backward conservative, respectively).

*Example 5.5*

It easy to give an example of a s/s system which is forward passive, even forward conservative, but not passive. We take  $\mathcal{X} = \{0\}$ ,  $\mathcal{U} = \mathbb{C}$ , and  $\mathcal{Y} = \mathbb{C}^2$ , and define the input/output map by

$$\begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(n), \quad n \in \mathbb{Z}^+$$

This system is forward conservative with respect to the inner product in  $\mathcal{W} = \mathbb{C}^3$  induced by the quadratic form

$$[w, w]_{\mathbb{C}^3} = -|y_1|^2 + \alpha|y_2|^2 + |u|^2$$

where  $\alpha$  is an arbitrary non-zero real number. The system is backward passive if and only if  $\alpha < 0$ , so by taking, e.g.  $\alpha = 1$  we have found an example which is forward conservative but not passive.

*Theorem 5.6*

A s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is passive (or conservative) if and only if  $V$  is a maximal non-negative (or Lagrangean, respectively) subspace of the node space  $\mathfrak{R}$ .

*Proof*

By assertion (5) in Proposition 2.2,  $V$  is a maximal non-negative subspace of  $\mathfrak{R}$  if and only if  $V$  is non-negative and  $V^{\perp}$  is non-positive in  $\mathfrak{R}$ . It is not difficult to see that  $V^{\perp}$  is non-positive in  $\mathfrak{R}$  if and only if  $V_*$  is non-negative in  $\mathfrak{R}_*$ . Similarly,  $V$  is Lagrangean if and only if both  $V$  and  $V^{\perp}$  are neutral in  $\mathfrak{R}$ , and this is equivalent to the statement that  $V$  is neutral in  $\mathfrak{R}$  and  $V_*$  is neutral in  $\mathfrak{R}_*$ . □

*Theorem 5.7*

Let the s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be forward passive, i.e. suppose that all its trajectories satisfy (3). Then the following conditions are equivalent:

- (1)  $\Sigma$  is passive;
- (2) At least one fundamental decomposition  $\mathcal{W} = -\mathcal{W}_- [ + ] \mathcal{W}_+$  of  $\mathcal{W}$  is admissible for  $\Sigma$ .
- (3) The canonical input space

$$\mathcal{U}_0 = \left\{ w \in \mathcal{W} \left| \begin{bmatrix} z \\ 0 \\ w \end{bmatrix} \in V \text{ for some } z \in \mathcal{X} \right. \right\} \tag{43}$$

is a maximal non-negative subspace of  $\mathcal{W}$ .

When these conditions hold, then every fundamental decomposition  $\mathcal{W} = -\mathcal{W}_- [ + ] \mathcal{W}_+$  of  $\mathcal{W}$  is admissible for  $\Sigma$ .

*Proof*

Proof of (1)  $\Rightarrow$  (2): Suppose that (1) holds, and let  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  be a fundamental decomposition of  $\mathcal{W}$ . Then  $\mathfrak{R}$  has the fundamental decomposition

$$\mathfrak{R} = -\mathfrak{R}_-[\dot{+}]\mathfrak{R}_+, \quad \text{where } \mathfrak{R}_- = \begin{bmatrix} \mathcal{X} \\ 0 \\ \mathcal{W}_- \end{bmatrix} \quad \text{and} \quad \mathfrak{R}_+ = \begin{bmatrix} 0 \\ \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix} \quad (44)$$

By Theorem 5.6,  $V$  is maximal non-negative, and by part (1) of Proposition 2.2,  $V$  has a representation

$$V = \begin{bmatrix} K_+ \\ 1_{\mathfrak{R}_+} \end{bmatrix} \mathfrak{R}_+ \quad (45)$$

where  $\mathfrak{R}_+$  is defined as in (44) and  $K_+$  is a linear contraction from  $\mathcal{D}_+$  to  $\mathfrak{R}_-$ . This operator has a four block decomposition  $K_+ = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}\left(\begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix}, \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_- \end{bmatrix}\right)$ , and we may rewrite (45) in the form

$$V = \begin{bmatrix} A & B \\ 1_{\mathcal{X}} & 0 \\ C & D \\ 0 & 1_{\mathcal{W}_+} \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix} = \left\{ \begin{bmatrix} Ax + Bw_+ \\ x \\ Cx + Dw_+ + w_+ \end{bmatrix} \middle| x \in \mathcal{X}, w_+ \in \mathcal{W}_+ \right\} \quad (46)$$

But this is an i/s/o representation of  $\Sigma$  of the type (23) with input space  $\mathcal{W}_+$  and output space  $\mathcal{W}_-$ , and thus the decomposition  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  is admissible for  $\Sigma$ . This proves (2), and at the same time it proves the final statement that *every* fundamental decomposition is admissible.

Proof of (2)  $\Rightarrow$  (1): If (2) holds, then  $V$  has representation (46). The forward passivity of  $\Sigma$  implies that  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is a contraction, and by part (1) of Proposition 2.2,  $V$  is maximal non-negative. Condition (1) now follows from Theorem 5.6.

Proof of (2)  $\Rightarrow$  (3): If (2) holds, then from (46) we get the graph representation

$$\mathcal{U}_0 = \begin{bmatrix} D \\ 1_{\mathcal{U}} \end{bmatrix} \mathcal{W}_+ \quad (47)$$

of  $\mathcal{U}_0$ , where  $D$  is the same operator as in (46). The forward passivity of  $\Sigma$  implies that  $D$  is a contraction. By part (1) of Proposition 2.2, this implies that  $\mathcal{U}_0$  is maximal non-negative.

Proof of (3)  $\Rightarrow$  (2): If  $\mathcal{U}_0$  is maximal non-negative, then  $\mathcal{U}_0$  has a graph representation (47) for some contraction  $D$  from  $\mathcal{W}_+$  to  $\mathcal{W}_-$ . By Lemma I.5.6, this implies that the decomposition  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  is admissible for  $\Sigma$ .  $\square$

We have the following two immediate corollaries.

*Corollary 5.8*

A forward passive s/s system  $\Sigma$  is passive if and only if it has a scattering representation.

This follows from the definition of a scattering representation given in the introduction (see also Section 6).

*Corollary 5.9*

A forward passive s/s system  $\Sigma$  is passive if and only if the zero section  $\mathfrak{B}(0)$  of the behaviour of  $\Sigma$  is maximal non-negative.

This is true because the zero section  $\mathfrak{B}(0)$  of a realizable behaviour  $\mathfrak{B}$  is equal to the canonical input space  $\mathcal{U}_0$  of each of its realizations.

*Corollary 5.10*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s signal system which has an admissible fundamental decomposition  $\mathcal{W} = -\mathcal{Y}[+]\mathcal{U}$  of the signal space. Let  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  be the corresponding scattering representation of  $\Sigma$ . Then the following claims are true:

- (1)  $\Sigma$  is passive if and only if  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is a contraction from  $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$  to  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ .
- (2)  $\Sigma$  is passive and forward conservative if and only if  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is an isometry from  $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$  to  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ .
- (3)  $\Sigma$  is conservative if and only if  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is a unitary operator from  $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$  to  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ .

*Proof*

All of these claims follow from Proposition 2.2, Theorem 5.6 and the graph representation (46) of  $V$  by means of the operator  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ . □

*Remark 5.11*

It follows from Theorem 5.7 that a necessary condition for the passivity of a s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is that the dimension of every admissible input space (including  $\mathcal{U}_0$ ) must be equal to  $\text{ind}_+ \mathcal{W}$ , and the dimension of every admissible output space must be equal to  $\text{ind}_- \mathcal{W}$ . Both of these conditions are violated in Example 5.5 when  $\alpha > 0$ .

A closer inspection of the proof of Theorem 5.7 shows that our assumption in Definition 5.1 that  $(V, \mathcal{X}, \mathcal{W})$  is a s/s node is essentially redundant:

*Proposition 5.12*

Let  $V$  be a maximal non-negative subspace of the Kreĩn space  $\mathfrak{K} = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ , where  $\mathcal{X}$  is a Hilbert space and  $\mathcal{W}$  is a Kreĩn space. Then  $(V, \mathcal{X}, \mathcal{W})$  is a passive s/s node.

*Proof*

We introduce the same fundamental decompositions of  $\mathcal{W}$  and  $\mathfrak{K}$  as in the proof of Theorem 5.7. By part (1) of Proposition 2.2,  $V$  has a graph representation of the type (46) for some contraction  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}: \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_- \end{bmatrix}$ . But this means that  $V$  has an i/s/o representation, and according to Theorem I.5.1,  $(V; \mathcal{X}, \mathcal{W})$  is a s/s node. By Theorem 5.6,  $\Sigma$  is passive. □

*Corollary 5.13*

Let  $V$  be a subspace of the Kreĩn space  $\mathfrak{K} = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ , where  $\mathcal{X}$  is a Hilbert space and  $\mathcal{W}$  is a Kreĩn space. Then  $(V, \mathcal{X}, \mathcal{Y})$  is a passive and forward conservative (or conservative) s/s node if and only if  $V$  is a maximal non-negative and neutral (or Lagrangean, respectively) subspace of  $\mathfrak{K}$ .

This follows from Proposition 2.2, Theorem 5.7 and Proposition 5.12.

*Lemma 5.14*

Let  $\Sigma = (V, \mathcal{X}, \mathcal{W})$  be a s/s system with driving variable representation  $\Sigma_{\text{dv/s/s}} := \left( \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$  and output nulling representation  $\Sigma_{\text{s/s/on}} := \left( \begin{bmatrix} A'' & B'' \\ C'' & D'' \end{bmatrix}; \mathcal{X}, \mathcal{W}, \mathcal{H} \right)$ .

(1) The following conditions are equivalent:

(a)  $\Sigma$  is forward passive (or forward conservative);

$$(b) \begin{bmatrix} (A')^* & 1_{\mathcal{X}} & (C')^* \\ (B')^* & 0 & (D')^* \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \geq 0 \text{ (or } = 0 \text{)}.^{**}$$

(2) The following conditions are equivalent:

(a)  $\Sigma$  is backward passive (or backward conservative);

$$(b) \begin{bmatrix} A'' & 1_{\mathcal{X}} & B'' \\ C'' & 0 & D'' \end{bmatrix} \begin{bmatrix} -1_{\mathcal{X}} & 0 & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} \begin{bmatrix} (A'')^* & (C'')^* \\ (B'')^* & (D'')^* \end{bmatrix} \geq 0 \text{ (or } = 0 \text{)}.^{**}$$

(3) The following conditions are equivalent:

(a)  $\Sigma$  is passive (or conservative);

(b) Both (1)(b) and (2)(b) hold.

*Proof*

This equivalence follows from (21), (22), Proposition 4.10, and Definitions 5.1 and 5.4.  $\square$

## 6. PASSIVE SCATTERING REPRESENTATIONS AND SCATTERING MATRICES

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a passive s/s system, and let  $\mathcal{W} = -\mathcal{Y}[+]\mathcal{U}$  be a fundamental decomposition of  $\mathcal{W}$ . By Theorem 5.7, this is an admissible input/output decomposition of  $\mathcal{W}$ , and by Corollary 5.10, if we denote the corresponding scattering representation by  $\Sigma_{\text{i/s/o}} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ , then  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B} \left( \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \right)$  is a contraction. This is equivalent to the statement that all its trajectories  $(x(\cdot), u(\cdot), y(\cdot))$  satisfy (9). If  $\Sigma$  is, in addition, forward conservative, then (10) holds.

<sup>\*\*</sup>The left-hand side should be non-negative in the Hilbert space  $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ . In addition to identifying the duals of the Hilbert spaces  $\mathcal{X}$  and  $\mathcal{L}$  with themselves we have here also identified the dual of the Krein space  $\mathcal{W}$  with  $\mathcal{W}$  itself (instead of using the anti-dual described at the beginning of Section 4).

<sup>\*\*</sup>The left-hand side should be non-negative in the Hilbert space  $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix}$ .

*Definition 6.1*

An i/s/o system  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ , where  $\mathcal{X}$ ,  $\mathcal{U}$  and  $\mathcal{Y}$  are Hilbert spaces, is called a *passive scattering i/s/o system* if all its trajectories  $(x(\cdot), u(\cdot), y(\cdot))$  satisfy inequalities (9). The input/output transfer function (26) is called its *scattering matrix*. If the trajectories of  $\Sigma_{i/s/o}$  satisfy Equations (10), then we call  $\Sigma_{i/s/o}$  a *forward conservative i/s/o scattering system*, and if both  $\Sigma_{i/s/o}$  and the adjoint i/s/o system  $\Sigma_{i/s/o}^* = \left( \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}; \mathcal{X}, \mathcal{Y}, \mathcal{U} \right)$  are forward conservative, then we call  $\Sigma_{i/s/o}$  a *conservative i/s/o scattering system*.<sup>¶¶</sup>

The above definition makes no reference to backward passivity. This is due to the fact that for a scattering i/s/o system already the forward inequalities (9) are strong enough to imply that the system is passive, i.e. also the adjoint system satisfies the corresponding inequalities, since in the case of Hilbert input and output spaces  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  is a contraction if and only if  $\begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$  is a contraction.

Given any passive (or forward conservative or conservative) i/s/o system  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  we can define a signal space  $\mathcal{W}$  by  $\mathcal{W} = -\mathcal{Y}[+] \mathcal{U}$ , after which we get a passive (or passive and forward conservative or conservative, respectively) s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  for which  $\Sigma_{i/s/o}$  is an passive scattering representation. Conversely, by Corollary 5.10, every scattering representation  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_- \right)$  of a passive s/s system  $\Sigma$  corresponding to some fundamental decomposition  $\mathcal{W} = -\mathcal{W}_- [+] \mathcal{W}_+$  of  $\mathcal{W}$  is a passive scattering i/s/o system. Thus, the *existence of a passive scattering representation is a necessary and sufficient condition for the passivity of a state signal system* (cf. Corollary 5.8). By the *family of scattering matrices* of a s/s system  $\Sigma$  we mean the family of all the scattering matrices of its scattering representations.

The theory of discrete time passive scattering systems has been developed, e.g. in the works [25–32] (continuous time versions are given in, e.g. References [20, 22–24, 33–39]). Among others, the following facts are known (cf. Corollary 5.10):

- (1) An i/s/o system  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ , where  $\mathcal{X}$ ,  $\mathcal{U}$ , and  $\mathcal{Y}$  are Hilbert spaces, is passive (or forward conservative or conservative) scattering system if and only if the operator  $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B} \left( \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} \right)$  is contractive (or isometric or unitary, respectively).
- (2) The scattering matrix  $\mathfrak{D}$  of a passive scattering i/s/o system  $\Sigma_{i/s/o}$  is defined (at least) in the open unit disk  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ , and  $\mathfrak{D}|_{\mathbb{D}}$  belongs to the Schur class  $S(\mathbb{D}; \mathcal{U}, \mathcal{Y})$  of  $\mathcal{B}(\mathcal{U}; \mathcal{Y})$ -valued contractive holomorphic function on  $\mathbb{D}$ .
- (3) Every function in  $S(\mathbb{D}; \mathcal{U}, \mathcal{Y})$  has a representation as the restriction to  $\mathbb{D}$  of the scattering matrix of some passive scattering i/s/o system  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ . This system is called a *passive scattering realization* of the given Schur function.

It is possible to take the realization  $\Sigma_{i/s/o}$  in (3) to be minimal (i.e. controllable and observable), and all such minimal realizations are pseudo-similar to each other [14, 27, 28]. Another alternative is to take  $\Sigma_{i/s/o}$  in (3) to be controllable and forward conservative (or observable and backward conservative), and such a realization is unique up to a unitary similarity transformation in the state space [25, Theorems 2.2.1 and 2.2.2]. A third possibility is to take it to be conservative and simple, and also this realization is unique up to a unitary similarity

<sup>¶¶</sup>References [20–24] use the term ‘energy preserving’ for the systems that we call forward conservative scattering systems.



transformation in the state space [25, Theorem 2.3.1]. In particular, this implies the following result.

*Proposition 6.2*

Let  $\varphi$  belong to the Schur class  $S(\mathbb{D}; \mathcal{U}, \mathcal{Y})$  for some Hilbert spaces  $\mathcal{U}$  and  $\mathcal{Y}$ . Then there exists a simple conservative s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  such that  $\mathcal{W} = -\mathcal{W}[\dot{+}]\mathcal{W}$  is a fundamental decomposition of  $\mathcal{W}$ , and such that the corresponding scattering matrix  $\mathfrak{D}$  of  $\Sigma$  satisfies  $\mathfrak{D}|_{\mathbb{D}} = \varphi$ . This system is determined uniquely by  $\varphi$  up to a unitary similarity transformation in the state space.

The general Theorems I.5.9 and I.6.5 on the relationships between different i/s/o representations of a s/s system  $\Sigma$  and their transfer functions remain valid in the following form:

*Theorem 6.3*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a passive s/s system. Let  $\mathcal{W} = -\mathcal{W}'_-[ \dot{+} ]\mathcal{W}'_+ = -\mathcal{W}'_-[\dot{+}]\mathcal{W}'_+$  be two fundamental decompositions of  $\mathcal{W}$ , and denote the corresponding scattering representations of  $\Sigma$  by  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{W}'_+, \mathcal{W}'_- \right)$ , respectively,  $\Sigma_{i/s/o}^1 = \left( \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}; \mathcal{X}, \mathcal{W}'_+, \mathcal{W}'_- \right)$ . We denote the four block transfer functions by  $\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} (1_x - zA)^{-1} & z(1_x - zA)^{-1}B \\ (1_x - zA)^{-1}C & zC(1_x - zA)^{-1}B + D \end{bmatrix}$  and  $\begin{bmatrix} \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ \mathfrak{C}_1(z) & \mathfrak{D}_1(z) \end{bmatrix}$ , respectively (so that the scattering matrices are  $\mathfrak{D}$  and  $\mathfrak{D}_1$ , respectively). Finally, we define<sup>||</sup>

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{W}'_-|\mathcal{W}'_-} & P_{\mathcal{W}'_-|\mathcal{W}'_+} \\ P_{\mathcal{W}'_+|\mathcal{W}'_-} & P_{\mathcal{W}'_+|\mathcal{W}'_+} \end{bmatrix} \tag{48}$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} \\ \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} = \begin{bmatrix} P_{\mathcal{W}'_-|\mathcal{W}'_-} & P_{\mathcal{W}'_-|\mathcal{W}'_+} \\ P_{\mathcal{W}'_+|\mathcal{W}'_-} & P_{\mathcal{W}'_+|\mathcal{W}'_+} \end{bmatrix} \tag{49}$$

- (1) The operators  $\Theta_{21}D + \Theta_{22}$  and  $\tilde{\Theta}_{11} - D\tilde{\Theta}_{21}$  have bounded inverses.
- (2) The operators  $A_1, B_1, C_1$ , and  $D_1$  are given by

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} A & B \\ \Theta_{11}C & \Theta_{11}D + \Theta_{12} \end{bmatrix} \begin{bmatrix} 1_x & 0 \\ \Theta_{21}C & \Theta_{21}D + \Theta_{22} \end{bmatrix}^{-1} \tag{50}$$

or equivalently,

$$\begin{aligned} A_1 &= A - B(\Theta_{21}D + \Theta_{22})^{-1}\Theta_{21}C \\ B_1 &= B(\Theta_{21}D + \Theta_{22})^{-1} \\ C_1 &= \Theta_{11}C - (\Theta_{11}D + \Theta_{12})(\Theta_{21}D + \Theta_{22})^{-1}\Theta_{21}C \\ D_1 &= (\Theta_{11}D + \Theta_{12})(\Theta_{21}D + \Theta_{22})^{-1} \end{aligned} \tag{51}$$

<sup>||</sup>The projections in (48) are orthogonal with respect to the original Krein space inner product of  $\mathcal{W}$  and also with respect to Hilbert space inner product  $(\cdot, \cdot)_{\mathcal{W}'_- \oplus \mathcal{W}'_+}$ , but not, in general, with respect to the Hilbert space inner product  $(\cdot, \cdot)_{\mathcal{W}'_- \oplus \mathcal{W}'_+}$ . A similar comment applies to the projections in (49).

and they are also given by the formulas

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & -B\tilde{\Theta}_{21} \\ 0 & \tilde{\Theta}_{11} - D\tilde{\Theta}_{21} \end{bmatrix}^{-1} \begin{bmatrix} A & B\tilde{\Theta}_{22} \\ C & -\tilde{\Theta}_{12} + D\tilde{\Theta}_{22} \end{bmatrix} \quad (52)$$

or equivalently,

$$\begin{aligned} A_1 &= A + B\tilde{\Theta}_{21}(\tilde{\Theta}_{11} - D\tilde{\Theta}_{21})^{-1}C \\ B_1 &= B\tilde{\Theta}_{22} + B\tilde{\Theta}_{21}(\tilde{\Theta}_{11} - D\tilde{\Theta}_{21})^{-1}(-\tilde{\Theta}_{12} + D\tilde{\Theta}_{22}) \\ C_1 &= (\tilde{\Theta}_{11} - D\tilde{\Theta}_{21})^{-1}C \\ D_1 &= (\tilde{\Theta}_{11} - D\tilde{\Theta}_{21})^{-1}(-\tilde{\Theta}_{12} + D\tilde{\Theta}_{22}) \end{aligned} \quad (53)$$

(3) For all  $z \in \Lambda_A$  the following conditions are equivalent:

- (a)  $z \in \Lambda_{A_1}$ ,
- (b) The operator  $\Theta_{21}\mathfrak{D}(z) + \Theta_{22}$  has a bounded inverse.
- (c) The operator  $\begin{bmatrix} 1_{\mathcal{X}-zA} & -zB \\ \Theta_{21}C & \Theta_{21}D + \Theta_{22} \end{bmatrix}$  has a bounded inverse.
- (d) The operator  $\tilde{\Theta}_{11} - \mathfrak{D}(z)\tilde{\Theta}_{21}$  has a bounded inverse.
- (e) The operator  $\begin{bmatrix} 1_{\mathcal{X}-zA} & -zB\tilde{\Theta}_{21} \\ -C & \tilde{\Theta}_{11} - D\tilde{\Theta}_{21} \end{bmatrix}$  has a bounded inverse.

(4) For all  $z \in \Lambda_A \cap \Lambda_{A_1}$  (in particular, for all  $z \in \mathbb{D}$ ),

$$\begin{aligned} \begin{bmatrix} \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ \mathfrak{C}_1(z) & \mathfrak{D}_1(z) \end{bmatrix} &= \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \Theta_{11}\mathfrak{C}(z) & \Theta_{11}\mathfrak{D}(z) + \Theta_{12} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{21}\mathfrak{C}(z) & \Theta_{21}\mathfrak{D}(z) + \Theta_{22} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \Theta_{11}C & \Theta_{11}D + \Theta_{12} \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}-zA} & -zB \\ \Theta_{21}C & \Theta_{21}D + \Theta_{22} \end{bmatrix}^{-1} \end{aligned} \quad (54)$$

or equivalently,

$$\begin{aligned} \mathfrak{A}_1(z) &= \mathfrak{A}(z) - \mathfrak{B}(z)(\Theta_{21}\mathfrak{D}(z) + \Theta_{22})^{-1}\Theta_{21}\mathfrak{C}(z) \\ \mathfrak{B}_1(z) &= \mathfrak{B}(z)(\Theta_{21}\mathfrak{D}(z) + \Theta_{22})^{-1} \\ \mathfrak{C}_1(z) &= \Theta_{11}\mathfrak{C}(z) - (\Theta_{11}\mathfrak{D}(z) + \Theta_{12})(\Theta_{21}\mathfrak{D}(z) + \Theta_{22})^{-1}\Theta_{21}\mathfrak{C}(z) \\ \mathfrak{D}_1(z) &= (\Theta_{11}\mathfrak{D}(z) + \Theta_{12})(\Theta_{21}\mathfrak{D}(z) + \Theta_{22})^{-1} \end{aligned} \quad (55)$$

(5) For all  $z \in \Lambda_A \cap \Lambda_{A_1}$  (in particular, for all  $z \in \mathbb{D}$ ),

$$\begin{aligned} \begin{bmatrix} \mathfrak{A}_1(z) & \mathfrak{B}_1(z) \\ \mathfrak{C}_1(z) & \mathfrak{D}_1(z) \end{bmatrix} &= \begin{bmatrix} 1_x & -\mathfrak{B}(z)\tilde{\Theta}_{21} \\ 0 & \tilde{\Theta}_{11} - \mathfrak{D}(z)\tilde{\Theta}_{21} \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z)\tilde{\Theta}_{22} \\ \mathfrak{C}(z) & -\tilde{\Theta}_{12} + \mathfrak{D}(z)\tilde{\Theta}_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1_x - zA & -zB\tilde{\Theta}_{21} \\ -C & \tilde{\Theta}_{11} - D\tilde{\Theta}_{21} \end{bmatrix}^{-1} \begin{bmatrix} 1_x & zB\tilde{\Theta}_{22} \\ 0 & -\tilde{\Theta}_{12} + D\tilde{\Theta}_{22} \end{bmatrix} \end{aligned} \quad (56)$$

or equivalently,

$$\begin{aligned} \mathfrak{A}_1(z) &= \mathfrak{A}(z) + \mathfrak{B}(z)\tilde{\Theta}_{21}(\tilde{\Theta}_{11} - \mathfrak{D}(z)\tilde{\Theta}_{21})^{-1}\mathfrak{C}(z) \\ \mathfrak{B}_1(z) &= \mathfrak{B}(z)\tilde{\Theta}_{22} + \mathfrak{B}(z)\tilde{\Theta}_{21}(\tilde{\Theta}_{11} - \mathfrak{D}(z)\tilde{\Theta}_{21})^{-1}(-\tilde{\Theta}_{12} + \mathfrak{D}(z)\tilde{\Theta}_{22}) \\ \mathfrak{C}_1(z) &= (\tilde{\Theta}_{11} - \mathfrak{D}(z)\tilde{\Theta}_{21})^{-1}\mathfrak{C}(z) \\ \mathfrak{D}_1(z) &= (\tilde{\Theta}_{11} - \mathfrak{D}(z)\tilde{\Theta}_{21})^{-1}(-\tilde{\Theta}_{12} + \mathfrak{D}(z)\tilde{\Theta}_{22}) \end{aligned} \quad (57)$$

*Proof*

Most of this follows from Theorem 5.7 and Theorems I.5.9 and I.6.5 (use Schur complements to verify the additional claims in part (3) and in formulas in (54) and (56)).  $\square$

*Corollary 6.4*

Let  $\Sigma(V; \mathcal{X}, \mathcal{W})$  be a passive s/s system.

- (1) To each fundamental decomposition  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  there corresponds a unique scattering matrix  $\mathfrak{D}$ , and  $\mathfrak{D}|_{\mathbb{D}}$  belongs to the Schur class  $S(\mathbb{D}; \mathcal{W}_+, \mathcal{W}_-)$ .
- (2) The set of all the scattering matrices of  $\Sigma$  can be parameterized in the following way. Let  $\mathfrak{D}$  be a fixed scattering matrix corresponding to some fundamental decomposition  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$ . Then the scattering matrix  $\mathfrak{D}_1$  corresponding to an arbitrary fundamental decomposition  $\mathcal{W} = -\mathcal{W}'_-[\dot{+}]\mathcal{W}'_+$  of  $\mathcal{W}$  is given by the linear fractional transformations

$$\begin{aligned} \mathfrak{D}_1(z) &= (\Theta_{11}\mathfrak{D}(z) + \Theta_{12})(\Theta_{21}\mathfrak{D}(z) + \Theta_{22})^{-1} \\ &= (\tilde{\Theta}_{11} - \mathfrak{D}(z)\tilde{\Theta}_{21})^{-1}(-\tilde{\Theta}_{12} + \mathfrak{D}(z)\tilde{\Theta}_{22}) \end{aligned} \quad (58)$$

whose coefficient matrices  $\Theta$  and  $\tilde{\Theta}$  are given by (48) and (49).

This follows directly from Theorem 6.3.

## 7. ORTHOGONAL DILATIONS AND COMPRESSIONS

In Section 8 of Part I we introduced the notions of a dilation and a compression of a s/s systems and studied some of their properties. In the development of a passive s/s systems theory it is necessary to work with *orthogonal* dilations and compressions.

*Definition 7.1*

The s/s system  $\tilde{\Sigma} = (\tilde{V}; \tilde{\mathcal{X}}, \mathcal{W})$  is an *orthogonal dilation* of the s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$ , or equivalently, the s/s system  $\Sigma$  is an *orthogonal compression onto  $\mathcal{X}$*  of the s/s system  $\tilde{\Sigma}$ , if the following conditions hold:

- (1)  $\mathcal{X}$  is a closed subspace of  $\tilde{\mathcal{X}}$ .
- (2) If  $(\tilde{x}(\cdot), w(\cdot))$  is a trajectory of  $\tilde{\Sigma}$  on  $\mathbb{Z}^+$  with  $\tilde{x}(0) \in \mathcal{X}$ , then  $(P_{\mathcal{X}}\tilde{x}(\cdot), w(\cdot))$  is a trajectory of  $\Sigma$  on  $\mathbb{Z}^+$ .
- (3) There is at least one decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  of  $\mathcal{W}$  which is admissible for both  $\tilde{\Sigma}$  and  $\Sigma$ .

In other words,  $\tilde{\Sigma} = (\tilde{V}; \tilde{\mathcal{X}}, \mathcal{W})$  is an orthogonal dilation of  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  if it is a dilation along  $\mathcal{L} = \tilde{\mathcal{X}} \ominus \mathcal{X}$  of  $\Sigma$  in the sense of Definition I.8.1. We recall from Lemma I.8.2 that the two systems  $\tilde{\Sigma}$  and  $\Sigma$  in Definition 7.1 always are externally equivalent. In particular, it is possible to replace condition (3) above by the following condition, which is equivalent to (3) whenever (1) and (2) hold:

- (3') The behaviours of  $\tilde{\Sigma}$  and  $\Sigma$  have the same zero sections.

All results obtained for general (non-orthogonal) dilations and compressions in Section I.8 remain valid in the orthogonal case with the simplification that the space  $\mathcal{L}$  in the direct sum decomposition  $\tilde{\mathcal{X}} = \mathcal{X} \dot{+} \mathcal{L}$  along which the dilations and compressions were taken is now fixed to be  $\mathcal{L} = \tilde{\mathcal{X}} \ominus \mathcal{X}$ . This applies also to orthogonal dilations and compressions of i/s/o systems.

In particular, the i/s/o system  $\tilde{\Sigma}_{i/s/o} = \left( \begin{bmatrix} \tilde{A} & \tilde{C} \\ \tilde{C} & \tilde{D} \end{bmatrix}; \tilde{\mathcal{X}}, \mathcal{U}, \mathcal{Y} \right)$  is an orthogonal dilation of the i/s/o system  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ , or equivalently,  $\Sigma_{i/s/o}$  is an orthogonal compression onto  $\mathcal{X}$  of  $\tilde{\Sigma}_{i/s/o}$ , if  $\mathcal{X}$  is a closed subspace of  $\tilde{\mathcal{X}}$  and the following condition holds: For each  $x_0 \in \mathcal{X}$  and each input sequence  $u(\cdot) \in \mathcal{U}^{\mathbb{Z}^+}$  the corresponding trajectories  $(\tilde{x}(\cdot), u(\cdot), \tilde{y}(\cdot))$  and  $(x(\cdot), u(\cdot), y(\cdot))$  of  $\tilde{\Sigma}_{i/s/o}$ , respectively,  $\Sigma_{i/s/o}$ , with initial state  $\tilde{x}(0) = x(0) = x_0$ , satisfy  $x(\cdot) = P_{\mathcal{X}}\tilde{x}(\cdot)$  and  $\tilde{y}(\cdot) = y(\cdot)$ . These notions play an essential role in passive i/s/o systems theory, as can be seen from, e.g. References [27, 28, 40–43].\*\*\*

For the convenience of the reader, let us repeat the following definition and recall the following theorem from Part I.

*Definition 7.2*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system.

- (1) A closed subspace  $\mathcal{L}$  of  $\mathcal{X}$  is *outgoing invariant* for  $\Sigma$  if to each  $x_0 \in \mathcal{L}$  there is a (unique) trajectory  $(x(\cdot), 0)$  of  $\Sigma$  with  $x(0) = x_0$  satisfying  $x(n) \in \mathcal{L}$  for all  $n \in \mathbb{Z}^+$ .
- (2) A closed subspace  $\mathcal{L}$  of  $\mathcal{X}$  is *strongly invariant* for  $\Sigma$  if every trajectory  $(x(\cdot), w(\cdot))$  of  $\Sigma$  with  $x(0) \in \mathcal{L}$  satisfies  $x(n) \in \mathcal{L}$  for all  $n \in \mathbb{Z}^+$ .

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\*\*\*In passive systems theory only orthogonal dilations are relevant, and the word ‘orthogonal’ is therefore usually not written out explicitly.

*Theorem 7.3*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system. We denote the reachable subspace of  $\Sigma$  by  $\mathfrak{R}$  and define  $\mathfrak{D} = \mathfrak{U}^\perp$ , where  $\mathfrak{U}$  is the unobservable subspace of  $\Sigma$ . Define

$$\mathcal{X}_\circ = \overline{P_\Sigma \mathfrak{R}}, \quad \mathcal{X}_\bullet = \overline{P_\mathfrak{R} \mathfrak{D}}$$

$$\mathfrak{R}_\circ = \begin{bmatrix} -\mathcal{X}_\circ \\ \mathcal{X}_\circ \\ \mathcal{W} \end{bmatrix}, \quad \mathfrak{R}_\bullet = \begin{bmatrix} -\mathcal{X}_\bullet \\ \mathcal{X}_\bullet \\ \mathcal{W} \end{bmatrix} \quad (59)$$

and let  $V_\circ$  and  $V_\bullet$  be the following subspaces of  $\mathfrak{R}_\circ$  and  $\mathfrak{R}_\bullet$ , respectively:

$$V_\circ = \left\{ \left[ \begin{array}{c} P_{\mathcal{X}_\circ} z \\ x \\ w \end{array} \right] \middle| x \in \mathcal{X}_\circ, \left[ \begin{array}{c} z \\ x \\ w \end{array} \right] \in V \right\}$$

$$V_\bullet = \left\{ \left[ \begin{array}{c} P_{\mathcal{X}_\bullet} z \\ x \\ w \end{array} \right] \middle| x \in \mathcal{X}_\bullet, \left[ \begin{array}{c} z \\ x \\ w \end{array} \right] \in V \right\} \quad (60)$$

Then both  $\Sigma_\circ = (V_\circ; \mathcal{X}_\circ, \mathcal{W})$  and  $\Sigma_\bullet = (V_\bullet; \mathcal{X}_\bullet, \mathcal{W})$  are minimal s/s systems which are orthogonal compressions of  $\Sigma$ .

*Proof*

Apply Theorem I.8.18 to the orthogonal setting. □

*Remark 7.4*

In Part I we defined a s/s system to be minimal if it did not have any non-trivial compression, without requiring this compression to be orthogonal, and we also showed that this is equivalent to the system being both controllable and observable. We can now conclude that every *non-minimal s/s system has, in fact, even a non-trivial orthogonal compression which is minimal*: if  $\Sigma$  is not minimal, then at least one of the orthogonal compressions  $\Sigma_\circ$  and  $\Sigma_\bullet$  in Theorem 7.3 is non-trivial, and they are both minimal.

We shall not here rewrite the rest of Section I.8 in an orthogonal setting. Instead, we shall develop that theory further by studying aspects related to adjoint systems and to passive and conservative systems.

*Theorem 7.5*

Let the s/s system  $\tilde{\Sigma} = (\tilde{V}; \tilde{\mathcal{X}}, \mathcal{W})$  be an orthogonal dilation of the s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$ . Then the adjoint  $\tilde{\Sigma}_* = (\tilde{V}_*; \tilde{\mathcal{X}}, \mathcal{W}_*)$  of  $\tilde{\Sigma}$  is an orthogonal dilation of the adjoint  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  of  $\Sigma$ .

*Proof*

Define  $\tilde{\Sigma}_*$  and  $\Sigma_*$  to be the adjoints of  $\tilde{\Sigma}$  and  $\Sigma$ , respectively. Recall from Lemma I.8.2 that  $\tilde{\Sigma}$  and  $\Sigma$  are externally equivalent, so in particular, their behaviours have the same zero section  $\mathfrak{B}(0)$ . By Proposition 4.4, the zero section  $\mathfrak{B}_*(0)$  of the behaviour of the adjoint system is given by  $\mathfrak{B}_*(0) = \mathfrak{B}(0)^{\langle \perp \rangle}$ , and therefore the zero sections of the behaviours  $\mathfrak{B}_*$ , respectively,  $\mathfrak{B}_*$  must coincide.

Let  $(\tilde{x}_*(\cdot), w_*(\cdot))$  be a trajectory of  $\tilde{\Sigma}_*$  with  $\tilde{x}_*(0) \in \mathcal{X}$ . We claim that  $(x_*(\cdot), w_*(\cdot))$  with  $x_*(\cdot) = P_{\mathcal{X}}\tilde{x}_*(\cdot)$  is a trajectory of  $\Sigma_*$ . According to Proposition 4.6, to prove this it suffices to show that (34) holds for all trajectories  $(x(\cdot), w(\cdot))$  of  $\Sigma$  and all  $n \in \mathbb{Z}^+$ . Since  $\tilde{\Sigma}$  is an orthogonal dilation of  $\Sigma$ , to each trajectory  $(x(\cdot), w(\cdot))$  there corresponds a (unique) trajectory  $(\tilde{x}(\cdot), w(\cdot))$  of  $\tilde{\Sigma}$  with  $\tilde{x}(0) = x(0)$  such that  $x(\cdot) = P_{\mathcal{X}}\tilde{x}(\cdot)$ . By Proposition 4.5 (applied to the system  $\tilde{\Sigma}$  and its adjoint), for all  $n \in \mathbb{Z}^+$ ,

$$-(\tilde{x}(n+1), x_*(0))_{\tilde{\mathcal{X}}} + (x(0), \tilde{x}_*(n+1))_{\tilde{\mathcal{X}}} + \sum_{k=0}^n \langle w(k), w_*(n-k) \rangle_{\langle \mathcal{W}, \mathcal{W}_* \rangle} = 0$$

This is equivalent to (34) since  $x(\cdot) = P_{\mathcal{X}}\tilde{x}(\cdot)$  and  $x_*(\cdot) = P_{\mathcal{X}}\tilde{x}_*(\cdot)$ . Thus,  $(x_*(\cdot), w_*(\cdot))$  with  $x_*(\cdot) = P_{\mathcal{X}}\tilde{x}_*(\cdot)$  is, indeed, a trajectory of  $\Sigma_*$  for every trajectory  $(\tilde{x}_*(\cdot), w_*(\cdot))$  of  $\tilde{\Sigma}_*$  with  $\tilde{x}_*(0) \in \mathcal{X}$ .  $\square$

*Proposition 7.6*

Let  $\tilde{\Sigma} = (\tilde{V}; \tilde{\mathcal{X}}, \mathcal{W})$  be an orthogonal dilation of the s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$ . Suppose that  $\mathcal{L}_o \subset \tilde{\mathcal{X}} \ominus \mathcal{X}$  is an outgoing invariant subspace for  $\tilde{\Sigma}$  such that  $\mathcal{L}_o \oplus \mathcal{X}$  is a strongly invariant subspace for  $\Sigma$ .<sup>†††</sup> Define  $\mathcal{L}_{o*} = \tilde{\mathcal{X}} \ominus (\mathcal{X} \oplus \mathcal{L}_o)$ . Then  $\mathcal{L}_{o*}$  is an outgoing invariant subspace for  $\tilde{\Sigma}_*$  such that  $\mathcal{L}_{o*} \oplus \mathcal{X}$  is a strongly invariant subspace for  $\Sigma_*$ .

*Proof*

We begin by proving that  $\mathcal{L}_{o*}$  is outgoing invariant. By Lemma I.8.6,  $\mathcal{L}_o \oplus \mathcal{X} \subset \mathfrak{R}$ , where  $\mathfrak{R}$  is the reachable subspace of  $\tilde{\Sigma}$ . Hence, by Proposition 4.7,  $\mathcal{L}_{o*} \subset \mathfrak{R}^\perp = \mathcal{U}_*$ , where  $\mathcal{U}_*$  is the unobservable subspace of  $\tilde{\Sigma}_*$ . This implies that for every  $\tilde{x}_* \in \mathcal{L}_{o*}$  there exists an unobservable trajectory  $(\tilde{x}_*(\cdot), 0)$  of  $\tilde{\Sigma}_*$  with  $\tilde{x}_*(0) = \tilde{x}_*$ . To show that  $\mathcal{L}_{o*}$  is outgoing invariant we must still show that each such trajectory satisfies  $\tilde{x}_*(n) \in \mathcal{L}_{o*}$  for all  $n \geq 1$ , or equivalently,  $\tilde{x}_*(n) \perp (\mathcal{L}_o \oplus \mathcal{X})$  for all  $n \geq 1$ . Fix an arbitrary  $\tilde{x}_0 \in \mathcal{L}_o \oplus \mathcal{X}$ , and let  $(\tilde{x}(\cdot), w(\cdot))$  be a trajectory of  $\tilde{\Sigma}$  with  $\tilde{x}(0) = \tilde{x}_0$ . Then  $\tilde{x}(n) \in \mathcal{L}_o \oplus \mathcal{X}$ ,  $n \in \mathbb{Z}^+$ , since  $\mathcal{L}_o \oplus \mathcal{X}$  is strongly invariant. Thus, since  $x_*(0) \in \mathcal{L}_{o*}$ , we have  $\tilde{x}(n) \perp x_*(0)$  for every such trajectory  $(\tilde{x}(\cdot), w(\cdot))$  and every  $n \in \mathbb{Z}^+$ . By using Proposition 4.5 (with  $\Sigma$  replaced by  $\tilde{\Sigma}$ ) we then find that

$$(\tilde{x}_0, \tilde{x}_*(n+1))_{\tilde{\mathcal{X}}} = 0, \quad n \in \mathbb{Z}^+$$

This being true for all  $\tilde{x}_0 \in \mathcal{L}_o \oplus \mathcal{X}$ , we must have  $\tilde{x}_*(n) \in \tilde{\mathcal{X}} \ominus (\mathcal{X} \oplus \mathcal{L}_o) = \mathcal{L}_{o*}$  for all  $n \geq 1$ . This proves that  $\mathcal{L}_{o*}$  is outgoing invariant.

The proof of the claim that  $\mathcal{L}_{o*} \oplus \mathcal{X}$  is a strongly invariant subspace for  $\Sigma_*$  is similar to the one above, and we leave it to the reader.  $\square$

<sup>†††</sup>The existence of such a subspace follows from the orthogonal version of Theorem I.8.7.

*Proposition 7.7*

Any orthogonal compression of a passive s/s system is passive.

*Proof*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be an orthogonal compression of the passive s/s system  $\tilde{\Sigma} = (\tilde{V}; \tilde{\mathcal{X}}, \mathcal{W})$ .

Then, for every  $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V$  we know that  $z$  is of the form  $z = P_{\mathcal{X}} \tilde{z}$  for some  $\tilde{z} \in \tilde{\mathcal{X}}$  such that  $\begin{bmatrix} \tilde{z} \\ x \\ w \end{bmatrix} \in \tilde{V}$ .

The non-negativity of  $\tilde{V}$  gives us  $-\|\tilde{z}\|_{\tilde{\mathcal{X}}}^2 + \|x\|_{\mathcal{X}}^2 + [w, w]_{\mathcal{W}} \geq 0$ . As  $\|z\|_{\mathcal{X}} \leq \|\tilde{z}\|_{\tilde{\mathcal{X}}}$  this implies that also  $-\|z\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + [w, w]_{\mathcal{W}} \geq 0$ . Thus,  $V$  is a non-negative subspace of the node space

$\mathfrak{R} = \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ . Since the two systems are externally equivalent they have the same zero section  $\mathcal{U}_0$  defined in (43) (this subspace coincides with the common zero section of the behaviours of the two systems), and this subspace is maximal non-negative since  $\tilde{\Sigma}$  is passive. By Theorem 5.7,  $\Sigma$  is passive.  $\square$

*Corollary 7.8*

Let  $\Sigma$  be a passive s/s system, and let  $\Sigma_{\circ}$  and  $\Sigma_{\bullet}$  be the orthogonal compressions of  $\Sigma$  defined in Theorem 7.3. Then  $\Sigma_{\circ}$  and  $\Sigma_{\bullet}$  are minimal passive s/s systems.

*Proof*

See Theorem 7.3 and Proposition 7.7.  $\square$

*Theorem 7.9*

Every passive s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  has a conservative orthogonal dilation. It is possible to choose this dilation to be minimal in the sense that it does not have any non-trivial orthogonal compression which is still a conservative orthogonal dilation of  $\Sigma$ . Any two such minimal conservative orthogonal dilations of  $\Sigma$  are similar to each other with a unitary similarity operator  $Q$ , which satisfies the extra condition  $Q|_{\mathcal{X}} = 1_{\mathcal{X}}$ .

*Proof*

Let  $\mathcal{W} = -\mathcal{W}_- \dot{+} \mathcal{W}_+$  be a fundamental decomposition of  $\mathcal{W}$ , and let  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_- \right)$  be the corresponding i/s/o representation of  $\Sigma$  (see Theorem 5.7). This representation is scattering passive. By, for example, [40, Proposition 9] or [28, Theorem 2.1],

$\Sigma_{i/s/o}$  has an orthogonal scattering conservative i/s/o dilation  $\tilde{\Sigma}_{i/s/o} = \left( \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}; \tilde{\mathcal{X}}, \mathcal{W}_-, \mathcal{W}_+ \right)$ ,

and it can be chosen to be minimal in the sense that it does not have any non-trivial orthogonal i/s/o compression which is still a conservative orthogonal dilation of  $\Sigma_{i/s/o}$ . Moreover, any two such minimal conservative orthogonal i/s/o dilations of  $\Sigma$  are similar to each other with a unitary similarity operator  $Q$  which satisfies the extra condition  $Q|_{\mathcal{X}} = 1_{\mathcal{X}}$ . This fact combined with Lemma I.8.22 gives us Theorem 7.9.  $\square$

## 8. PASSIVE BEHAVIOURS AND THEIR REALIZATIONS

We begin by defining what we mean by a passive behaviour.

*Definition 8.1*

By a *passive behaviour* we mean a behaviour  $\mathfrak{B}$  on a Kreĭn space  $\mathcal{W}$  with following two properties:

- (1)  $\sum_{n=0}^T [w(n), w(n)] \geq 0$  for all  $w(\cdot) \in \mathfrak{B}$  and all  $T \in \mathbb{Z}^+$ .
- (2) The zero section  $\mathfrak{B}(0) = \{w(0) | w \in \mathfrak{B}\}$  is a maximal non-negative subspace of  $\mathcal{W}$ .

*Proposition 8.2*

The behaviour of a passive system is passive.

*Proof*

This follows from the inequality

$$\|x(n+1)\|_x^2 - \|x(0)\|_x^2 \leq \sum_{k=0}^n [w(k), w(k)]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+ \quad (61)$$

which is an immediate consequence of (3), together with Theorem 5.7, and the fact that the zero section  $\mathfrak{B}(0)$  coincides with the canonical input space  $\mathcal{U}_0$ .  $\square$

A generalization of Proposition 8.2 will be given in Proposition 9.5, and a partial converse to Proposition 8.2 will be given in Proposition 8.11.

*Proposition 8.3*

Let  $\Sigma$  be a forward passive s/s system, and let  $\mathfrak{B}$  be its behaviour. Then

- (1)  $\mathfrak{B}$  satisfies condition (1) in Definition 8.1.
- (2)  $\Sigma$  is passive if and only if  $\mathfrak{B}$  is passive.

*Proof*

This, too, follows from (61), Theorem 5.7, and the fact that the zero section  $\mathfrak{B}(0)$  coincides with the canonical input space  $\mathcal{U}_0$ .  $\square$

Theorem 8.6 below contains another partial converse to Proposition 8.2: every passive behaviour can be realized by a passive s/s system. The proof of this result is based on the following lemma.

*Lemma 8.4*

Let  $\mathfrak{B}$  be a behaviour on  $\mathcal{W}$  satisfying condition (2) in Definition 8.1, and let  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  be a fundamental decomposition of  $\mathcal{W}$ . Then, for every sequence  $w_+(\cdot) \in \mathcal{W}_+^{\mathbb{Z}^+}$  there exists at least one sequence  $w(\cdot) \in \mathfrak{B}$  such that  $w_+(n) = P_{\mathcal{W}_+} w(n)$  for all  $n \in \mathbb{Z}^+$  (that is, the orthogonal projection of  $\mathfrak{B}$  onto  $\mathcal{W}_+^{\mathbb{Z}^+}$  is surjective).

*Proof*

Fix some arbitrary  $w_+(\cdot) \in \mathcal{W}_+^{\mathbb{Z}^+}$ . We shall construct the needed sequence  $w(\cdot)$  recursively.

By assumption (2) and part (1) of Proposition 2.2, there is some  $w(0) \in \mathfrak{B}(0)$  such that  $P_{\mathcal{W}_+} w(0) = w_+(0)$ . The condition  $w(0) \in \mathfrak{B}(0)$  means that  $w(0) = w_0(0)$  for some sequence  $w_0(\cdot) \in \mathcal{W}$ .



Suppose that we have been able to find  $w(0), \dots, w(n)$  such that the sequence  $\{w(k)\}_{k=0}^n$  is the restriction to  $[0, n]$  of some  $w_n(\cdot) \in \mathfrak{B}$ , and such that  $P_{\mathcal{W}_+} w(k) = w_+(k)$  for all  $k \leq n$ . We then choose some  $w(n+1) \in \mathfrak{B}(0)$  such that  $P_{\mathcal{W}_+} w(n+1) = w_+(n+1) - P_{\mathcal{W}_+} w_n(n+1)$ . By the right-shift invariance of  $\mathfrak{B}$ , there is a sequence  $v_{n+1}(\cdot) \in \mathfrak{B}$  such that  $v_{n+1}(k) = 0$  for  $k \leq n$ , and such that  $v_{n+1}(n+1) = w(n+1)$ . Define  $w_{n+1}(\cdot) = w_n(\cdot) + v_{n+1}(\cdot)$ . Then the sequence  $\{w(k)\}_{k=0}^{n+1}$  is the restriction to  $[0, n+1]$  of  $w_{n+1}(\cdot) \in \mathfrak{B}$ , and  $P_{\mathcal{W}_+} w(k) = w_+(k)$  for all  $k \leq n+1$ . By induction we get a sequence  $w(\cdot)$  with the property that  $P_{\mathcal{W}_+} w(k) = w_+(k)$  for all  $k \in \mathbb{Z}^+$ , as well as a sequence  $\{w_n(\cdot)\}_{n=0}^\infty$ , where each  $w_n(\cdot) \in \mathfrak{B}$ , such that  $w(k) = w_n(k)$  for all  $k \leq n$ . Clearly,  $w_n(\cdot) \rightarrow w(\cdot)$  in  $\mathcal{W}^{\mathbb{Z}^+}$  as  $n \rightarrow \infty$ , and since  $\mathfrak{B}$  is closed and each  $w_n \in \mathfrak{B}$ , we must have  $w(\cdot) \in \mathfrak{B}$ .  $\square$

### Proposition 8.5

If the behaviour  $\mathfrak{B}$  of a s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  is passive, then every fundamental decomposition of  $\mathcal{W}$  is admissible for  $\Sigma$ .

### Proof

Let  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  be a fundamental decomposition of  $\mathcal{W}$ , and denote  $\mathcal{U} = \mathcal{W}_+$  and  $\mathcal{Y} = \mathcal{W}_-$ . Let  $u \in \ell^2(\mathbb{Z}^+; \mathcal{U})$ . By Lemma 8.4, there exists some  $y(\cdot) \in \mathcal{Y}^{\mathbb{Z}^+}$  such that the sequence  $w(\cdot) = (u(\cdot), y(\cdot))$  belongs to  $\mathfrak{B}$ . The passivity of  $\mathfrak{B}$  and the fact that  $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$  is a fundamental decomposition of  $\mathcal{W}$  implies that for all  $n \in \mathbb{Z}^+$ ,

$$\sum_{k=0}^n \|y(k)\|_{\mathcal{Y}}^2 \leq \sum_{k=0}^n \|u(k)\|_{\mathcal{U}}^2 \quad (62)$$

By Definition I.7.4, the decomposition  $\mathcal{W} = -\mathcal{Y}[\dot{+}]\mathcal{U}$  is admissible for  $\mathfrak{B}$ , hence by Theorem I.7.5, it is also admissible for  $\Sigma$ .  $\square$

### Theorem 8.6

Let  $\mathfrak{B}$  be a passive behaviour on  $\mathcal{W}$ .

- (1)  $\mathfrak{B}$  has a simple conservative realization, which is unique up to a unitary similarity transformation in the state space.
- (2)  $\mathfrak{B}$  has a controllable passive and forward conservative realization, which is unique up to a unitary similarity transformation in the state space.
- (3)  $\mathfrak{B}$  has an observable passive and backward conservative realization, which is unique up to a unitary similarity transformation in the state space.

### Proof

Let  $\mathcal{W} = -\mathcal{W}_-[\dot{+}]\mathcal{W}_+$  be a fundamental decomposition of  $\mathcal{W}$ , and denote  $\mathcal{U} = \mathcal{W}_+$  and  $\mathcal{Y} = \mathcal{W}_-$ . Let  $u \in \ell^2(\mathbb{Z}^+; \mathcal{U})$ . Then by (62),  $y(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{Y})$ . Inequality (62) together with the fact  $\mathfrak{B}$  is a linear subspace of  $\mathcal{W}^{\mathbb{Z}^+}$  implies that the sequence  $y(\cdot)$  is unique, and that the mapping  $\mathfrak{D}$  from  $u(\cdot)$  to  $y(\cdot)$  is a linear contraction from  $\ell^2(\mathbb{Z}^+; \mathcal{U})$  to  $\ell^2(\mathbb{Z}^+; \mathcal{Y})$ . The right-shift invariance of  $\mathfrak{B}$  implies that  $\mathfrak{D}$  is right-shift invariant, too. The symbol  $\mathfrak{D}$  of  $\mathfrak{D}$  is a Schur function (a contractive analytic function) in the unit disk  $\mathbb{D}$  (see, e.g. Reference [44, Lemma 3.2, p. 195] or [23, Theorem 10.3.5 and the remark on p. 703]). We can now use a standard i/s/o result to realize  $\mathfrak{D}$  as the transfer function of a simple conservative (or controllable passive and forward conservative, or observable passive and backward conservative) i/s/o system  $\Sigma_{i/s/o}$ , and all of

these realizations are unique up to a unitary similarity transformation in the state space; see, e.g. Reference [25, Theorems 2.2.1, 2.2.2, and 2.3.1]. The input/output map of the resulting system will be equal to  $\mathfrak{D}$ , and the graph of  $\mathfrak{D}$  is the given passive behaviour  $\mathfrak{B}$ . The system  $\Sigma_{i/s/o}$  can be interpreted as a scattering representation of a passive s/s system  $\Sigma$  which is conservative and simple (or controllable passive and forward conservative, or observable passive and backward conservative). This system is still unique up to a unitary similarity transformation in the state space.  $\square$

The realizations given in Theorem 8.6 need not be minimal. Minimal passive realizations also exist (see Proposition 8.9 below). Not all of these minimal realizations have the property that their minimal conservative orthogonal dilation (given in Theorem 7.9) is simple. This follows from Corollary I.5.5. and Lemma I.8.22 combined with the fact that the same claim is true for orthogonal scattering conservative dilations of passive i/s/o systems, as shown by an example in Reference [29, Section 6].

*Proposition 8.7*

The adjoint  $\mathfrak{B}_*$  of a passive behaviour  $\mathfrak{B}$  is passive.

*Proof*

Let  $\Sigma$  be one of the passive realization of  $\mathfrak{B}$  given in Theorem 8.6. Then also the adjoint s/s system  $\Sigma_*$  are passive. By Theorem 4.15,  $\mathfrak{B}_*$  is the behaviour induced by  $\Sigma_*$ , and by Proposition 8.2,  $\mathfrak{B}_*$  is passive.  $\square$

*Definition 8.8*

A passive s/s system has *minimal losses* if its minimal conservative orthogonal dilation is simple.

Passive s/s systems with minimal losses are important, e.g. for the theory of Darlington representations. We shall return to this elsewhere.

*Proposition 8.9*

Every passive behaviour  $\mathfrak{B}$  has a minimal passive s/s realization with minimal losses. More precisely, if  $\Sigma$  is the simple conservative realization in part (1) of Theorem 8.6, then the two compressions  $\Sigma_{\circ}$  and  $\Sigma_{\bullet}$  constructed in Theorem 7.3 are minimal passive s/s systems with minimal losses which also realize the same behaviour  $\mathfrak{B}$ .

*Proof*

Define  $\Sigma$ ,  $\Sigma_{\circ}$ , and  $\Sigma_{\bullet}$  as described above. By Remark 7.4, both  $\Sigma_{\circ}$  and  $\Sigma_{\bullet}$  are minimal and passive, and they have minimal losses since they have a simple conservative dilation, namely  $\Sigma$ . Finally, by Lemma I.8.2,  $\Sigma_{\circ}$  and  $\Sigma_{\bullet}$  have the same behaviour as  $\Sigma$ .  $\square$

*Remark 8.10*

The realization in part (2) of Theorem 8.6 can be obtained from the one in part (1) through an orthogonal compression onto the reachable subspace. This means that the system  $\Sigma_{\circ}$  in Proposition 8.9 is the orthogonal compression of the system in part (2) of Theorem 8.6 to the orthogonal companion of the unobservable subspace. Analogously, the realization in part (3) of Theorem 8.6 can be obtained from the one in part (1) through an orthogonal compression onto

the orthogonal companion of the unobservable subspace, and the system  $\Sigma_{\bullet}$  in Proposition 8.9 is the orthogonal compression of the system in part (3) of Theorem 8.6 to the reachable subspace.

As the following proposition shows, the behaviour of a s/s system  $\Sigma$  can be passive even if  $\Sigma$  itself is not passive.

*Proposition 8.11*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system.

- (1) If  $\Sigma$  is pseudo-similar to a passive system, then the behaviour of  $\Sigma$  is passive.
- (2) If  $\Sigma$  is minimal and the behaviour of  $\Sigma$  is passive, then  $\Sigma$  is pseudo-similar to a passive system.

*Proof*

Assertion (1) follows from Propositions 3.3 and 8.2. Conversely, suppose that  $\Sigma$  is minimal and that the behaviour  $\mathfrak{B}$  of  $\Sigma$  is passive. By Proposition 8.9,  $\mathfrak{B}$  has a minimal passive realization  $\Sigma_1$ , and by Proposition 3.3,  $\Sigma$  is pseudo-similar to  $\Sigma_1$ . □

We shall say more about this in the next section.

### 9. $H$ -PASSIVE STATE/SIGNAL SYSTEMS

In this section we extend the notion of passivity by allowing a non-trivial storage function in the state space. This storage function is induced by a positive self-adjoint operator  $H$  in the state space  $\mathcal{X}$  (i.e.  $H$  is self-adjoint and  $(x, Hx) > 0$  for every non-zero  $x \in \mathcal{D}(H)$ ). Every such operator has a unique positive self-adjoint square root, which we denote by  $H^{1/2}$ . The inverse of this square root is denoted by  $H^{-1/2}$ .

*Definition 9.1*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system with a Hilbert state space  $\mathcal{X}$  and a Kreĭn signal space  $\mathcal{W}$ , and let  $H$  be a positive self-adjoint operator in  $\mathcal{X}$ .

- (1)  $\Sigma$  is *forward  $H$ -passive* if for any trajectory  $(x(\cdot), w(\cdot))$  of  $\Sigma$  with  $x(0) \in \mathcal{D}(H^{1/2})$  we have  $x(n) \in \mathcal{D}(H^{1/2})$  and

$$\|H^{1/2}x(n+1)\|_{\mathcal{X}}^2 - \|H^{1/2}x(n)\|_{\mathcal{X}}^2 \leq [w(n), w(n)]_{\mathcal{W}} \tag{63}$$

for all  $n \in \mathbb{Z}^+$ .

- (2)  $\Sigma$  is *backward  $H$ -passive* if the adjoint s/s system  $\Sigma_*$  is forward  $H^{-1}$ -passive, i.e. if for any trajectory  $(x_*(\cdot), w_*(\cdot))$  of  $\Sigma_* = (V_*; \mathcal{X}, \mathcal{W}_*)$  with  $x_*(0) \in \mathcal{D}(H^{-1/2})$  we have  $x_*(n) \in \mathcal{D}(H^{-1/2})$  and

$$\|H^{-1/2}x_*(n+1)\|_{\mathcal{X}}^2 - \|H^{-1/2}x_*(n)\|_{\mathcal{X}}^2 \leq [w_*(n), w_*(n)]_{\mathcal{W}_*} \tag{64}$$

for all  $n \in \mathbb{Z}^+$ .

- (3)  $\Sigma$  is  *$H$ -passive* if it is both forward and backward  $H$ -passive.

In particular, if  $\Sigma$  is forward  $H$ -passive and if  $(x(\cdot), w(\cdot))$  is an externally generated trajectory of  $\Sigma$  (i.e.  $x(0) = 0$ ), then  $x(n) \in \mathcal{D}(H^{1/2})$  for all  $n \in \mathbb{Z}^+$ . Likewise, if  $\Sigma$  is backward  $H$ -passive and if  $(x_*(\cdot), w_*(\cdot))$  is an externally generated trajectory of  $\Sigma_*$ , then  $x_*(n) \in \mathcal{D}(H^{-1/2})$  for all  $n \in \mathbb{Z}^+$ . Also note that  $1_{\mathcal{X}}$ -passivity is equivalent to passivity, and that  $\Sigma$  is  $H$ -passive if and only if the adjoint system  $\Sigma_*$  is  $H^{-1}$ -passive.

*Lemma 9.2*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system with a Hilbert state space  $\mathcal{X}$  and a Krein signal space  $\mathcal{W}$ , and let  $H$  be a positive self-adjoint operator in  $\mathcal{X}$ .

- (1)  $\Sigma$  is forward  $H$ -passive if and only if the conditions  $\begin{bmatrix} z \\ x \end{bmatrix} \in V$  and  $x \in \mathcal{D}(H^{1/2})$  imply that  $z \in \mathcal{D}(H^{1/2})$ , and

$$\|H^{1/2}z\|_{\mathcal{X}}^2 - \|H^{1/2}x\|_{\mathcal{X}}^2 \leq [w, w]_{\mathcal{W}}, \quad \begin{bmatrix} z \\ x \end{bmatrix} \in V, \quad x \in \mathcal{D}(H^{1/2}) \quad (65)$$

- (2)  $\Sigma$  is backward  $H$ -passive if and only if the conditions  $\begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in V_*$  and  $x_* \in \mathcal{D}(H^{-1/2})$  imply that  $z_* \in \mathcal{D}(H^{-1/2})$ , and

$$\|H^{-1/2}z_*\|_{\mathcal{X}}^2 - \|H^{-1/2}x_*\|_{\mathcal{X}}^2 \leq -[w_*, w_*]_{\mathcal{W}}, \quad \begin{bmatrix} z_* \\ x_* \\ w_* \end{bmatrix} \in V, \quad x_* \in \mathcal{D}(H^{-1/2}) \quad (66)$$

We leave the easy proof of this lemma to the reader.

*Definition 9.3*

A positive self-adjoint solution  $H$  of (65) is called a *generalized solution of the s/s forward KYP inequality* for the s/s system  $\Sigma$ , and a positive self-adjoint solution  $H$  of (66) is called a *generalized solution of the backward s/s KYP inequality*. By a *generalized solution of the s/s KYP inequality* we mean a positive self-adjoint operator  $H$  that satisfies both (65) and (66).

We denote the set of all generalized solutions of the forward s/s KYP inequality for  $\Sigma$  by  $M_{\Sigma}^+$ , we denote the set of all generalized solutions of the backward s/s KYP inequality for  $\Sigma$  by  $M_{\Sigma}^-$ , and we define  $M_{\Sigma} = M_{\Sigma}^+ \cap M_{\Sigma}^-$ . The connection between these sets will be explained in Proposition 9.9 below.

*Lemma 9.4*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s system with a Hilbert state space  $\mathcal{X}$  and a Krein signal space  $\mathcal{W}$ , and let  $H$  be a positive self-adjoint operator in  $\mathcal{X}$ . Then  $\Sigma$  is forward  $H$ -passive if and only if  $H \in M_{\Sigma}^+$ ,  $\Sigma$  is backward  $H$ -passive if and only if  $H \in M_{\Sigma}^-$ , and  $\Sigma$  is  $H$ -passive if and only if  $H \in M_{\Sigma}$ .

*Proof*

This follows immediately from Lemma 9.2 and Definition 9.3.  $\square$

*Proposition 9.5*

The behaviour of an  $H$ -passive system is passive.

*Proof*

The  $H$ -passivity of a s/s system  $\Sigma$  means that it is both forward and backward  $H$ -passive. It follows from (65) that the zero section  $\mathfrak{B}(0)$  of the behaviour  $\mathfrak{B}$  of  $\Sigma$  is non-negative in  $\mathcal{W}$ , and it follows from (66) that the zero section  $\mathfrak{B}_*(0)$  of the adjoint behaviour  $\mathfrak{B}_*$  induced by  $\Sigma_*$  is non-negative in  $\mathcal{W}_*$ . We recall from Proposition 4.4 that  $\mathfrak{B}_*(0) = \mathfrak{B}(0)^{\langle \perp \rangle}$ , hence  $\mathfrak{B}(0)^{[\perp]} = \mathcal{S}\mathfrak{B}_*(0)$  is non-positive in  $\mathcal{W}$ . By part (5) of Proposition 2.2,  $\mathfrak{B}(0)$  is maximal non-negative in  $\mathcal{W}$ . By iterating (63) (starting with  $n = 0$  and  $x(0) = 0$ ) we find that  $\mathfrak{B}$  also has property (1) in Definition 8.1. Thus,  $\mathfrak{B}$  is passive.  $\square$

Theorem 9.12 given below is a partial converse to Proposition 9.5.

The above KYP inequalities can be reformulated by using different representations of  $\Sigma$  (driving variable, output nulling, or i/s/o). For example, suppose that  $\mathcal{W}$  has an orthogonal admissible decomposition  $\mathcal{W} = -\mathcal{Y}[+] \mathcal{U}$ , where  $\mathcal{Y}$  and  $\mathcal{U}$  are Krein spaces in the inner product inherited from  $\mathcal{W}$  (this is a fundamental decomposition if  $\mathcal{Y}$  and  $\mathcal{U}$  are Hilbert spaces). We denote the corresponding i/s/o representation of  $\Sigma$  by  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{Y}, \mathcal{U} \right)$ . In terms of this representation,  $H \in M_{\Sigma}^+$  if and only if

$$A\mathcal{D}(H^{1/2}) \subset \mathcal{D}(H^{1/2}), \quad B\mathcal{U} \subset \mathcal{D}(H^{1/2}) \quad (67)$$

and

$$\begin{aligned} & \|H^{1/2}(Ax + Bu)\|_{\mathcal{X}}^2 - \|H^{1/2}x\|_{\mathcal{X}}^2 \leq [u, u]_{\mathcal{U}} \\ & - [Cx + Du, Cx + Du]_{\mathcal{Y}}, \quad x \in \mathcal{D}(H^{1/2}), \quad u \in \mathcal{U} \end{aligned} \quad (68)$$

By the same argument combined with Proposition 4.11,  $H \in M_{\Sigma}^-$  if and only if

$$A^*\mathcal{R}(H^{1/2}) \subset \mathcal{R}(H^{1/2}), \quad C^*\mathcal{Y} \subset \mathcal{R}(H^{1/2}) \quad (69)$$

and

$$\begin{aligned} & \|H^{-1/2}(A^*x_* + C^*y_*)\|_{\mathcal{X}}^2 - \|H^{-1/2}x_*\|_{\mathcal{X}}^2 \leq [y_*, y_*]_{\mathcal{Y}} \\ & - [B^*x_* + D^*y_*, C^*x_* + D^*y_*]_{\mathcal{U}}, \quad x_* \in \mathcal{R}(H^{1/2}), \quad y_* \in \mathcal{Y} \end{aligned} \quad (70)$$

Thus, the set of positive self-adjoint solutions of (67)–(68) is the same as the set  $M_{\Sigma}^+$  of positive self-adjoint solutions of (65), and the set of positive self-adjoint solutions of (69)–(70) is the same as the set  $M_{\Sigma}^-$  of positive self-adjoint solutions of (66). In the case where  $H$  is bounded the condition (67) holds automatically, and condition (68) may be rewritten as inequality (12) in the Krein space  $\mathcal{X}[+] \mathcal{U}$ .

More generally, it is possible to study the solutions  $H$  of (67)–(68) and (69)–(70) corresponding to an arbitrary i/s/o system  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  without any reference to an underlying s/s system. Here we allow  $\mathcal{Y}$  and  $\mathcal{U}$  to be Krein spaces. In this case, we refer to (67)–(68) and (69)–(70) as the forward, respectively, backward, generalized *transmission KYP inequalities*. We call  $\Sigma_{i/s/o}$  *forward or backward  $H$ -passive* if  $H$  is a generalized solution of the

forward or backward transmission KYP inequality, and we call  $\Sigma_{i/s/o}$  *H-passive* if it is both forward and backward *H-passive*. When  $\mathcal{Y}$  and  $\mathcal{U}$  are *Hilbert* spaces we replace the word ‘transmission’ by ‘scattering’.

We denote the set of all generalized solutions of the forward transmission (or scattering) KYP inequality for  $\Sigma_{i/s/o}$  by  $M_{\Sigma_{i/s/o}}^+$ , we denote the set of all generalized solutions of the backward transmission KYP inequality for  $\Sigma_{i/s/o}$  by  $M_{\Sigma_{i/s/o}}^-$ , and we define  $M_{\Sigma_{i/s/o}} = M_{\Sigma_{i/s/o}}^+ \cap M_{\Sigma_{i/s/o}}^-$ .

The above discussion may be summarized as follows.

*Proposition 9.6*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s signal system which has an admissible orthogonal decomposition  $\mathcal{W} = -\mathcal{Y}[+] \mathcal{U}$  of the signal space. Let  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  be the corresponding transmission representation of  $\Sigma$ . Then the following claims are true:

- (1)  $\Sigma$  is forward *H-passive* if and only if  $\Sigma_{i/s/o}$  is forward *H-passive*, i.e.  $M_{\Sigma}^+ = M_{\Sigma_{i/s/o}}^+$ .
- (2)  $\Sigma$  is backward *H-passive* if and only if  $\Sigma_{i/s/o}$  is backward *H-passive*, i.e.  $M_{\Sigma}^- = M_{\Sigma_{i/s/o}}^-$ .
- (3)  $\Sigma$  is *H-passive* if and only if  $\Sigma_{i/s/o}$  is *H-passive*, i.e.  $M_{\Sigma} = M_{\Sigma_{i/s/o}}$ .

Thus, in particular, the sets  $M_{\Sigma_{i/s/o}}^+$ ,  $M_{\Sigma_{i/s/o}}^-$ , and  $M_{\Sigma_{i/s/o}}$  are independent of the particular orthogonal decomposition  $\mathcal{W} = -\mathcal{Y}[+] \mathcal{U}$ .

As our next proposition shows, if the decomposition  $\mathcal{W} = -\mathcal{Y}[+] \mathcal{U}$  is fundamental, so that  $\Sigma_{i/s/o}$  is a *scattering representation* of  $\Sigma$ , then forward and backward *H-passivity* are equivalent, so that in this case  $M_{\Sigma}^+ = M_{\Sigma}^- = M_{\Sigma}$ . This was first proved in a i/s/o setting in Reference [14].

*Proposition 9.7*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a s/s signal system which has an admissible *fundamental* decomposition  $\mathcal{W} = -\mathcal{Y}[+] \mathcal{U}$  of the signal space. Then the following claims are equivalent:

- (1)  $\Sigma$  is forward *H-passive*,
- (2)  $\Sigma$  is backward *H-passive*,
- (3)  $\Sigma$  is *H-passive*.

*Proof*

Let  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  be the scattering representation of  $\Sigma$  corresponding to the decomposition  $\mathcal{W} = -\mathcal{Y}[+] \mathcal{U}$ . Then, by Proposition 9.6,  $\Sigma$  is forward *H-passive* if and only if  $\Sigma_{i/s/o}$  is forward *H-passive*. By Reference [24, Proposition 4.6],  $\Sigma_{i/s/o}$  is forward *H-passive* if and only if it is backward *H-passive*. Thus, (1) and (2) are equivalent.  $\square$

*Theorem 9.8*

Let the s/s system  $\Sigma = (V; \mathcal{X}, \mathcal{Y})$  be forward *H-passive*. Then the following conditions are equivalent:

- (1)  $\Sigma$  is *H-passive*;
- (2) At least one fundamental decomposition  $\mathcal{W} = -\mathcal{Y}[+] \mathcal{U}$  of  $\mathcal{W}$  is an admissible input/output decomposition for  $\Sigma$ .
- (3) The zero section  $\mathfrak{B}(0)$  of the behaviour  $\mathfrak{B}$  of  $\Sigma$  is a maximal non-negative subspace of  $\mathcal{W}$ .
- (4) The behaviour  $\mathfrak{B}$  of  $\Sigma$  is passive.

If these conditions hold, then every fundamental decomposition  $\mathcal{W} = -\mathcal{D}[+] \mathcal{U}$  of  $\mathcal{W}$  is admissible.

*Proof*

By Proposition 9.5, (1)  $\Rightarrow$  (4). Trivially, (4)  $\Rightarrow$  (3), and by Corollary 5.9, (3)  $\Rightarrow$  (1). By Proposition 8.5, (4)  $\Rightarrow$  (1), and by Proposition 9.7, (2)  $\Rightarrow$  (1).  $\square$

*Proposition 9.9*

If both  $M_{\Sigma}^+$  and  $M_{\Sigma}^-$  are non-empty, then  $M_{\Sigma}$  is non-empty as well, and  $M_{\Sigma}^+ = M_{\Sigma}^- = M_{\Sigma}$ .

*Proof*

Let  $H \in M_{\Sigma}^+$  and  $H_1 \in M_{\Sigma}^-$ . It follows from (65) that the zero section  $\mathfrak{B}(0)$  of the behaviour  $\mathfrak{B}$  of  $\Sigma$  is non-negative on  $\mathcal{W}$ , and it follows from (66) with  $H$  replaced by  $H_1$  that the zero section  $\mathfrak{B}_*(0)$  of the adjoint behaviour  $\mathfrak{B}_*$  induced by  $\Sigma_*$  is non-negative in  $\mathcal{W}_*$ . Continuing in the same way as we did in the proof of Proposition 9.5 we find that  $\mathfrak{B}(0)$  is maximal non-negative in  $\mathcal{W}$ . By Theorem 9.8,  $H \in M_{\Sigma}$ . Thus,  $M_{\Sigma}^+ \subset M_{\Sigma}$ . The same argument applied to the adjoint system shows that also  $M_{\Sigma}^- \subset M_{\Sigma}$ .  $\square$

*Remark 9.10*

It is possible that one of the two sets  $M_{\Sigma}^+$  and  $M_{\Sigma}^-$  is empty and the other non-empty. This is true, for example, in Example 5.5, where  $M_{\Sigma}^+ \neq \emptyset$  (it contains the identity) but  $M_{\Sigma}^- = \emptyset$  (it must be empty, since otherwise the system would be passive).

*Proposition 9.11*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a minimal s/s system with Hilbert state space  $\mathcal{X}$  and Krein signal space  $\mathcal{W}$ .

- (1) If  $\Sigma$  is  $H$ -passive for some  $H$ , then  $\Sigma$  is pseudo-similar to a unique passive system  $\Sigma_H = (V_H; \mathcal{X}, \mathcal{W})$  with pseudo-similarity operator  $H^{1/2}$ .
- (2) Conversely, if  $\Sigma$  is pseudo-similar to a passive system  $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$  with similarity operator  $Q$ , then  $\Sigma$  is  $H$ -passive with  $H = Q^*Q$ , and  $\Sigma_1$  is unitarily similar to the system  $\Sigma_H$  in assertion (1).

*Proof*

*Proof of (1):* Let  $\mathcal{W} = -\mathcal{D}[+] \mathcal{U}$  be a fundamental decomposition of  $\mathcal{W}$ . By Proposition 9.7, this decomposition is admissible for  $\Sigma$ . Let  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  be the corresponding scattering representation of  $\Sigma$ . By Proposition 9.6,  $\Sigma_{i/s/o}$  is  $H$ -passive. According to Reference [14, Proposition 4.2], there exists a (unique) scattering passive i/s/o system  $\Sigma_{i/s/o}^H = \left( \begin{bmatrix} A_H & B_H \\ C_H & D_H \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  such that  $\Sigma_{i/s/o}$  is pseudo-similar to  $\Sigma_{i/s/o}^H$  with similarity operator  $H^{1/2}$  (here  $A_H$  is the closure of  $H^{1/2}AH^{-1/2}$ ,  $C_H$  is the closure of  $CH^{1/2}$ ,  $B_H = H^{1/2}B$ , and  $D_H = D$ ). Let  $\Sigma_H = (V_H; \mathcal{X}, \mathcal{W})$  be the corresponding s/s system. Then  $\Sigma$  is pseudo-similar to  $\Sigma_H$  with pseudo-similarity operator  $H^{1/2}$ . The uniqueness of  $\Sigma_H$  is immediate: the system  $\Sigma_H$  is uniquely determined by its trajectories, and the trajectories of  $\Sigma_H$  are determined uniquely by the trajectories of  $\Sigma$  and the pseudo-similarity  $H^{1/2}$ .

*Proof of (2):* Let  $H = Q^*Q$ . Then  $H$  is a positive self-adjoint operator in  $\mathcal{X}$  with  $\mathcal{D}(H^{1/2}) = \mathcal{D}(Q)$ , and  $Q$  has a polar decomposition  $Q = UH^{1/2}$ , where  $U$  is a unitary operator

mapping  $\mathcal{X}$  onto  $\mathcal{X}_1$  (see, e.g. Reference [45, p. 334]). Let  $V_H = \left\{ \begin{bmatrix} U^{-1}z \\ U^{-1}x \\ w \end{bmatrix} \middle| \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V_1 \right\}$ . Then  $\Sigma_1$  is unitarily similar to the minimal passive system  $\Sigma_H = (V_H; \mathcal{X}, \mathcal{W})$ , and the trajectories of  $\Sigma_H$  are of the form  $(U^{-1}x_1(\cdot), w(\cdot))$  where  $(x_1(\cdot), w(\cdot))$  is a trajectory of  $\Sigma_1$ . Moreover, the operator  $H^{1/2}$  is a pseudo-similarity between  $\Sigma$  and  $\Sigma_H$ . The passivity of  $\Sigma_H$  now implies that  $\Sigma$  is  $H$ -passive.  $\square$

*Theorem 9.12*

Let  $\Sigma = (V; \mathcal{X}, \mathcal{W})$  be a minimal s/s system with Hilbert state space  $\mathcal{X}$  and Kreĭn signal space  $\mathcal{W}$ . If the behaviour of  $\Sigma$  is passive, there exists a positive self-adjoint operator  $H$  with respect to which  $\Sigma$  is  $H$ -passive, or equivalently,  $\Sigma$  is pseudo-similar to a passive system  $\Sigma_H = (V_H; \mathcal{X}, \mathcal{W})$  with pseudo-similarity operator  $H^{1/2}$ . Moreover, it is possible to choose  $H$  in such a way that  $\Sigma_H$  is minimal, or equivalently,

$$\overline{H^{1/2}\mathfrak{R}_\infty} = \mathcal{X}, \quad \overline{H^{-1/2}\mathfrak{R}_{*\infty}} = \mathcal{X} \tag{71}$$

where

$$\mathfrak{R}_\infty = \bigcup_{n=1}^\infty \{x(n) | (x(\cdot), w(\cdot)) \text{ is an externally generated trajectory of } \Sigma\}^{\dagger\dagger\dagger}$$

and  $\mathfrak{R}_{*\infty}$  is the corresponding subspace for  $\Sigma_*$ .

*Proof*

By Proposition 8.9, there exists minimal passive s/s system  $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$  which realizes the same behaviour  $\mathfrak{B}$  as  $\Sigma$ . According to Proposition 3.3, the systems  $\Sigma$  and  $\Sigma_1$  are pseudo-similar. By assertion (2) of Proposition 9.11,  $\Sigma$  is  $H$ -passive where  $H = Q^*Q$ , and  $H^{1/2}$  is a pseudo-similarity operator between  $\Sigma$  and  $\Sigma_H$ . Thus, for any trajectory  $(x(\cdot), w(\cdot))$  of  $\Sigma$  with  $x(0) \in \mathcal{D}(H^{1/2})$  we have  $x(n) \in \mathcal{D}(H^{1/2})$  for all  $n \in \mathbb{Z}^+$ , and  $(H^{1/2}x(\cdot), w(\cdot))$  is a trajectory of  $\Sigma$ . The converse is also true: for any trajectory  $(x_H(\cdot), w(\cdot))$  of  $\Sigma_H$  with  $x_H(0) \in \mathcal{D}(H^{1/2})$  we have  $x_H(n) \in \mathcal{D}(H^{1/2})$  for all  $n \in \mathbb{Z}^+$ , and  $(H^{-1/2}x_H(\cdot), w(\cdot))$  is a trajectory of  $\Sigma$ . This connection holds, in particular, for all externally generated trajectories (since  $x(0) = 0$  and  $x_H(0) = 0$ ). This implies that  $\mathfrak{R}_{\infty H} = \overline{H^{1/2}\mathfrak{R}_\infty}$ , where  $\mathfrak{R}_{\infty H}$  is the analogue of  $\mathfrak{R}_\infty$  for  $\Sigma_H$ . The controllability of  $\Sigma_H$  implies that  $\overline{H^{1/2}\mathfrak{R}_\infty} = \mathfrak{R}_{\infty H} = \mathcal{X}$ . By applying the same argument to the dual system we get  $\overline{H^{-1/2}\mathfrak{R}_{*\infty}} = \mathcal{X}$  (recall that, by Theorem 4.13,  $H^{1/2}$  is a similarity between  $\Sigma_*$  and  $(\Sigma_H)_*$ ).  $\square$

It is possible to compare two generalized positive self-adjoint solutions of the KYP inequality to each other by using the standard partial ordering of non-negative self-adjoint operators, which is defined as follows (see Reference [14] for details). If  $H_1$  and  $H_2$  are two non-negative self-adjoint operators on the Hilbert space  $\mathcal{X}$ , then we write  $H_1 \preceq H_2$  whenever  $\mathcal{D}(H_2^{1/2}) \subset \mathcal{D}(H_1^{1/2})$  and  $\|H_1^{1/2}x\| \leq \|H_2^{1/2}x\|$  for all  $x \in \mathcal{D}(H_2^{1/2})$ . For bounded non-negative operators  $H_1$  and  $H_2$  with  $\mathcal{D}(H_2) = \mathcal{D}(H_1) = \mathcal{X}$  this ordering coincides with the standard ordering of bounded self-adjoint operators on  $\mathcal{X}$ .

$\dagger\dagger\dagger$  Thus,  $\mathfrak{R}_\infty$  consists of all those states of  $\Sigma$  that can be reached in some finite time, starting from the zero state.



**Theorem 9.13**

Let  $\Sigma$  be a minimal s/s system with a passive behaviour, and let  $M_{\Sigma}^{\min}$  be the set of all solutions of the KYP inequality for  $\Sigma$  that satisfy (71). Then  $M_{\Sigma}^{\min}$  is non-empty, and there exist unique  $H_{\circ} \in M_{\Sigma}^{\min}$  and  $H_{\bullet} \in M_{\Sigma}^{\min}$  such that

$$H_{\circ} \leq H \leq H_{\bullet}, \quad H \in M_{\Sigma}^{\min}$$

**Proof**

By Theorem 9.12, the set  $M_{\Sigma}^{\min}$  is non-empty. Let  $\mathcal{W} = -\mathcal{Y} \dot{+} \mathcal{U}$  be a fundamental decomposition of  $\mathcal{W}$ . By Proposition 8.5, this decomposition is admissible for  $\Sigma$ . Let  $\Sigma_{i/s/o}$  be the corresponding scattering representation of  $\Sigma$ . Then, by Proposition 9.6,  $M_{\Sigma} = M_{\Sigma_{i/s/o}}$ . The existence of a minimal solution  $H_{\circ}$  and a maximal solution  $H_{\bullet}$  in  $M_{\Sigma}^{\min}$  now follows from Reference [14, Theorem 5.11 and Proposition 5.15].  $\square$

**Remark 9.14**

Above we studied the set  $M_{\Sigma}^{\min}$  by reducing the problem to the corresponding problem for i/s/o systems  $\Sigma_{i/s/o}$  and scattering supply rate  $j(y, u) = \|u\|_{\mathcal{U}}^2 - \|y\|_{\mathcal{Y}}^2$ , which was solved in Reference [14]. This reduction was based on the following facts:

- (1) A decomposition  $\mathcal{W} = -\mathcal{Y} \dot{+} \mathcal{U}$  is fundamental if and only if the inner product  $[\cdot, \cdot]_{\mathcal{W}}$  in  $\mathcal{W}$  induces the scattering supply rate  $j(y, u) = -\|y\|_{\mathcal{Y}}^2 + \|u\|_{\mathcal{U}}^2$  on  $[\mathcal{Y}]$ , where  $\|y\|_{\mathcal{Y}}^2 = -[y, y]_{\mathcal{W}} \geq 0$  and  $\|u\|_{\mathcal{U}}^2 = [u, u]_{\mathcal{W}} \geq 0$  for all  $\begin{bmatrix} y \\ u \end{bmatrix} \in [\mathcal{W}]$ ,
- (2) every fundamental decomposition of  $\mathcal{W}$  is admissible for an  $H$ -passive system  $\Sigma$  with Krein signal space  $\mathcal{W}$ ,
- (3) if  $\Sigma_{i/s/o}$  is a scattering representation of a s/s system  $\Sigma$ , then  $M_{\Sigma}^{\pm} = M_{\Sigma}^{-} = M_{\Sigma}$ , and this set coincides with the corresponding sets for  $\Sigma_{i/s/o}$ , i.e.  $M_{\Sigma_{i/s/o}}^{+} = M_{\Sigma_{i/s/o}}^{-} = M_{\Sigma_{i/s/o}} = M_{\Sigma}$ . If, in addition,  $\Sigma$  is minimal, then the same statement remains true for  $M_{\Sigma}^{\min}$ .

This approach is not restricted to i/s/o systems with a *scattering* supply rate. The same argument applies to an arbitrary i/s/o system  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  with Hilbert input and output spaces  $\mathcal{U}$  and  $\mathcal{Y}$  and with an arbitrary supply rate  $j(y, u)$  as long as *this supply rate defines a Krein space inner product in the signal space  $\mathcal{W} = [\mathcal{Y}]$* . Each such supply rate has a representation of the form  $j(y, u) = \left( \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y \\ u \end{bmatrix} \right)_{\mathcal{Y} \oplus \mathcal{U}}$  for some self-adjoint operator  $J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{Y} \oplus \mathcal{U})$  with a bounded inverse. The decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  is orthogonal if and only if  $J$  is block diagonal, i.e.  $J_{12} = 0$  and  $J_{21} = 0$ . In the case of a bounded operator  $H$  the (forward) KYP inequality for  $\Sigma_{i/s/o}$  is given by

$$\begin{bmatrix} A^*HA - H - C^*J_{11}C & A^*HB - C^*J_{12} - C^*J_{11}D \\ B^*HA - J_{21}C - D^*J_{11}C & B^*HB - D^*J_{12} - J_{21}D - D^*J_{11}D - J_{22} \end{bmatrix} \leq 0 \quad (72)$$

and  $M_{\Sigma_{i/s/o}}^{+}$  consists of generalized solutions of this inequality. The adjoint supply rate  $j_*(y, u)$  which is applied to the adjoint i/s/o system  $\Sigma_{i/s/o}^* = \left( \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}; \mathcal{X}, \mathcal{Y}, \mathcal{U} \right)$  is given by  $j_*(y, u) = -\left( \begin{bmatrix} y \\ u \end{bmatrix}, J^{-1} \begin{bmatrix} y \\ u \end{bmatrix} \right)_{\mathcal{Y} \oplus \mathcal{U}}$ , and  $M_{\Sigma_{i/s/o}}^{-}$  consists of the inverses of all generalized solutions of the KYP inequality for  $\Sigma_{i/s/o}^*$ . The system  $\Sigma_{i/s/o}$  can be interpreted as an i/s/o representation of a

s/s system  $\Sigma$  with the signal space  $\mathcal{W} = \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$  with Kreĭn space inner product  $\left[ \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix} \right]_{\mathcal{W}} = \left( \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y' \\ u' \end{bmatrix} \right)_{\mathcal{Y} \oplus \mathcal{U}}$  for all  $\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix} \in \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$ , and with this interpretation  $M_{\Sigma}^{+} = M_{\Sigma_{i/s/o}^{+}}$  and  $M_{\Sigma}^{-} = M_{\Sigma_{i/s/o}^{-}}$ . Property (3) above is still valid in the weaker form presented in Proposition 9.9.

It is also possible to proceed in the opposite direction, and to consider the  $H$ -passive s/s system  $\Sigma$  to be the primary object, from which we can construct various  $H$ -passive i/s/o systems with different supply rates. If  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  is an admissible decomposition for  $\Sigma$ , then the corresponding i/s/o system  $\Sigma_{i/s/o}$  is  $H$ -passive with respect to the supply rate on  $\begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$  inherited from the inner product  $[\cdot, \cdot]_{\mathcal{W}}$ .<sup>§§§</sup> Thus, in the family of i/s/o systems  $\Sigma_{i/s/o} = \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix}; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$  that we get from  $\Sigma$  by varying the decomposition  $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$  the coefficients  $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$  vary, and so do the supply rates  $j(y, u)$ , but *the set of solutions  $M_{\Sigma_{i/s/o}}^{+}$  of the generalized KYP inequality (72) stays the same.* The same comment applies to the sets  $M_{\Sigma_{i/s/o}}^{-}$  and  $M_{\Sigma_{i/s/o}}$ , too.

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<sup>§§§</sup>The transmission supply rate  $j(y, u) = -[y, y]_{\mathcal{Y}} + [u, u]_{\mathcal{U}}$  corresponds to an *orthogonal* admissible decomposition  $\mathcal{W} = -\mathcal{Y} \dot{+} \mathcal{U}$  of the signal space  $\mathcal{W}$ , and this case is discussed in Proposition 9.6.

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