WELL-POSED LINEAR SYSTEMS IN L^1 AND L^∞ ARE REGULAR

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Abstract. We study well-posed linear system with locally L^p inputs and outputs, where $1 \leq p \leq \infty$, and whose input, state and output spaces are Banach spaces. Like in the usual Hilbert space theory, we call such a system regular if each of its step responses has a right Cesaro limit at zero. In this case, the system has a feedthrough operator: the Cesaro limit is given by the feed-through operator applied to the constant value of the input step. Regular systems have a simple representation. We show that the system is regular if $p = \infty$, or if $p = 1$ and the state space is reflexive. We also present an application to a quadratic cost minimization problem for a parabolic equation.

Key Words. Well-posed linear system, regular linear system, feedthrough operator, parabolic equation, LQ-problem, spectral factorization.

1. Introduction

Many infinite-dimensional systems can be described by the equations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

on a triple of Banach spaces, namely, the input space U, the state space X, and the output space Y. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operators A, B and C are usually unbounded. It is often convenient to use the "integral" representation of the system, which consists of the four operators from the initial state x_0 and the input function u to the final state $x(t)$ and the output function y:

$$\begin{aligned} x(t) &= \mathcal{A}^t x_0 + \mathcal{B}\tau^t \pi_+ u, \\ y &= \mathcal{C}\, x_0 + \mathcal{D}\pi_+ u. \end{aligned} \tag{2}$$

Here, $\mathcal{A}^t$ is the semigroup generated by A (which maps the initial state x_0 into the final state $x(t)$), $\mathcal{B}\tau^t\pi_+$ is the map from the input u to the final state $x(t)$, $\mathcal{C}$ is the map from the initial state x_0 to the output y, and $\mathcal{D}\pi_+$ is the input-output map from u to y (see the notation in Section 2).

The *well-posedness* assumption is that (2) behaves well in an L^p-setting, i.e., $x(t) \in X$ and $y \in L^p_{\mathrm{loc}}(\mathbf{R}^+; Y)$ depend continuously on $x_0 \in X$ and on $u \in L^p_{\mathrm{loc}}(\mathbf{R}^+; U)$. If this is the case, we call the operators $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ a *well-posed linear system*. There is an almost one-to-one correspondence between (1) and (2): most well-posed linear systems can be represented as in (1). These systems are called *regular*. They are characterized by the fact that their transfer function has a strong limit at $+\infty$ (along the real axis), see Weiss [22]. However, there do exist irregular well-posed systems, and this may happen, for example, in the most commonly studied case where $p = 2$ and U, X, Y are Hilbert spaces. For example, any realization of the transfer function

$$G(s) = \cos(\log s)$$

is irregular (this example is due to Kirsten Morris). Such realizations exist with $p = 2$. We show that in the case $p = \infty$, every well-posed linear system has a representation of the form (1) (for a restricted class of inputs u). A similar statement is true when $p = 1$ (in this case, there is no extra limitation on the input function, but the state space should be reflexive). In particular, the transfer function G defined above has no realizations with $p = 1$ or $p = \infty$.

A simple example to which this theory applies is a system whose semigroup $\mathcal{A}^t$ is analytic on a reflexive state space and whose control operator B and observation operator C are not too unbounded, i.e., there exists some $\alpha < 1$ such that $C(\gamma I - A)^{-\alpha} B$ is a bounded linear operator for some (hence for every) γ in the resolvent set of A. In this case it is possible to use any value of p, $1 \leq p \leq \infty$, as long as the state space X is adapted to the value of p. The values $p = 1$ and $p = \infty$

are especially useful in the proof of the regularity of the solution to an optimal control problem for a stable parabolic system.

Many results presented in Sections 2–3 are briefly stated (as remarks) at the end of various sections in the paper [22]. Full proofs of these results will be given in the book [16]. The proofs of the results in Section 4 are given in [15]. A preprint of [15] is available from http://www.abo.fi/~staffans/.

2. Well-posed linear systems

As already outlined in Section 1, it is possible to define a well-posed linear system $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ without any reference to the system of equations (1). For this, we have to introduce some spaces and some simple operators. We denote $\mathbf{R}^+ = [0, \infty)$, $\mathbf{R}^- = (-\infty, 0)$,

$$(\pi_J u)(s) = \begin{cases} u(s), & s \in J, \\ 0, & s \notin J, \end{cases} \quad \text{for all } J \subset \mathbf{R},$$

$$\pi_+ u = \pi_{\mathbf{R}^+}, \qquad \pi_- u = \pi_{\mathbf{R}^-},$$

$$(\tau^t u)(s) = u(t+s), \quad -\infty < t, s < \infty.$$

The space $L^p_{c,\mathrm{loc}}(\mathbf{R}; U)$ consists of all the functions $u : \mathbf{R} \to U$ that are locally in L^p and whose support is bounded to the left. We interpret $L^p_{\mathrm{loc}}(\mathbf{R}^+; U)$ as the subspace of functions in $L^p_{c,\mathrm{loc}}(\mathbf{R}; U)$ which vanish on $\mathbf{R}^-$. A sequence of functions u_n converges in $L^p_{c,\mathrm{loc}}(\mathbf{R}; U)$ to a function u if the common support of all the functions u_n is bounded to the left and u_n converges to u in the L^p sense on every bounded time interval. The continuity of $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ in the following definition is with respect to this convergence.

Definition 1. *Let U, X, and Y be Banach spaces, and let $1 \le p \le \infty$. An* L^p-well-posed linear system *Ψ on (Y, X, U) is a quadruple $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ of continuous linear operators satisfying the following conditions:*

(i) $t \mapsto \mathcal{A}^t$ is a strongly continuous semigroup of operators on X;

(ii) $\mathcal{B}\colon L^p_{c,\mathrm{loc}}(\mathbf{R}; U) \to X$ satisfies $\mathcal{A}^t\mathcal{B}u = \mathcal{B}\tau^t\pi_- u$, for all $u \in L^p_{c,\mathrm{loc}}(\mathbf{R}; U)$ and all $t \in \mathbf{R}^+$;

(iii) $\mathcal{C}\colon X \to L^p_{c,\mathrm{loc}}(\mathbf{R}; Y)$ satisfies $\mathcal{C}\mathcal{A}^t x = \pi_+\tau^t\mathcal{C}x$, for all $x \in X$ and all $t \in \mathbf{R}^+$;

(iv) $\mathcal{D}\colon L^p_{c,\mathrm{loc}}(\mathbf{R}; U) \to L^p_{c,\mathrm{loc}}(\mathbf{R}; Y)$ satisfies $\tau^t\mathcal{D}u = \mathcal{D}\tau^t u$, $\pi_-\mathcal{D}\pi_+ u = 0$, and $\pi_+\mathcal{D}\pi_- u = \mathcal{C}\mathcal{B}u$, for all $u \in L^p_{c,\mathrm{loc}}(\mathbf{R}; U)$ and all $t \in \mathbf{R}$.

The different components of Ψ are called as follows: U is the input space, X is the state space, Y is the output space, $\mathcal{A}$ is the semigroup, $\mathcal{B}$ is the controllability map, $\mathcal{C}$ is the observability map, and $\mathcal{D}$ is the input-output map. The state $x(t) \in X$ at time $t \in \mathbf{R}^+$ and the output $y \in L^2_{\mathrm{loc}}(\mathbf{R}^+; Y)$ of Ψ with initial time zero, initial state $x_0 \in X$ and input function $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+; U)$ are given by (2).

For more details, explanations and examples we refer the reader to [5], [10]–[11], [12]–[15], [18]–[24] and the references therein. Most of the available literature deals with Hilbert spaces and $p = 2$.

The case $p = \infty$ differs from the other cases in the sense that some of the results that we give below are not valid for all $u \in L^\infty_{\mathrm{loc}}(\mathbf{R}^+; U)$ but only for regulated u. A function defined on a real interval is called *regulated* if it has right and left limits at every point. Since we identify functions which are equal almost everywhere, we may assume without loss of generality that regulated functions are right continuous at every point. Such functions are sometimes also called "cadlag".

For any interval $J \subset \mathbf{R}$, the space $Reg(J; U)$ of bounded regulated U-valued functions on J is a closed subspace of $L^\infty(J; U)$, and in fact it is the closure of the space of step functions. (A step function is a function constant on every interval of a locally finite partition of J into subintervals.) We refer to Chapter 7 of Dieudonné [6] for details (see also Remark 6.10 in [20] for a duality property of *Reg*). Shift-invariant operators on $Reg(\mathbf{R})$ were studied by Baker [1]. We denote by $Reg_{c,\mathrm{loc}}(\mathbf{R}; U)$ the space of regulated functions on $\mathbf{R}$ whose support is bounded to the left. The convergence in this space is uniform convergence on bounded intervals, combined with a uniform bound to the left on the supports. We denote by $Reg_{\mathrm{loc}}(\mathbf{R}^+; U)$ the space of regulated functions on $\mathbf{R}^+$ (this is a subspace of $L^\infty_{\mathrm{loc}}(\mathbf{R}^+; U)$).

Definition 2. *Let U, X, and Y be Banach spaces. A* Reg-well-posed linear system *Ψ on (U, X, Y) is a quadruple $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ of continuous linear operators satisfying the same conditions as in Definition 1, but with $L^p_{c,\mathrm{loc}}$ replaced by $Reg_{c,\mathrm{loc}}$. The different components of Ψ are given the same names as in Definition 1, and the state and output are also defined in the same way.*

By a well-posed linear system we mean a system which is either *Reg*-well-posed or L^p-well-posed for some p, $1 \le p \le \infty$. There is a simple relationship between L^∞-well-posed and *Reg*-well-posed linear systems:

Theorem 1. *If $\left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is an L^∞-well-posed linear system on (U, X, Y), and if we restrict the domains of $\mathcal{B}$ and $\mathcal{D}$ to $Reg_{c,\mathrm{loc}}(\mathbf{R}; U)$, then the resulting system is Reg-well-posed.*

Our proof of this innocent looking theorem is surprisingly complicated. It is based on the fact that (as we shall see below) L^∞-well-posed linear systems have a differential representation of the form (1), valid for all $u \in Reg_{\rm loc}(\mathbf{R}^+;U)$.

Before introducing the operators B and C in (1), we need two auxiliary spaces X_1 and X_{-1}. Choose any γ in the resolvent set of the generator A of $\mathcal{A}$. We let X_1 be the domain of A, with the norm $\|x\|_{X_1} = \|(\gamma I - A)x\|_X$, and X_{-1} is the completion of X with the norm $\|x\|_{X_{-1}} = \|(\gamma I - A)^{-1}x\|_X$. The semigroup $\mathcal{A}$ can be extended to a strongly continuous semigroup on X_{-1}, which we denote by the same symbol. We denote the space of bounded linear operators from U to Y by $\mathcal{L}(U;Y)$.

Proposition 1. *Every well-posed linear system* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *has a unique* control operator $B \in \mathcal{L}(U;X_{-1})$*, determined by the fact that the state* $x(t)$ *of* Ψ *defined in* (2) *is given by the standard variation of constants formula for* (1)*, i.e.,*

$$x(t) = \mathcal{A}^t x_0 + \int_0^t \mathcal{A}^{t-s} B u(s)\, ds. \qquad (3)$$

This formula is valid for all $x_0 \in X$ *and all* $u \in L^p_{\rm loc}(\mathbf{R}^+;U)$ *if* Ψ *is* L^p*-well-posed for some* $p < \infty$*, and for all* $u \in Reg_{\rm loc}(\mathbf{R}^+;U)$ *if* Ψ *is* L^∞*-well-posed or Reg-well-posed. It also has a unique* observation operator $C \in \mathcal{L}(X_1;Y)$*, which is determined by the fact that the state-output map* $\mathcal{C}$ *in* (2) *is given by (for almost all* $t \in \mathbf{R}^+$*)*

$$(\mathcal{C}x_0)(t) = C\mathcal{A}^t x_0, \qquad \forall x_0 \in X_1.$$

In other words, x is the *strong solution* of the equation $x'(t) = Ax(t) + Bu(t)$ with initial time zero, initial state x_0, and input function u.

The existence of a control operator B for $p < \infty$ is proved in Weiss [19], and the existence of an observation operator for all values of $p \in [1,\infty]$ is proved in Weiss [20]. The *Reg*-case is described in [22, Remark 5.9], without proof. In Weiss [21, Theorem 3.4] a counterexample appears which shows that (3) cannot in general be extended to all $u \in L^\infty_{\rm loc}(\mathbf{R}^+;U)$ in the L^∞-case. (See also [11] and [22, Remark 2.4].)

The control operator B is said to be *bounded* if the range of B lies in X, in which case $B \in \mathcal{L}(U;X)$. The observation operator C is said to be *bounded* if it is continuous with respect to the norm of X, i.e., if it can be extended to an operator in $\mathcal{L}(X;Y)$.

Proposition 2. *If* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *is* L^∞*-well-posed or Reg-well-posed, then the observation operator* C *is bounded. If* Ψ *is* L^1*-well-posed and* X *is reflexive, then the control operator* B *is bounded.*

This result is the union of [19, Theorem 4.8] and of [20, Proposition 6.5].

3. Regularity

As explained in Section 1, regular systems are a subclass of the well-posed linear systems. The concept of regularity was introduced in Weiss [18], and several equivalent characterizations of it are available, see [22], out of which one was mentioned in Section 1. We consider the following characterization to be the most basic one.

For each $v \in U$ we define v_+ to be the function

$$v_+(s) = \begin{cases} v, & s \ge 0, \\ 0, & s < 0. \end{cases}$$

Then $v_+ \in Reg(\mathbf{R}^+;U) \subset L^p_{\rm loc}(\mathbf{R}^+;U)$, so that $\mathcal{D}v_+$ belongs to $L^p_{\rm loc}(\mathbf{R}^+;Y)$.

Definition 3. *The well-posed linear system* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *is* regular *if for any* $v \in U$ *the limit*

$$Dv = \lim_{\tau \to 0+} \frac{1}{\tau} \int_0^\tau (\mathcal{D}v_+)(s)\, ds \qquad (4)$$

exists. In this case the operator $D \in \mathcal{L}(U;Y)$ *is called the* feedthrough operator *of* Ψ *(or of* $\mathcal{D}$*).*

If the above holds, we also say that $\mathcal{D}$ is regular. Note that $\mathcal{D}v_+$ is the so-called step response of Ψ and the limit in (4) is a strong Cesaro limit of order one. The fact that $D \in \mathcal{L}(U;Y)$ follows from the uniform boundedness principle. If $p < \infty$ or if $p = \infty$ and $u \in Reg_{\rm loc}(\mathbf{R}^+,U)$, then regularity implies that the output function can be expressed as in (1). More precisely, if we introduce the Λ-extension of C,

$$C_\Lambda x = \lim_{\alpha \to +\infty} C\alpha(\alpha I - A)^{-1}x$$

(see [20] and [23] for details), then for almost every $t \ge 0$, the function y from (2) is given by

$$y(t) = C_\Lambda x(t) + Du(t). \qquad (5)$$

(For $p = \infty$, C is bounded and hence $C_\Lambda = C$.)

The transfer function of Ψ (or of $\mathcal{D}$) is

$$G(s) = C_\Lambda(sI - A)^{-1}B + D.$$

This transfer function represents $\mathcal{D}$ in the frequency domain, in the following sense: If we denote the Laplace transforms of u and y by $\hat{u}$ and $\hat{y}$, and if we assume that $x_0 = 0$, then

$$\hat{y}(s) = G(s)\hat{u}(s)$$

for $\Re s$ sufficiently large; this is true for all $u \in L^p(\mathbf{R}^+;U)$ if $p < \infty$, and for all $u \in Reg(\mathbf{R}^+;U)$ if $p = \infty$. This formula need not be true for all $u \in L^\infty(\mathbf{R}^+;U)$ when $p = \infty$, see [21].

Proposition 3. *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a well-posed linear system on (Y, X, U) which has a bounded control operator B or a bounded observation operator C. Then $\mathcal{D}$ is regular.*

This follows from [22, Theorem 5.8 and Remark 5.9]. The following proposition is an easy consequence of Proposition 3 and of (5) (see also Remark 2.4 in [22]).

Proposition 4. *Let $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a well-posed linear system on (Y, X, U) with a bounded control operator B or with a bounded observation operator C. Then there is a unique $D \in \mathcal{L}(U; Y)$ such that, for (almost) every $t \in \mathbf{R}$, the output y of Ψ (defined in (2)) is given by (5). This holds for all $u \in L^p_{\mathrm{loc}}(\mathbf{R}^+; U)$ if Ψ is L^p-well-posed with $p < \infty$, and for all $u \in Reg_{\mathrm{loc}}(\mathbf{R}^+; U)$ if Ψ is L^∞-well-posed or Reg-well-posed.*

In particular, this means that every L^∞-well-posed linear system has a representation of the form (1), valid for regulated inputs.

By combining Propositions 2 and 3, we see that L^∞-well-posed linear systems are regular, and so are L^1-well-posed linear systems if the state space X is reflexive. In fact, we know that L^1-well-posed linear systems are regular even if their state space is not reflexive, if the output space Y is finite dimensional. This follows from the representation theorem for multipliers on L^1, see Brainerd and Edwards [2]. This theorem says that any shift-invariant operator $\mathcal{D}$ on $L^1(\mathbf{R})$ is a convolution with a bounded measure. In particular, the operator is causal if the measure is supported on $[0, \infty)$. The transfer function of $\mathcal{D}$ is, of course, the Laplace transform of this measure. Other applications of this result are given in [3].

The built-in regularity of the two extreme cases $p = 1$ and $p = \infty$ make them especially interesting in some applications. In addition, the uniform convergence that is used in the *Reg*-well-posed case can be very handy in the study of systems with an additional nonlinear part, for example a nonlinear feedback controller.

4. The LQ problem and a parabolic example

Our present interest in the *Reg*-case arose in our study [12, 14, 15], [24] of the LQ (linear quadratic) optimal control problem. There it is natural to work with $p = 2$, and to let U, X, and Y be Hilbert spaces. It is quite easy to show that the optimal solution to a coercive quadratic cost minimization problem can be written as a (closed loop) L^2-well-posed linear system; see [12, Theorem 27] or [14, Theorem 4.4]. The technique is roughly the following. First one uses a spectral factorization (or equivalently, an inner-outer factorization) to create an extra (state) feedback output for the original system, and then this output is connected back to the input to give a state feedback representation of the optimal solution.

The extended system and the closed loop system are always L^2-well-posed, but they need not be regular. In particular, the input-output map from the original input to the (open loop) state feedback output need not be be regular. The regularity of this input-output map is needed in the construction of the appropriate algebraic Riccati equation satisfied by the Riccati operator (the optimal cost operator), and in the proof of the fact that the optimal feedback operator can be computed from the Riccati operator. This is a significant restriction on the applicability of the results presented in [12], [14], and [24]. (An alternative approach, which is not based on a regularity assumption of this type, can be found in [7].)

There are some cases where the regularity of the extended system in the LQ problem is guaranteed; see the discussion in [4]. One of them is the following parabolic example studied in [15].

Let A generate an analytic semigroup $\mathcal{A}$ on a Hilbert space H. Choose some γ in the resolvent set of A. Then we can define the fractional powers $(\gamma I - A)^\alpha$, $\alpha \in \mathbf{R}$, in the standard way [9, Section 2.6]. We let H_α be the domain of $(\gamma I - A)^\alpha$, with norm $\|x\|_{H_\alpha} = \|(\gamma I - A)^\alpha x\|_H$. Then the restrictions of A to H_α for $\alpha > 0$ and the extensions of A to H_α for $\alpha < 0$ (which we still denote by A) generate analytic semigroups in H_α, for all $\alpha \in \mathbf{R}$. These semigroups are all similar to each other, and they commute with A^β for all $\beta \in \mathbf{R}$. We therefore denote all of them by the same letter $\mathcal{A}$. The generator of the semigroup $\mathcal{A}$ in H_α is then $A \in \mathcal{L}(H_{\alpha+1}; H_\alpha)$. See, e.g., [8], [9], or [15] for details.

This time we build a well-posed linear system $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ from its generating operators A, B, C, and D. The operator A was already presented above. We let $B \in \mathcal{L}(U; H_{\alpha_B})$, $C \in \mathcal{L}(H_{\alpha_C}; Y)$, and $D \in \mathcal{L}(U; Y)$. Here α_B and α_C are two fixed numbers satisfying $\alpha_C < \alpha_B + 1$. For each $x_0 \in H_{\alpha_B}$ and $u \in L^1_{c,\mathrm{loc}}(\mathbf{R}; U)$ we define

$$\begin{aligned} (\mathcal{B}u)(t) &= \int_{-\infty}^{t} \mathcal{A}^{t-s} B u(s)\, ds, && t \in \mathbf{R}, \\ (\mathcal{C}x_0)(t) &= C\mathcal{A}^t x_0, && t \in \mathbf{R}^+, \\ (\mathcal{D}u)(t) &= C(\mathcal{B}u)(t) + Du(t), && t \in \mathbf{R}. \end{aligned} \tag{6}$$

We claim that this results in an L^p-well-posed linear system for all $p \in [1, \infty]$:

Proposition 5. *Let A generate an analytic semigroup $\mathcal{A}$ in H, and let $B \in \mathcal{L}(U; H_{\alpha_B})$, $C \in \mathcal{L}(H_{\alpha_C}; Y)$, and $D \in \mathcal{L}(U; Y)$, where $\alpha_C < \alpha_B + 1$. Define $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ as in (6). Then, for each*

$p \in [1,\infty]$, $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is an L^p-well-posed linear system on (U,X,Y), where $X = H_\beta$ with $\alpha_C - 1/p < \beta < \alpha_B - 1/p + 1$ (we define $1/\infty = 0$). If $p = 1$, then we can also take $X = H_{\alpha_B}$, and if $p = \infty$ then we can also take $X = H_{\alpha_C}$.

The easy proof of this proposition is given in [15]. It is based on Young's inequality [17, p. 178] (the convolution of an L^p-function with an L^q-function belongs to L^r with $1/r = 1/p + 1/q - 1$) and the well-known fact [9, Theorem 6.13] that for each $\alpha \geq 0$, there exist a constant $K > 0$ and $\omega \in \mathbf{R}$ such that

$$\|A^\alpha \mathcal{A}^t\| \leq K t^{-\alpha} \mathrm{e}^{\omega t}, \quad t > 0,$$

where the norm represents the operator norm in any one of the spaces H_γ.

This result was used in [15] to prove the existence of a regular spectral factor for the factorization problem arising from a quadratic cost minimization problem for an exponentially stable parabolic system with control and observation operator B and C satisfying $CA^{-\alpha}B \in \mathcal{L}(U;Y)$ for some $\alpha < 1$ (the exponential stability of $\mathcal{A}$ implies that A is invertible). The idea is to work in three different state spaces $W \subset X \subset V$. The space X is chosen so that Ψ is L^2-well-posed on (U,X,V). For example, we may choose $X = H_\beta$, where $\beta = (\alpha_B + \alpha_C)/2$. The space W is chosen so that Ψ is *Reg*-well-posed on (U,W,Y). For example, $W = H_{\beta_C}$ will do. The space V is chosen so that Ψ is L^1-well-posed on V. For example, take $V = H_{\alpha_B}$. The L^2-well-posedness in the state space X is needed for the application of the L^2-theory in [12], [14], and [24]. The *Reg*-well-posedness in the state space W is needed in the proof of the regularity of the spectral factor, and the L^1-well-posedness in the state space V is needed in the proof of the regularity of the adjoint of the spectral factor. See [15] for details.

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