The i/s/o resolvent set and the i/s/o resolvent matrix of an i/s/o system in continuous time

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Outline of Talk

- Resolvents of (multi-valued) operators
- I/s/o resolvents of i/s/o (pseudo-)systems
- I/s/o pseudo-resolvents and resolvent linear systems
- Extension of time domain notions to the frequency domain

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- **Resolvents of (multi-valued) operators**
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The Resolvent of an Operator

- Every generator A of a C_0 semigroup defines a linear autonomous dynamical system in continuous time:

$$\Sigma\colon \begin{cases} x(t) \in \operatorname{dom}(A), \\ \dot{x}(t) = Ax(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{1}$$

  We call x a classical trajectory of (1) on $\mathbb{R}^+$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$ and (1) holds.
- By taking Laplace transforms in (1) we see that the Laplace transform $\hat{x}$ of x satisfies

$$\lambda\hat{x}(\lambda) - x_0 = A\hat{x}(\lambda), \tag{2}$$

  for sufficiently large $\Re\lambda$ (proof: multiply by $\mathrm{e}^{-\lambda t}$ and integrate by parts.)
- By definition, λ belongs to the resolvent set $\rho(A)$ of A if it is true for every $x_0 \in \mathcal{X}$ that the equation (2) has a unique solution $\hat{x}(\lambda)$, and this solution depends continuously on x_0.
- $\hat{x}(\lambda) = (\lambda - A)^{-1}x_0$, $\lambda \in \rho(A)$. Here $(\lambda - A)^{-1} \in \mathcal{L}(\mathcal{X})$.

Linear Multi-Valued Operators (= Linear Relations)

- By a multi-valued linear operator in $\mathcal{X}$ we mean a multi-valued (set-valued) map $A\colon \mathcal{X} \to \mathcal{X}$ whose graph

$$\mathrm{graph}\,(A) = \{\left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right] \mid y \in Ax\}$$

  is a linear subspace of $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right]$.
- A is closed if $\mathrm{graph}\,(A)$ is closed.
- $\mathrm{dom}\,(A) = \{x \in \mathcal{X} \mid Ax \neq \emptyset\}$.

The Resolvent of a Multi-Valued Operator

- Also every closed multi-valued linear operator A generates a linear autonomous dynamical system in continuous time:

$$\Sigma\colon \begin{cases} x(t) \in \mathrm{dom}\,(A)\,, \\ \dot{x}(t) \in Ax(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{3}$$

  We call x a classical trajectory of (3) on $\mathbb{R}^+$ if $x \in C^1(\mathbb{R}^+; \mathcal{X})$ and (3) holds. (This system may, of course be rather trivial.)
- By taking Laplace transforms in (3) we see that for all those $\lambda \in \mathbb{C}$ for which the Laplace transforms of x and $\dot{x}$ converge we have

$$\lambda \hat{x}(\lambda) - x_0 \in A\hat{x}(\lambda) \tag{4}$$

  (proof: multiply by $\mathrm{e}^{-\lambda t}$ and integrate by parts).
- By definition, λ belongs to the resolvent set $\rho(A)$ of A if it is true for every $x_0 \in \mathcal{X}$ that the inclusion (4) has a unique solution $\hat{x}(\lambda)$, and this solution depends continuously on x_0.
- $\hat{x}(\lambda) = (\lambda - A)^{-1}x_0$, $\lambda \in \rho(A)$. Here $(\lambda - A)^{-1} \in \mathcal{L}(\mathcal{X})$.

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Linear I/S/O Pseudo-Systems in Continuous Time

- By adding inputs and outputs we get a linear i/s/o system in continuous time. In the multi-valued case such a system may be written in the form

$$\Sigma\colon \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (5)$$

  Here $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is a closed linear multi-valued operator ($\mathcal{X}$ = state space, $\mathcal{U}$ = input space, $\mathcal{Y}$ = output space). By a classical trajectory of (5) we mean a triple of functions $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ where $x \in C^1(\mathbb{R}^+; \mathcal{X})$, $u \in C(\mathbb{R}^+; \mathcal{U})$, and $y \in C(\mathbb{R}^+; \mathcal{Y})$ which satisfies (5).
- The single-valued case looks the same, except that "$\in$" in the bottom line is replaced by "$=$".
- i/s/o pseudo-system: S may be multi-valued,
  i/s/o system: S is assumed to be single-valued, dense domain.

The Resolvent Set of a Linear I/S/O Pseudo-System

If x, $\dot{x}$, u, and y in (5) are Laplace transformable, then it follows from (5) (since we assume S to be closed) that the Laplace transforms $\hat{x}$, $\hat{u}$, and $\hat{y}$ of x, u, and y satisfy

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} \in S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \tag{6}$$

(proof: multiply by $\mathrm{e}^{-\lambda t}$ and integrate by parts.)

Definition

1. $\lambda \in \mathbb{C}$ belongs to the resolvent set $\rho(\Sigma)$ of Σ (or alternatively, to the i/s/o resolvent set $\rho_{\mathrm{iso}}(S)$ of S) if for every $x_0 \in \mathcal{X}$ and for every $\hat{u}(\lambda) \in \mathcal{U}$ there is a unique pair of vectors $\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ satisfying (6).
2. For each $\lambda \in \rho(\Sigma)$ we define the i/s/o resolvent matrix $\widehat{\mathfrak{S}}(\lambda)$ of Σ at λ to be the linear operator $\begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix} \to \begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix}$.

- It follows from the closed graph theorem that the i/s/o resolvent matrix $\widehat{\mathfrak{S}}(\lambda)$ is a bounded linear operator for each $\lambda \in \rho(\Sigma)$.
- In particular, this implies that $\widehat{\mathfrak{S}}(\lambda)$ has a block matrix representation

$$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}, \quad \lambda \in \rho(\Sigma),$$

  where each of the components $\widehat{\mathfrak{A}}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$, $\widehat{\mathfrak{C}}(\lambda)$, $\widehat{\mathfrak{D}}(\lambda)$ is a bounded linear operator.
- $\widehat{\mathfrak{S}}(\lambda)$ is actually even an analytic function of λ, hence so are $\widehat{\mathfrak{A}}(\lambda)$, $\widehat{\mathfrak{B}}(\lambda)$, $\widehat{\mathfrak{C}}(\lambda)$, and $\widehat{\mathfrak{D}}(\lambda)$.
- $\widehat{\mathfrak{S}}$ satisfies the i/s/o resolvent identity

$$\widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = (\mu - \lambda) \begin{bmatrix} \widehat{\mathfrak{A}}(\mu) \\ \widehat{\mathfrak{C}}(\mu) \end{bmatrix} \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \end{bmatrix} \tag{7}$$

  for all μ, $\lambda \in \rho_{\mathrm{iso}}(S)$.

Components of the I/S/O Resolvent Matrix

Definition

The components $\widehat{\mathfrak{A}}$, $\widehat{\mathfrak{B}}$, $\widehat{\mathfrak{C}}$, and $\widehat{\mathfrak{D}}$ of the i/s/o resolvent matrix $\widehat{\mathfrak{S}}$ are called as follows:

1. $\widehat{\mathfrak{A}}$ is the *s/s (state/state) resolvent function* of Σ,
2. $\widehat{\mathfrak{B}}$ is the *i/s (input/state) resolvent function* of Σ,
3. $\widehat{\mathfrak{C}}$ is the *s/o (state/output) resolvent function* of Σ,
4. $\widehat{\mathfrak{D}}$ is the *i/o (input/output) resolvent function* of Σ,

- $\widehat{\mathfrak{A}}$ is the(standard) resolvent of the main operator A of S (both in the single-valued case and the multi-valued case).
- The i/o resolvent function $\widehat{\mathfrak{D}}$ is know as the under different names, such as "transfer function", or "characteristic function", or "Weyl function".
- In operator theory the i/s resolvent function $\widehat{\mathfrak{B}}$ is sometimes called the Γ-field.

Regular I/S/O Systems and Pseudo-Systems

Definition

1. By a regular i/s/o pseudo-system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean an i/s/o pseudo-system of the type (5) where the (possibly multi-valued) generating operator S has a nonempty i/s/o resolvent set (but its domain need not be dense in $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$).
2. By a regular i/s/o system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean an i/s/o pseudo-system whose generator S is single-valued and has dense domain in $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ (plus a nonempty i/s/o resolvent set).

It turns out that regular i/s/o systems have been known for a long time under a different name:

Theorem

An i/s/o system Σ is regular if and only if its generating operator S is an operator node Moreover, if Σ is regular, then $\rho(\Sigma) = \rho(A)$, where A is the main operator of S. (See next page for definitions!)

Operator Nodes

Definition

By an operator node on a triple of Hilbert spaces $(\mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean a (possibly unbounded) linear operator $S\colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ with the following properties. We define the main operator $A\colon \mathrm{dom}\,(A) \to \mathcal{X}$ of S by $Ax = P_{\mathcal{X}} S \begin{bmatrix} x \\ 0 \end{bmatrix}$ for all $x \in \mathrm{dom}\,(A) := \left\{ x \in \mathcal{X} \,\middle|\, \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathrm{dom}\,(S) \right\}$, and require the following conditions to hold:

1. S is closed as an operator from $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ to $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$.
2. $P_{\mathcal{X}} S$ is closed as an operator from $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}$ to $\mathcal{X}$.
3. $\mathrm{dom}\,(A)$ is dense in $\mathcal{X}$ and $\rho(A) \neq \emptyset$.
4. For every $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ with $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{dom}\,(S)$.

We call S a system node if, in addition, A generates a C_0 semigroup.

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I/S/O Pseudo-Resolvents

Recall that the i/s/o resolvent $\widehat{\mathfrak{S}}$ of an i/s/o pseudo-system satisfies the i/s/o resolvent identity

$$\widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = (\mu - \lambda) \begin{bmatrix} \widehat{\mathfrak{A}}(\mu) \\ \widehat{\mathfrak{C}}(\mu) \end{bmatrix} \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \end{bmatrix} \tag{7}$$

for all μ, $\lambda \in \rho_{\mathrm{iso}}(S)$.

Definition

Let Ω be an open subset of the complex plane $\mathbb{C}$. An analytic $\mathcal{L}(\left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{Y} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right])$-valued function $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ defined in Ω is called an *i/s/o pseudo-resolvent in* $(\mathcal{X}, \mathcal{U}, \mathcal{Y}; \Omega)$ if it satisfies the identity (7) for all μ, $\lambda \in \Omega$.

Thus, the i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ of a regular i/s/o pseudo-system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is an i/s/o pseudo-resolvent in $\rho(\Sigma)$.

- In (Opm05) Mark Opmeer makes systematic use of the notion of an i/s/o pseudo-resolvent. He instead uses the name resolvent linear system for this notion.
- In the same article he also investigates what can be said about time domain trajectories (in the distribution sense) of resolvent linear systems satisfying some additional conditions.
- One of these additional set of conditions is that Ω should contain some right-half plane and that $\widehat{\mathfrak{S}}$ should satisfy a polynomial growth bound in this right-half plane.

Every I/S/O Pseudo-Resolvent is an I/S/O Resolvent!

Theorem

Let Ω be an open subset of the complex plane $\mathbb{C}$. Then every i/s/o pseudo-resolvent $\widehat{\mathfrak{S}}$ in $(\mathcal{X},\mathcal{U},\mathcal{Y};\Omega)$ is the restriction to Ω of the i/s/o resolvent of some i/s/o pseudo-system $\Sigma = (S;\mathcal{X},\mathcal{U},\mathcal{Y})$ satisfying $\rho(\Sigma) \supset \Omega$. The i/s/o pseudo-system Σ is determined uniquely by $\widehat{\mathfrak{S}}$, and $\widehat{\mathfrak{S}}$ has a unique extension to $\rho(\Sigma)$. This extension is maximal in the sense that $\widehat{\mathfrak{S}}$ cannot be extended to an i/s/o pseudo-resolvent on any larger open subset of $\mathbb{C}$.

- This result is well-known in the case where the system has no input and no output, and $\widehat{\mathfrak{A}}(\lambda)$ is injective and has dense range for some $\lambda \in \Omega$; see, e.g., (Paz83, Theorem 9.3, p. 36).
- A multi-valued version of this theorem, still with no input and output, is found in (DdS87, Remark, pp. 148–149).
- It follows from the above theorem that Opmeer's Resolvent Linear Systems $\equiv$ regular i/s/o pseudo-systems!

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Extensions of Time Domain Notions

Many of (if not all?) the important notions in linear control theory can be defined in terms of the behavior of classical (or generalized) time domain trajectories of the system. This is true, for example, for the notions of

- stability, stabilizability, and detectability
- controllability and observability
- minimality
- passivity and conservativity
- restrictions and projections
- compressions and dilations
- intertwinements.

By replacing time domain trajectories by frequency domain trajectories it is possible to extend (to generalize) these notions. (See next slide!)

Frequency Domain Trajectories

- The inclusion

$$\Sigma\colon \begin{cases} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}(S), \\ \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (5)$$

  describes the evolution of time domain trajectories of Σ.

- From (5) we get the frequency domain inclusion

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} \in S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}. \qquad (6)$$

  by taking (formal) Laplace transforms as explained above.

- It is possible to introduce the notion of a frequency domain trajectory by replacing (5) by (6), and at the same time replacing the time domain interval $\mathbb{R}^+$ by some open subset Ω of the complex (frequency domain) plane $\mathbb{C}$.

Definition

Let $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be an i/s/o pseudo-system, and let Ω be an open subset of $\mathbb{C}$. By a (frequency domain) Ω-trajectory of Σ with initial state $x_0 \in \mathcal{X}$ we mean a triple of analytic functions $\begin{bmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{bmatrix}$ defined in Ω with values in $\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$ which satisfy

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \end{bmatrix} \in S \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}, \qquad \lambda \in \Omega. \tag{6}$$

If Σ is regular and $\Omega \subset \rho(\Sigma)$, then

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \widehat{\mathfrak{S}}(\lambda) \begin{bmatrix} x_0 \\ \hat{u}(\lambda) \end{bmatrix} = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda)x_0 + \widehat{\mathfrak{B}}(\lambda)\hat{u}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda)x_0 + \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda) \end{bmatrix}, \quad \lambda \in \Omega. \tag{8}$$

- By replacing time domain trajectories by frequency domain trajectories it is possible to extend (to generalize) these notions.
- See the full MTNS article and (AS14) for details.

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