H-Passive Linear Discrete Time Invariant State/Signal Systems

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Summary

- Discrete time-invariant i/s/o systems
- H-passivity with different supply rates
- State/signal systems
- H-passive s/s systems
- The KYP inequality
- Signal behaviors
- Passive S/S Systems $\leftrightarrow$ Passive Behaviors
- Realization theory

Discrete time-invariant i/s/o systems

Discrete Time-Invariant I/S/O System

Linear discrete-time-invariant systems are typically modeled as i/s/o (input/state/output) systems of the type

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0, \\ y(n) &= Cx(n) + Du(n), \qquad n \in \mathbb{Z}^+. \end{aligned} \tag{1}$$

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By a trajectory of this system we mean a triple of sequences (u, x, y) satisfying (1).

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The system (1) is H-passive if all trajectories satisfy the condition

$$E_H(x(n+1)) - E_H(x(n)) \le j(u(n), y(n)), \qquad n \in \mathbb{Z}^+, \tag{2}$$

where E_H is a positive storage function (Lyapunov function)

$$E_H(x) = \langle Hx, x \rangle_{\mathcal{X}}, \quad H > 0,$$

and j is an indefinite quadratic supply rate

$$j(u, y) = \langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], J \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \rangle_{\mathcal{Y} \oplus \mathcal{U}}$$

determined by a signature operator J $(= J^* = J^{-1})$.

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(ii) The impedance supply rate $j_{\mathrm{imp}}(u,y) = 2\Re\langle y, \Psi u\rangle_{\mathcal{U}}$ with signature operator $J_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$, where Ψ is a unitary operator $\mathcal{U} \to \mathcal{Y}$.

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(iii) The transmission supply rate $j_{\rm tra}(u,y) = -\langle y, J_{\mathcal{Y}} y\rangle_{\mathcal{Y}} + \langle u, J_{\mathcal{U}} u\rangle_{\mathcal{U}}$ with signature operator $J_{\rm tra} = \left[\begin{smallmatrix} -J_{\mathcal{Y}} & 0 \\ 0 & J_{\mathcal{U}} \end{smallmatrix}\right]$, where $J_{\mathcal{Y}}$ and $J_{\mathcal{U}}$ are signature operators in $\mathcal{Y}$ and $\mathcal{U}$, respectively.

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It is possible to combine all these cases into one single setting, called the s/s (state/signal) setting. The idea is to introduce a class of systems which does not distinguish between inputs and outputs.

State/Signal Systems

State/Signal System: Definition

A linear discrete time-invariant s/s system Σ is modelled by a system of equations

$$x(n+1) = F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0, \tag{3}$$

Here F is a bounded linear operator with a closed domain $\mathcal{D}(F) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ ($\mathbb{Z}^+ = 0, 1, 2, \ldots$) and certain additional properties.

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In the case of an i/s/o system we take $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, $F \left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right] = Ax + Bu$, and $\mathcal{D}(F) = \left\{ \left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right] \;\middle|\; y = Cx + Du \right\}$.

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(ii) The trajectory (x, w) is determined uniquely by x_0 and w.

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This system is determined by the fact that $(x_*(\cdot), w_*(\cdot))$ is a trajectory of Σ_* if and only if

$$-\langle x(n+1), x_*(0)\rangle_{\mathcal{X}} + \langle x(0), x_*(n+1)\rangle_{\mathcal{X}} + \sum_{k=0}^{n}[w(k), w_*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+,$$

for all trajectories $(x(\cdot), w(\cdot))$ of Σ.

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for all trajectories $(x(\cdot), w(\cdot))$ of Σ.

The adjoint of Σ_* is the original system Σ.

Controllability and Observability

A s/s system Σ is controllable if the set of all states $x(n)$, $n \geq 1$, which appear in some trajectory $(x(\cdot), w(\cdot))$ of Σ with $x(0) = 0$ (i.e., an externally generated trajectory) is dense in $\mathcal{X}$.

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Σ is minimal if Σ is both controllable and observable.

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(iv) passive if it is $1_{\mathcal{X}}$-passive ($1_{\mathcal{X}}$ is the identity operator in $\mathcal{X}$).

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The S/S KYP Inequality

It is not difficult to see that a s/s system Σ whose trajectories are defined by (3) is forward H-passive if and only if $H > 0$ is a solution of the generalized s/s KYP (Kalman–Yakubovich–Popov) inequality[2]

$$\|H^{1/2}F\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]\|_{\mathcal{X}}^2 - \|H^{1/2}x\|_{\mathcal{X}}^2 \le [w, w]_{\mathcal{W}}, \quad \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F), \quad x \in \mathcal{D}(H^{1/2}). \tag{4}$$

[2] In particular, in order for the first term in this inequality to be well-defined we require F to map $\{\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F) \mid x \in \mathcal{D}(H^{1/2})\}$ into $\mathcal{D}(H^{1/2})$.

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This inequality is named after Kalman [Kal63], Yakubovich [Yak62], and Popov [Pop61] (who at that time restricted themselves to the finite-dimensional input/state/output case).

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There is a rich literature on this version of the KYP inequality and the corresponding equality; see, e.g., [PAJ91], [IW93], and [LR95], and the references mentioned there.

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Infinite-Dimensional I/S/O KYP Inequality: History

In the seventies the classical results on the i/s/o KYP inequalities were extended to systems with $\dim \mathcal{X} = \infty$ by Yakubovich and his students and collaborators (see [Yak74, Yak75, LY76] and the references listed there).

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There is now a rich literature also on this subject; see, e.g., the discussion in [Pan99] and the references cited there.

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There is now a rich literature also on this subject; see, e.g., the discussion in [Pan99] and the references cited there.

However, it is (almost) always assumed that H or H^{-1} is bounded. The only exception is the article [AKP05] by Arov, Kaashoek and Pik.

Signal behaviors

Behaviors: Definition

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We call this the behavior induced by Σ, and refer to Σ as a s/s realization of $\mathfrak{W}$, or, in the case where Σ is minimal, as a minimal s/s realization of $\mathfrak{W}$.

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Two s/s systems Σ_1 and Σ_2 with the same signal space are externally equivalent if they induce the same behavior.

Pseudo-Similarity

Two s/s systems Σ and Σ_1 with the same signal space $\mathcal{W}$ and state spaces $\mathcal{X}$ and $\mathcal{X}_1$, respectively, are called pseudo-similar if there exists an injective densely defined closed linear operator $R\colon \mathcal{X} \to \mathcal{X}_1$ with dense range such that the following conditions hold:

$$(x(\cdot), w(\cdot)) \text{ is a trajectory of } \Sigma \Leftrightarrow (Rx(\cdot), w(\cdot)) \text{ is a trajectory of } \Sigma_1.$$

In particular, if Σ_1 and Σ_2 are pseudo-similar, then they are externally equivalent.

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Conversely, if Σ_1 and Σ_2 are minimal and externally equivalent, then they are necessarily pseudo-similar.

A realizable behavior $\mathfrak{W}$ on the signal space $\mathcal{W}$ has a minimal s/s realization, which is determined by $\mathfrak{W}$ up to pseudo-similarity. (See [AS05, Section 7] for details.)

The Adjoint Behavior

The adjoint of the behavior $\mathfrak{W}$ on $\mathcal{W}$ is a behavior $\mathfrak{W}_*$ on $\mathcal{W}_*$ defined as the set of sequences w_* satisfying

$$\sum_{k=0}^{n} [w(k), w_*(n-k)]_{\mathcal{W}} = 0, \qquad n \in \mathbb{Z}^+,$$

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If $\mathfrak{W}$ is induced by Σ, then $\mathfrak{W}_*$ is (realizable and) induced by Σ_*,

and the adjoint of $\mathfrak{W}_*$ is the original behavior $\mathfrak{W}$.[3]

[3] Is this statement true or false if $\mathfrak{W}$ is not realizable?

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(ii) backward passive if $\mathfrak{W}_*$ is forward passive,

(iii) passive if it is realizable[4] and both forward and backward passive.

[4] We do not know if the realizability assumption is redundant or not.

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Thus, if Σ is backward H_2-passive for at least one H_2, then forward H-passivity implies backward H-passivity for all $H > 0$.

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Ordering of Solutions of KYP Inequality

We denote the set of all solutions $H = H^* > 0$ of the KYP inequality by M_Σ, and we let $M_\Sigma^{\min}$ be the set of $H \in M_\Sigma$ for which the system Σ_H in assertion (ii) of Theorem 2 is minimal by $\mathcal{L}_\Sigma^{\min}$.

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Theorem 3. Let Σ be a minimal s/s system with a passive behavior. Then $M_\Sigma^{\min} \neq \emptyset$ and $M_\Sigma^{\min}$ contains a minimal element $H_\circ$ and a maximal element $H_\bullet$, i.e., $H_\circ \preceq H \preceq H_\bullet$ for every $H \in M_\Sigma^{\min}$.

$H_1 \preceq H_2 \Leftrightarrow \mathcal{D}(\sqrt{H_2}) \subset \mathcal{D}(\sqrt{H_1})$ and $\|\sqrt{H_1}x\| \leq \|\sqrt{H_2}x\| \; \forall x \in \mathcal{D}(\sqrt{H_2})$.

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$E_{H_\circ}(\cdot)$ is the available storage, and $E_{H_\bullet}(\cdot)$ is the required supply (Willems).

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$H_\circ$ is the optimal and $H_\bullet$ is the $*$-optimal solution of the KYP inequality (Arov).

Further Extensions

Instead of working with energy inequalities we can also work with energy balance equations. In this case the system will be forward conservative or even conservative.

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Analogous results also hold for the quadratic cost minimization problem and its dual. The advantage with this approach is that we get rid of the finite cost condition. This is current joint work with Mark Opmeer.

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