Well-Posed State/Signal Systems in Continuous Time

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Outline

- Continuous time-invariant i/s/o systems
- State/signal nodes
- Well-posed state/signal nodes
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- Input/state/output representations
- Extensions
- Why use a differential formulation?

Continuous Time-Invariant I/S/O System (First Model)

The simplest model for a linear continuous-time-invariant system is of the type

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{1}$$

Here $\mathbb{R}^+ = [0, \infty)$ and A, B, C, D, are linear operators.

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$u(t) \in \mathcal{U}$ = the input space,
$x(t) \in \mathcal{X}$ = the state space,
$y(t) \in \mathcal{Y}$ = the output space (all Banach spaces).

Continuous Time-Invariant I/S/O System (Second Model)

In order to include partial differential equations we need A, B, C, and D to be unbounded, and typically their domains are not independent of each other. Therefore, we have to replace the model (1) by the more general model

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \tag{2}$$

Here S is a closed and typically unbounded operator $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.

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One way to avoid this problem is to ignore the distinction between an input and an output, and to replace the i/s/o model by a state/signal model.

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A state/signal system is the natural model of a possibly infinite-dimensional linear cirquit.

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We rewrite the model

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \tag{2}$$

in graph form to get rid of the explicit input $u(t)$ and output $y(t)$: It is equivalent to

$$\begin{bmatrix}\dot{x}(t)\\ x(t)\\ w(t)\end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \tag{3}$$

where $w(t) = \left[\begin{smallmatrix}y(t)\\ u(t)\end{smallmatrix}\right]$ and $V = \left\{\left[\begin{smallmatrix}z\\ x\\ w\end{smallmatrix}\right] \in \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right] \,\middle|\, w = \left[\begin{smallmatrix}y\\ u\end{smallmatrix}\right],\ \left[\begin{smallmatrix}z\\ y\end{smallmatrix}\right] = S\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\right\}$.

State/Signal Node

We end up studying state/signal models of the type

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We call this a state/signal node (the differential form of a state/signal system), and denote it by $\Xi = (V; \mathcal{X}, \mathcal{W})$.

Classical State/Signal Trajectories

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \tag{3}$$

By a classical trajectory of $\Xi = (V; \mathcal{X}, \mathcal{W})$ on $\mathbb{R}^+$ we mean a pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ satisfying (3). We denote this family of trajectories by $\mathfrak{V}$.

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Externally generated classical trajectories: $\mathfrak{V}_0[0,T] = \left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T] \;\middle|\; \left[\begin{smallmatrix} x(0) \\ w(0) \end{smallmatrix}\right] = 0 \right\}$.
(Trajectories in $\mathfrak{V}_0[0,T]$ start with an empty internal memory, and they are driven exclusively by the external signal.)

Generalized State/Signal Trajectories

$$\begin{bmatrix} \dot{x}_n(t) \\ x_n(t) \\ w_n(t) \end{bmatrix} \in V, \qquad t \in [0,T].$$

Fix some $p \in [1,\infty)$. The pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{smallmatrix}\right]$ is a generalized trajectory of Ξ on $[0,T]$ if there exists $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ such that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{smallmatrix}\right]$. We denote this family of trajectories by $\mathfrak{W}[0,T]$.

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The pair of functions $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ is a generalized trajectory of Ξ on $\mathbb{R}^+$ if the restriction of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to every finite interval $[0,T]$ is a generalized trajectory on $[0,T]$. We denote this family of trajectories by $\mathfrak{W}$.

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Externally generated generalized trajectories: $\mathfrak{W}_0[0,T] = \left\{\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[0,T] \;\middle|\; x(0) = 0\right\}$. (Trajectories in $\mathfrak{W}_0[0,T]$ start with an empty internal memory, and they are driven exclusively by the external signal.)

Conditions Required from a Node

We throughout require a s/s node to satisfy (at least) the following three conditions:

(i) V is a closed subspace of $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$.

(ii) If $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$ then $z = 0$.

(iii) There is a $T > 0$ such that for each $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$ there exists at least one classical trajectory $\begin{bmatrix} x \\ w \end{bmatrix}$ of Ξ on $[0, T]$ with $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$.

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Decompose the signal space $\mathcal{W}$ into a direct sum $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$. Let $P_{\mathcal{U}}^{\mathcal{Y}}$ be the projection onto $\mathcal{U}$ along $\mathcal{Y}$, i.e., $\mathcal{R}(P_{\mathcal{U}}^{\mathcal{Y}}) = \mathcal{U}$ and $\mathcal{N}(P_{\mathcal{U}}^{\mathcal{Y}}) = \mathcal{Y}$.

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Definition 1. The node $\Xi = (V; \mathcal{X}, \mathcal{W})$ is well-posed if there exists a $T > 0$ and a direct sum decomposition $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$ of $\mathcal{W}$ such that:

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(iv) The set $\{x(0) \mid \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]\}$ is dense in $\mathcal{X}$.

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(v) The set $\{\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \mid \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0[0,T]\}$ is dense in $L^p([0,T];\mathcal{U})$.

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(vi) there exists a finite constant K such that all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}([0,T])$ satisfy

$$\|x(t)\|_{\mathcal{X}} + \|w\|_{L^p([0,t];\mathcal{W})} \leq K\big(\|x(0)\|_{\mathcal{X}} + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|_{L^p([0,t];\mathcal{U})}\big) \tag{4}$$

for all $t \in [0,T]$.

Admissible I/O Decompositions

A decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of $\mathcal{W}$ satisfying conditions (iv)–(vi) above for some $T > 0$ is called an admissible i/o (input/output) pair for Ξ.

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In general a well-posed s/s node has more than one admissible i/o pair. The following result can be used to test when a given decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ is admissible for Ξ. (See next slide.)

Admissibility Theorem

Theorem 1. Let $\Xi = (V; \mathcal{X}, \mathcal{W})$ be a well-posed state/signal node, and let $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$. Then the following conditions are equivalent:

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(i) $(\mathcal{U}, \mathcal{Y})$ is an admissible i/o pair for Ξ, i.e., conditions (iv)–(vi) in Definition 1 hold for some $T > 0$ (or equivalently, for all $T > 0$).

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(ii) The map $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \to \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ is a bijection $\mathfrak{W}_0 \to L^p([0,T];\mathcal{U})$ for some $T > 0$ (or equivalently, for all $T > 0$).

Repetition

Recall: Every s/s node (well-posed or not) is required to satisfy (at least)

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$\mathfrak{V}_0[0,T] = \left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T] \;\middle|\; \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} = 0 \right\}$ (externally generated classical trajectories)

$\mathfrak{W}_0[0,T] = \left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[0,T] \;\middle|\; x(0) = 0 \right\}$ (externally generated generalized trajectories)

Well-Posedness Theorem

If $\Xi = (V; \mathcal{X}, \mathcal{W})$ is well-posed, then $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0[0,T]$ for all $T > 0$.

Under this assumption we can characterize well-posedness and admissibility of a s/s node in terms of generalized trajectories (as opposed to the family $\mathfrak{V}[0,T]$ of classical trajectories used in Definition 1). (See next slide.)

Well-Posedness Theorem

Theorem 2. Let $\Xi = (V; \mathcal{X}, \mathcal{W})$ be a s/s node. In addition suppose that $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0[0,T]$ for some $T > 0$. Let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be a direct sum decomposition of $\mathcal{W}$. Then the following conditions are equivalent:

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(i) Ξ is well-posed and $(\mathcal{U}, \mathcal{Y})$ is an admissible i/o pair for Ξ.

(ii) for some (or equivalently, for all) $T > 0$ the map $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \to \begin{bmatrix} x(0) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix}$ is a bijection $\mathfrak{W}[0,T] \to \left[\begin{smallmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{smallmatrix}\right]$.

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(iii) for some (or equivalently, for all) $T > 0$ the following two conditions hold:
  (a) for each $x_0 \in \mathcal{X}$ there exists at least one $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[0,T]$ such that $x(0) = x_0$.
  (b) the map $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \to \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ is a bijection $\mathfrak{W}_0 \to L^p([0,T];\mathcal{U})$.

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However, in many cases the familiy of generalized trajectories is more important than the family of classical trajectories.

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We therefore introduce the notion of a well-posed state/signal system:

Definition 3. By a well-posed state/signal system $\Sigma = (\mathfrak{W}; \mathcal{X}, \mathcal{W})$ we mean the family of generalized trajectories $\mathfrak{W}$ on $[0, \infty)$ of a some well-posed state/signal node $\Xi = (V; \mathcal{X}, \mathcal{W})$.

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If a decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ is admissible for some some well-posed s/s node $\Xi = (V; \mathcal{X}, \mathcal{W})$ that generates Σ, then it is also admissible for every other well-posed s/s node that generates Σ. In this case we call $(\mathcal{U}, \mathcal{Y})$ an admissible i/o pair for Σ.

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If a decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ is admissible for some some well-posed s/s node $\Xi = (V; \mathcal{X}, \mathcal{W})$ that generates Σ, then it is also admissible for every other well-posed s/s node that generates Σ. In this case we call $(\mathcal{U}, \mathcal{Y})$ an admissible i/o pair for Σ.

Moreover, there always exists a maximal generating node (see next slide):

Maximal Well-Posed State/Signal Nodes

Theorem 4. (i) Among all the nodes $(V;\mathcal{X},\mathcal{W})$ that generate a well-posed linear state/signal system $\Sigma = (\mathfrak{W};\mathcal{X},\mathcal{W})$ there is always a maximal one $(V_{\max};\mathcal{X},\mathcal{W})$. (Maximality of $(V_{\max};\mathcal{X},\mathcal{W})$ means that if both $(V;\mathcal{X},\mathcal{W})$ and $(V_{\max};\mathcal{X},\mathcal{W})$ generate the same system $(\mathfrak{W};\mathcal{X},\mathcal{W})$, then necessarily $V \subset V_{\max}$.)

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(ii) $\Xi = (V; \mathcal{X}, \mathcal{W})$ is maximal if and only if $\mathfrak{V} = \mathfrak{W} \cap \begin{bmatrix} C^1(\mathbb{R}^+; \mathcal{X}) \\ C(\mathbb{R}^+; \mathcal{W}) \end{bmatrix}$, i.e., every generalized trajectory (x, w) which has the smoothness of a classical trajectory is actually classical.

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Note, in particular, that $V_{\max}$ is uniquely determined by Σ, which is uniquely determined by the node $(V;\mathcal{X},\mathcal{W})$.

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Theorem 5. Let $\Sigma = (\mathfrak{W}; \mathcal{X}, \mathcal{W})$ be a well-posed state/signal system, and let $(\mathcal{U}, \mathcal{Y})$ be an admissible i/o pair for Σ. Then the map $(x_0, u) \to (x, \mathcal{P}^{\mathcal{U}}_{\mathcal{Y}} w)$ (where (x, w) is the trajectory described above) defines a well-posed linear i/s/o system $\Sigma_{i/s/o}$ in the sense of [Sta05], with $\mathcal{U}$ as input space and $\mathcal{Y}$ as output space.

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We call this $\Sigma_{i/s/o}$ the i/s/o representation of Σ corresponding to the i/o pair $(\mathcal{U}, \mathcal{Y})$.

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Theorem 6. (i) To each well-posed i/s/o system $\Sigma_{i/s/o}$ with input space $\mathcal{U}$ and output space $\mathcal{Y}$ there corresponds a unique well-posed state/signal system $\Sigma = (\mathfrak{W}; \mathcal{X}, \mathcal{U} \times \mathcal{Y})$ such that $\Sigma_{i/s/o}$ is the i/s/o representation of Σ corresponding to the i/o pair $(\mathcal{U}, \mathcal{Y})$.

(ii) The maximal generating subspace $V_{\max}$ of the underlying state/signal node $\Xi_{\max} = (V_{\max}; \mathcal{X}, \mathcal{W})$ is the graph of the i/o system node which generates $\Sigma_{i/s/o}$. (See, e.g., [Sta05] for the definition of an i/o system node.)

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Extensions

- Different representations exist, such as driving-variable and output-nulling representations.

- Interconnections of well-posed state/signal systems (in progress)

- Passive well-posed state/signal systems (the main motivation for studying state/signal systems in the first place).

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Anwer: The set of needed algebraic conditions becomes too complicated and non-intuitive! (This is how we originally started out.) It is possible to proceed in the 'standard' direction, starting with an 'integral' formulation, but already the definition of what we mean by a well-posed state/signal system becomes too complicated.

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Standard Definition: By a C_0 semigroup one means a family of operators $\mathfrak{A}^t$ in $\mathcal{B}(\mathcal{X})$ satisfying

(i) $\mathfrak{A}^0 = 1_{\mathcal{X}}$,

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The generator A of this semigroup is given by $Ax = \lim_{t \downarrow 0} \frac{1}{t}(\mathfrak{A}^t x - x)$, with domain $\mathcal{D}(A)$ consisting of those $x \in \mathcal{X}$ for which the above limit exists.

Construction of a C_0-Semigroup by Our Method

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(vi) There exist constants $T > 0$ and K_T such that all classical trajectories x satisfy $\sup_{0 \le t \le T} \|x(t)\|_{\mathcal{X}} \le K_T \|x(0)\|_{\mathcal{X}}$.

The Resulting Semigroup

If the above conditions (i)–(vi) hold, then the family $\mathfrak{A}^t\colon x_0 \mapsto x(t)$, where x is the generalized trajectory with $x(0) = x_0$, is a C_0 semigroup.

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Open Question: Do conditions (i)–(vi) imply that the domain of A is automatically maximal?

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