The State/Signal Resolvent Functions

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Outline of Talk

- Input/state/output systems in time and frequency domain
- The i/s/o resolvent set and i/s/o resolvent matrix
- State/signal systems in the time domain
- The s/s resolvent set and the characteristic node bundle
- Some examples
- The characteristic signal bundle

General I/S/O System

One way to model the dynamics of an i/s/o (input/state/output) system is to use an equation of the following form, where $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ is a closed linear operator:

$$\Sigma_{\text{iso}}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x^0. \tag{1}$$

$x(t) \in \mathcal{X}$ is the state,
$u(t) \in \mathcal{U}$ is the input,
$y(t) \in \mathcal{Y}$ is the output.
$\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces.
Different classes of i/s/o systems of this type are described in (Sta05).

Bounded I/S/O System

A general i/s/o system can be seen as an extension of a standard finite-dimensional i/s/o system. If S is bounded, the S can be written in block matrix form $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, and (1) becomes

$$\Sigma_{\mathrm{iso}}: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x^0. \qquad (2)$$

In this case we say that
A is the main operator,
B is the control operator,
C is the observation operator,
D is the feedthrough operator.
The case where A generates a C_0 semigroup and B, C, and D are bounded is described in the book (CZ95).

$$\Sigma_{\text{iso}}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x^0. \qquad ((1))$$

Definition

- By a classical future trajectory of Σ_{iso} we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ which satisfies (1) for all $t \in \mathbb{R}^+$, with x continuously differentiable with values in $\mathcal{X}$ and $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ continuous with values in $\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$.

The Frequency Domain

Let $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ be a classical future trajectory which is, for example, bounded. Multiplying the equation

$$\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x^0. \qquad (1)$$

by $\mathrm{e}^{-\lambda t}$ and integrating over $\mathbb{R}^+$ we find that the Laplace transforms $\begin{bmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{bmatrix}$ of $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ satisfy (since S is assumed to be closed)

$$\widehat{\Sigma}_{\rm iso}: \begin{cases} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \end{cases} \quad \Re\lambda > 0. \qquad (3)$$

- Let Ω be an open subset of $\mathbb{C}$. By a (frequency domain) Ω-trajectory of Σ we mean a quadruple $(\hat{x}, \hat{u}, \hat{y}, x^0)$, where $\hat{x}$, $\hat{u}$, and $\hat{y}$ are analytic functions in Ω and x^0 is a constant, which satisfy

$$\widehat{\Sigma}_{\rm iso}: \begin{cases} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \end{cases} \qquad \lambda \in \Omega. \tag{3}$$

On the previous page we have $\Omega = \mathbb{C}^+$. In the discrete time setting it is natural to choose $\Omega = \mathbb{D}$ (= unit disk).
Note that we have dropped the assumptions that $\hat{x}$, $\hat{u}$, and $\hat{y}$ are the Laplace transforms of some time-domain functions x, u, and y, and that x^0 is the value of the function x at zero.

Frequency Domain Inputs and Outputs

We arrived at the frequency domain equation

$$\widehat{\Sigma}_{\rm iso}: \begin{cases} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \end{cases} \quad \lambda \in \Omega, \tag{3}$$

by taking Laplace transforms in the time domain equation (1).
In the original time domain setting x^0 was the initial state, u was the input, x the "final" state, and y the output.
The analogous interpretation in the frequency domain would be to interpret x^0 and $\hat{u}$ as "given data" and $\hat{x}$ and $\hat{y}$ as "dependent data":

- x^0 and $\hat{u}(\lambda)$ should be "free" in the sense that x^0 can be an arbitrary vector in $\mathcal{X}$ and $\hat{u}$ can be an arbitrary analytic function in Ω with values in $\mathcal{U}$,
- $\hat{x}(\lambda)$ and $\hat{y}(\lambda)$ should be determined uniquely by x^0 and $\hat{u}(\lambda)$.

Definition

- A point $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(\Sigma)$ of Σ, or equivalently, to the i/s/o resolvent set $\rho_{i/s/o}(S)$ of S, if for every $x^0 \in \mathcal{X}$ and for every $\hat{u}(\lambda) \in \mathcal{U}$ there is a unique pair of vectors $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix}$ satisfying

$$\widehat{\Sigma}_{\rm iso}: \begin{cases} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \lambda\hat{x}(\lambda) - x^0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \end{cases} \qquad \lambda \in \Omega, \tag{3}$$

  and $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix}$ depends continuously on $\begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}$.

- The $\mathcal{L}(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix})$-valued matrix function $\widehat{\mathfrak{S}}\colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ with domain $\rho(\Sigma)$ which at the point λ maps $\begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}$ into $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix}$ is called the i/s/o resolvent matrix of Σ (or of S).

The I/S/O Resolvent Matrix

Since $\widehat{\mathfrak{S}}(\lambda) \in \mathcal{L}(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]; \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right])$ for every $\lambda \in \rho(\Sigma)$ this operator has a block matrix representation

$$\widehat{\mathfrak{S}}(\lambda) := \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}, \qquad \lambda \in \rho(\Sigma).$$

- $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$ is the i/s/o (input/state/output) resolvent matrix,
- $\widehat{\mathfrak{A}}$ is the s/s (state/state) resolvent function ($=$ standard resolvent),
- $\widehat{\mathfrak{B}}$ is the i/s (input/state) resolvent function ($=$ Gamma field),
- $\widehat{\mathfrak{C}}$ is the s/o (state/output) resolvent function,
- $\widehat{\mathfrak{D}}$ is the i/o (input/output) resolvent function ($=$ Weyl function),

The Classical Frequency Domain

In the classical case where $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ the equation (3) takes the form

$$\begin{aligned} \lambda\hat{x}(\lambda) - x^0 &= A\hat{x}(\lambda) + B\hat{u}(\lambda), \\ \hat{y}(\lambda) &= C\hat{x}(\lambda) + D\hat{u}(\lambda), \end{aligned} \qquad \lambda \in \Omega. \tag{4}$$

If $\lambda \in \rho(A)$, then we can solve $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$ in terms of $\left[\begin{smallmatrix} x^0 \\ \hat{u}(\lambda) \end{smallmatrix}\right]$ to get

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x^0 \\ \hat{u}(\lambda) \end{bmatrix}, \qquad \lambda \in \Omega, \tag{5}$$

where

$$\begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} = \begin{bmatrix} (\lambda - A)^{-1} & (\lambda - A)^{-1}B \\ C(\lambda - A)^{-1} & C(\lambda - A)^{-1}B + D \end{bmatrix}, \qquad \lambda \in \Omega. \tag{6}$$

Theorem

Let $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ *be an operator with dense domain. Then* $\rho_{i/s/o}(S) \neq \emptyset$ *if and only if* S *is of the following type.*

Definition

An operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ is an operator node (in the sense of (Sta05)) if it satisfies the following four conditions:

1. S is closed and $\mathrm{dom}\,(S)$ is dense in $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$,
2. $\rho(A) \neq \emptyset$, where A is the main operator of S, i.e., $Ax = \begin{bmatrix}1_{\mathcal{X}} & 0\end{bmatrix} S \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right]$ with $\mathrm{dom}\,(A) = \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right] \in \mathrm{dom}\,(S)\right\}$ ("top left corner of S").
3. $\begin{bmatrix}1_{\mathcal{X}} & 0\end{bmatrix} S$ (= the "top half" of S) can be extended to a bounded linear operator $\begin{bmatrix}A_{-1} & B\end{bmatrix} : \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \mathcal{X}_{-1}$, where $\mathcal{X}_{-1}$ is the so called extrapolation space induced by A.
4. $\mathrm{dom}\,(S) = \left\{\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \left[\begin{smallmatrix}\mathcal{U}\\ \mathcal{Y}\end{smallmatrix}\right] \;\middle|\; A_{-1}x + Bu \in \mathcal{X}\right\}$.

The Resolvent Identity

Lemma

- *The i/s/o resolvent matrix* $\widehat{\mathfrak{S}} = \begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$ *is analytic in* $\rho(\Sigma)$.
- $\widehat{\mathfrak{S}}$ *satisfies the i/s/o resolvent identity*

$$\widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = (\mu - \lambda) \begin{bmatrix} \widehat{\mathfrak{A}}(\mu) \\ \widehat{\mathfrak{C}}(\mu) \end{bmatrix} \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \end{bmatrix} \tag{7}$$

  for all μ, $\lambda \in \rho(\Sigma)$.
- *If* $\rho_{i/s/o}(S) \neq \emptyset$, *then* $\rho_{i/s/o}(S) = \rho(A)$, *where* A *is the main operator of* S.

Thus in particular, $\rho_{i/s/o}(S) \neq \emptyset$, then $\rho(A) \neq \emptyset$. However, $\rho(A) \neq \emptyset \nRightarrow \rho_{i/s/o}(S) \neq \emptyset$.

Example: Electrical Circuit

One option to model the dynamics of an electrical circuit with lumped elements is to use a finite-dimensional i/s/o system.

- The state $x(t)$ is an N-vector whose components are the currents in the coils and the voltages over the capacitors.
- If the circuit has M terminals, then we can, e.g., use the currents entering these terminals as inputs, and the voltages over the terminals as the outputs.
- The equation (1) describing the dynamics of the system can be deribed from the Kirchoff's and Ohm's laws.

However, we could just as well have picked the voltages to be the inputs and the currents to be the outputs. This would give a different i/s/o system, but the underlying physical system remains the same! Thus, every electrical circuit can be used to construct an infinite family of i/s/o systems (by choosing different combinations of voltages and currents as inputs and outputs.

Is there a simple "equation" which describes the curcuit itself (instead of an infinite family of i/s/o systems)?

Example: A Boundary Control System

A special case of an infinite-dimensional i/s/o system is the following boundary control system:

$$\Sigma_{\mathrm{iso}}\colon \begin{cases} \dot{x}(t) = Lx(t), \\ \Gamma_0 x(t) = u(t), \qquad t \in \mathbb{R}^+, \quad x(0) = x^0. \\ \Gamma_1 x(t) = y(t), \end{cases} \tag{8}$$

Here L is, e.g., a partial differential operator in some Lipschitz domain in $\mathbb{R}^n$, and Γ_0 and Γ_1 are two boundary mappings, e.g., Γ_0 = Neumann trace and Γ_1 = Dirichlet trace. See (Sta05) for details. Above we may interpret u as the input and y the output, or the other way around. Or we could replace Γ_0 and Γ_1 by some other boundary mappings. Different choices of inputs and outputs lead completely different i/s/o system of the type (1). Thus, to every boundary control system of the type (8) there corresponds an infinite family of i/s/o systems.

Is there a simple way to describe the boundary control system itself (instead of using an infinite family of i/s/o systems)?

Boundary Control System: I/S/O-Free Formulation

In the case of the boundary control system (8) the solution is obvious: We simply combine the two variables u and y into a common "interaction" signal $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ which contains both the input and the output, define $\Gamma = \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \end{bmatrix}$, and write (8) in the form

$$\Sigma_{\mathrm{iso}}: \begin{cases} \dot{x}(t) = Lx(t), \\ \Gamma x(t) = w(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x^0. \tag{9}$$

In the case of the electrical circuit

$$\Sigma_{\mathrm{iso}}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x^0. \tag{1}$$

the solution is less obvious, but such a solution still exists.

$$\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \dot x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \qquad (1)$$

First s/s formulation: Write $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, and move the output equation into the domain of a new generator $F\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \to \mathcal{X}$ (whose domain is no longer dense in $\mathcal{W}$):

$$\Sigma\colon \begin{cases} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \mathrm{dom}\,(F), \\ \dot x(t) = F\left(\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}\right), \end{cases} \qquad t \in \mathbb{R}^+, \qquad (10)$$

$$\mathrm{dom}\,(F) = \left\{ \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S),\ y_0 = \begin{bmatrix} 0 & 1_{\mathcal{Y}} \end{bmatrix} S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \right\},$$

$$F \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right].$$

How to Go from an I/S/O System to a S/S System?

$$\Sigma_{\text{iso}}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{1}$$

Second s/s formulation: Use graph Representation of (1), $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]$, $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$:

$$\Sigma: \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+. \tag{11}$$

where the generating subspace V is the (reordered) graph of S (or of F):

$$\begin{aligned} V &= \left\{ \begin{bmatrix} z_0 \\ x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \mathfrak{K} \;\middle|\; \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \operatorname{dom}(S), \; \left[\begin{smallmatrix} z_0 \\ y_0 \end{smallmatrix}\right] = S \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \right\} \\ &= \left\{ \begin{bmatrix} z_0 \\ x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \mathfrak{K} \;\middle|\; \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \in \operatorname{dom}(F), \; z_0 = F \begin{bmatrix} x_0 \\ \left[\begin{smallmatrix} u_0 \\ y_0 \end{smallmatrix}\right] \end{bmatrix} \right\}. \end{aligned}$$

Classical Future Trajectories

$$\Sigma_{\rm iso}: \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}\,(S), \\ \begin{bmatrix} \dot x(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \qquad t \in \mathbb{R}^+. \tag{1}$$

$$\Sigma: \begin{cases} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \in \mathrm{dom}\,(F), \\ \dot x(t) = F\left(\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}\right), \end{cases} \qquad t \in \mathbb{R}^+, \tag{10}$$

$$\Sigma: \begin{bmatrix} \dot x(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+. \tag{11}$$

- A classical future trajectory of (10) or (11) is a pair of continuous functions (x, w), with x continuously differentiable, which satisfies (10) or (11).
- (u, x, y) is a classical future trajectory of the i/s/o system $\Sigma_{\rm iso}$ if and only if $\left(x, \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]\right)$ is a classical future trajectory of the corresponding s/s system Σ.

S/S Systems in the Frequency Domain

$$\Sigma\colon \begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \tag{11}$$

If x, $\dot{x}$, and $w$in (11) are Laplace transformable, then it follows from (11) (since we assume V to be closed) that the Laplace transforms $\hat{x}$ and $\hat{w}$ of x and w y satisfy

$$\widehat{\Sigma}\colon \begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V \tag{12}$$

(proof: multiply by $\mathrm{e}^{-\lambda t}$ and integrate by parts in the $\dot{x}$-component.)

The Characteristic Node Bundle

$$\widehat{\Sigma}_{\rm iso}: \begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V. \tag{13}$$

This formula can be rewritten in the form

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) := \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V. \tag{14}$$

Definition

The family of subspaces $\widehat{\mathfrak{E}} : \{\widehat{\mathfrak{E}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathfrak{K} = \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ is called the characteristic node bundle. We refer to each of the subspaces $\widehat{\mathfrak{E}}(\lambda)$ as the fiber of $\widehat{\mathfrak{E}}$ att the point $\lambda \in \mathbb{C}$.

Thus, $\widehat{\mathfrak{E}}$ is an "analytic subspace-valued function" defined on $\mathbb{C}$.

Recall: Let $\Sigma_{\rm iso}$ be an i/s/o representation of Σ, and split the i/o signal w into $w = u + y$, where $u \in \mathcal{U}$ is the input and $y \in \mathcal{Y}$ is the output. Then

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \tag{14}$$

if and only if

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in V. \tag{13}$$

if and only if

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \qquad \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \mathrm{dom}\,(S)\,, \tag{3}$$

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \tag{14}$$

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} = S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \qquad \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \mathrm{dom}\,(S), \tag{3}$$

If $\lambda \in \rho_{\mathrm{iso}}(S)$, then

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \widehat{\mathfrak{S}}(\lambda) \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix},$$

which can be rewritten in the form

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \begin{bmatrix} \hat{u}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \begin{bmatrix} 0 \\ \widehat{\mathfrak{C}}(\lambda) \end{bmatrix} & \begin{bmatrix} 1_{\mathcal{U}} \\ \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix}.$$

Here $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ can be arbitrary, and we get (see next slide)

I/S/O Interpretation of the Characteristic Node Bundle

Lemma

Let $\Sigma_{\rm iso}$ be an i/s/o representation of the s/s system Σ, and suppose that $\lambda \in \rho(\Sigma_{\rm iso})$. Then the fiber $\widehat{\mathfrak{E}}(\lambda)$ of the characteristic node bundle $\widehat{\mathfrak{E}}$ at λ has the representation

$$\widehat{\mathfrak{E}}(\lambda) = \operatorname{im}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \begin{bmatrix} 0 \\ \widehat{\mathfrak{C}}(\lambda) \end{bmatrix} & \begin{bmatrix} 1_{\mathcal{U}} \\ \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \end{bmatrix}\right) \tag{15}$$

Note that this can be interpreted as a graph representation of $\widehat{\mathfrak{E}}(\lambda)$ over the first copy of $\mathcal{X}$ and the input space $\mathcal{U}$.

The Resolvent Set of a State/Signal System

Suppose that $\lambda \in \rho(\Sigma_{\rm iso})$ for some i/s/o representation $\Sigma_{\rm iso}$ of Σ. Then $\widehat{\mathfrak{E}}(\lambda)$ has the following properties:

1. $\begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) \Rightarrow x = 0$;
2. For every $z \in \mathcal{X}$ there exists some $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda)$.
3. The projection of $\widehat{\mathfrak{E}}(\lambda)$ onto its first and third components is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

Definition

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node with node bundle $\widehat{\mathfrak{E}}$. Then the *resolvent set* $\rho(\Sigma)$ of Σ consists of all those points $\lambda \in \mathbb{C}$ for which conditions (1)–(3) above hold.

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Theorem

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. Then $\rho(\Sigma)$ is the union of the resolvent sets of all i/s/o representations of Σ.

Example: Unbounded Block Matrix Operator

Example

Let $\mathcal{X}$ be a Hilbert space, and let T be a closed unbounded operator in $\mathcal{X}$ with a dense domain $\mathrm{dom}\,(T) \neq \mathcal{X}$ and a nonempty resolvent set, and define the operators $S_i \colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right]$, $i = 1, 2, \ldots, 16$, by

$$\begin{aligned}
S_1 &= \left[\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right], & S_2 &= \left[\begin{smallmatrix} 0 & 0 \\ T & 0 \end{smallmatrix}\right], & S_3 &= \left[\begin{smallmatrix} T & 0 \\ T & 0 \end{smallmatrix}\right], \\
S_4 &= \left[\begin{smallmatrix} 0 & T \\ 0 & 0 \end{smallmatrix}\right], & S_5 &= \left[\begin{smallmatrix} 0 & 0 \\ 0 & T \end{smallmatrix}\right], & S_6 &= \left[\begin{smallmatrix} 0 & T \\ 0 & T \end{smallmatrix}\right], \\
S_7 &= \left[\begin{smallmatrix} T & T \\ 0 & 0 \end{smallmatrix}\right], & S_8 &= \left[\begin{smallmatrix} 0 & 0 \\ T & T \end{smallmatrix}\right], & S_9 &= \left[\begin{smallmatrix} T & T \\ T & T \end{smallmatrix}\right], \\
S_{10} &= \left[\begin{smallmatrix} T & 0 \\ 0 & T \end{smallmatrix}\right], & S_{11} &= \left[\begin{smallmatrix} 0 & T \\ T & 0 \end{smallmatrix}\right], & S_{12} &= \left[\begin{smallmatrix} T & T \\ T & -T \end{smallmatrix}\right], \\
S_{13} &= \left[\begin{smallmatrix} T & T \\ T & 0 \end{smallmatrix}\right], & S_{14} &= \left[\begin{smallmatrix} T & T \\ 0 & T \end{smallmatrix}\right], \\
S_{15} &= \left[\begin{smallmatrix} T & 0 \\ T & T \end{smallmatrix}\right], & S_{16} &= \left[\begin{smallmatrix} 0 & T \\ T & T \end{smallmatrix}\right].
\end{aligned} \tag{16}$$

with their natural domains (see next page).

Example: The Domains

Example

The domains of the operators T_i, $i = 1, \dots, 16$ are

$$\begin{aligned} \operatorname{dom}(S_1) &= \operatorname{dom}(S_2) = \operatorname{dom}(S_2) = \begin{bmatrix} \operatorname{dom}(T) \\ \mathcal{X} \end{bmatrix}, \\ \operatorname{dom}(S_4) &= \operatorname{dom}(S_5) = \operatorname{dom}(S_6) = \begin{bmatrix} \mathcal{X} \\ \operatorname{dom}(T) \end{bmatrix}, \\ \operatorname{dom}(S_7) &= \operatorname{dom}(S_8) = \operatorname{dom}(S_9) = \left\{ \left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right] \;\middle|\; z_1 + z_2 \in \operatorname{dom}(T) \right\} \\ \operatorname{dom}(S_i) &= \begin{bmatrix} \operatorname{dom}(T) \\ \operatorname{dom}(T) \end{bmatrix}, \qquad 10 \le i \le 16. \end{aligned} \tag{17}$$

All of these $\Sigma^i_{\text{iso}} = (S_i, \mathcal{X}, \mathcal{X}, \mathcal{X})$ are i/s/o systems of the type (1).

Example

Only $\Sigma^1_{\rm iso}$, $\Sigma^3_{\rm iso}$, $\Sigma^7_{\rm iso}$, and $\Sigma^9_{\rm iso}$ have nonempty resolvent sets. These four nodes all have the same main operator T, and $\rho(\Sigma^1_{\rm iso}) = \rho(\Sigma^3_{\rm iso}) = \rho(\Sigma^7_{\rm iso}) = \rho(\Sigma^9_{\rm iso}) = \rho(T)$, and their i/s/o resolvent matrices are given by

$$
\begin{aligned}
\widehat{\mathfrak{S}}^1_{\rm iso}(\lambda) &= \begin{bmatrix} (\lambda - T)^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\
\widehat{\mathfrak{S}}^3_{\rm iso}(\lambda) &= \begin{bmatrix} (\lambda - T)^{-1} & 0 \\ T(\lambda - T)^{-1} & 0 \end{bmatrix}, \\
\widehat{\mathfrak{S}}^7_{\rm iso}(\lambda) &= \begin{bmatrix} (\lambda - T)^{-1} & T(\lambda - T)^{-1} \\ 0 & 0 \end{bmatrix}, \\
\widehat{\mathfrak{S}}^9_{\rm iso}(\lambda) &= \begin{bmatrix} (\lambda - T)^{-1} & T(\lambda - T)^{-1} \\ T(\lambda - T)^{-1} & \lambda T(\lambda - T)^{-1} \end{bmatrix},
\end{aligned}
\tag{18}
$$

respectively, for $\lambda \in \rho(T)$.

Example: State/Signal Spectra

Example

Let $\Sigma_i = (V_i, \mathcal{X}, \mathcal{W})$, $i = 1, 2, \ldots, 16$ be the s/s node induced by the i/s/o node $\Sigma^i_{\rm iso}$ in Example 11. Then

1. Σ_i is bounded (and hence $\rho(\Sigma_i) \neq \emptyset$) for $i =$ 5, 6, 8, and 9;
2. Σ_i is not bounded but $\rho(\Sigma_i) \neq \emptyset$ for $i =$1, 3, 7, 10, 12, 13, 14, and 15;
3. $\rho(T) \subset \rho(\Sigma_i)$ for $i =$ 1, 3, 7, 9, 10, 12, 13, 14, and 15, and $\mathbb{C} \setminus \{0\} \subset \rho(\Sigma_i)$ for $i =$ 5, 6, 8;
4. Σ_i is unbounded and $\rho(\Sigma_i) = \emptyset$ for $i =$ 2 and 4;
5. Σ_{11} is not bounded, and $\rho(\Sigma_i) \neq \emptyset$ if and only if T^2 is closed (in which case $\mathbb{C} \setminus \{0\} \subset \rho(\Sigma_{11})$);
6. Σ_{16} is not bounded, and $\rho(\Sigma_i) \neq \emptyset$ if and only if $\frac{1}{\lambda} T^2 + T$ is closed for some $\lambda \neq 0$ (in which case all $\lambda \neq 0$ for which $\frac{1}{\lambda} T^2 + T$ belongs to $\rho(\Sigma_{16})$).

Example: Characteristic Signal Bundles

The values of the different characteristic signal bundles are listed below under the additional assumption that $\lambda \in \rho(T)$ if $i = 1, 3, 7, 9, 10, 12, 13, 14, 15$, and $\lambda \neq 0$ if $i = 2, 4, 5, 6, 8, 11$, and 16 (in other words, λ belong to the resolvent set of the formal main operator (which is either T or 0) of the i/s/o node Σ^i_{iso}:

$$
\begin{aligned}
&\widehat{\mathfrak{F}}_1(\lambda) = \widehat{\mathfrak{F}}_2(\lambda) = \widehat{\mathfrak{F}}_3(\lambda) = \widehat{\mathfrak{F}}_7(\lambda) = \begin{bmatrix} \{0\} \\ \mathcal{X} \end{bmatrix}, \qquad \widehat{\mathfrak{F}}_4(\lambda) = \begin{bmatrix} \{0\} \\ \mathrm{dom}(T) \end{bmatrix}, \\
&\widehat{\mathfrak{F}}_5(\lambda) = \widehat{\mathfrak{F}}_6(\lambda) = \widehat{\mathfrak{F}}_8(\lambda) = \widehat{\mathfrak{F}}_{10}(\lambda) = \widehat{\mathfrak{F}}_{14}(\lambda) = \widehat{\mathfrak{F}}_{15}(\lambda) = \mathrm{graph}\,(T), \\
&\widehat{\mathfrak{F}}_9(\lambda) = \mathrm{graph}\left(\lambda T(\lambda - T)^{-1}\right), \\
&\widehat{\mathfrak{F}}_{11}(\lambda) = \mathrm{graph}\left(\tfrac{1}{\lambda} T^2\right), \\
&\widehat{\mathfrak{F}}_{12}(\lambda) = \mathrm{graph}\left(\lambda T(\lambda - T)^{-1} - 2T\right), \\
&\widehat{\mathfrak{F}}_{13}(\lambda) = \mathrm{graph}\left(\lambda T(\lambda - T)^{-1} - T\right), \\
&\widehat{\mathfrak{F}}_{16}(\lambda) = \mathrm{graph}\left(\tfrac{1}{\lambda} T^2 + T\right).
\end{aligned} \tag{19}
$$

Frequency Domain Input/Output Behavior

- In i/s/o systems theory one is often intersted in the "pure i/o behavior", which one gets by "ignoring the state". More precisely, one takes the inital state $x_0 = 0$, and only looks at the relationship between the input u and the output y, ignoring the state x.
- If we in the frequency domain setting take $x_0 = 0$ and ignore $\hat{x}$, then the full frequency domain relation

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix}$$

  is replaced by the i/o relation $\hat{y}(\lambda) = \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda)$, where $\widehat{\mathfrak{D}}(\lambda)$ is the i/o resolvent function.
- The same procedure can be carried out in the case of a s/s system: We take $x_0 = 0$ and ignore the values of $\hat{x}(\lambda)$. (next slide)

Recall the full frequency domain s/s signal behavior:

$$\begin{bmatrix} x_0 \\ \hat{x}(\lambda) \\ \hat{w}(\lambda) \end{bmatrix} \in \widehat{\mathfrak{E}}(\lambda) := \begin{bmatrix} -1_{\mathcal{X}} & \lambda & 0 \\ 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 1_{\mathcal{W}} \end{bmatrix} V. \qquad (14)$$

Taking $x_0 = 0$ and ignoring the value of $\hat{x}(\lambda)$ we see that $\hat{w}(\lambda) \in \widehat{\mathfrak{F}}(\lambda)$, where

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \;\middle|\; \begin{bmatrix} 0 \\ z \\ w \end{bmatrix} \in \mathfrak{E}(\lambda) \text{ for some } z \in \mathcal{X} \right\}. \qquad (20)$$

The Characteristic Signal Bundle

$$\widehat{\mathfrak{F}}(\lambda) = \left\{ w \in \mathcal{W} \;\middle|\; \begin{bmatrix} 0 \\ z \\ w \end{bmatrix} \in \mathfrak{E}(\lambda) \text{ for some } z \in \mathcal{X} \right\}. \qquad (20)$$

Definition

The family of subspaces $\widehat{\mathfrak{F}} : \{\widehat{\mathfrak{F}}(\lambda) \mid \lambda \in \mathbb{C}\}$ of $\mathcal{W}$ is called the characteristic signal bundle. We refer to each of the subspaces $\widehat{\mathfrak{F}}(\lambda)$ as the fiber of $\widehat{\mathfrak{F}}$ at the point $\lambda \in \mathbb{C}$.

Whereas the characteristic node bundle $\mathfrak{E}$ is analytic everywhere in $\mathbb{C}$ (i.e., the fibers depend on λ in an analytic way), the same is not true for the singal bundle $\widehat{\mathfrak{F}}$. Even the dimension of the fibers $\widehat{\mathfrak{F}}(\lambda)$ may change from one point to another.

Lemma

The characteristic signal bundle $\widehat{\mathfrak{F}}$ is analytic in $\rho(\Sigma)$.

Where to Read What?

- An "easy" introduction to what I have been talking about here is written down in (Sta14).
- Proofs are given in (AS14).
- The connection to boundary triplets and generalized boundary triplets is explained in (AKS12a; AKS12b).

[AKS12a] Damir Z. Arov, Mikael Kurula, and Olof J. Staffans, *Boundary control state/signal systems and boundary triplets*, Operator Methods for Boundary Value Problems, Cambridge University Press, 2012.

[AKS12b] ———, *Passive state/signal systems and conservative boundary relations*, Operator Methods for Boundary Value Problems, Cambridge University Press, 2012.

[AS14] Damir Z. Arov and Olof J. Staffans, *Linear stationary systems in continuous time*, 2014, Book manuscript, available at http://users.abo.fi/staffans/publ.html.

[CZ95] Ruth F. Curtain and Hans Zwart, *An introduction to infinite-dimensional linear systems theory*, Springer-Verlag, New York, 1995.

[Sta05] Olof J. Staffans, *Well-posed linear systems*, Cambridge University Press, Cambridge and New York, 2005.

[Sta14] ———, *The stationary state signal systems story*, Submitted, 2014.