H-Passive Linear Discrete Time Invariant State/Signal Systems

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***Abstract*— A linear state/signal system in discrete time has a state space $\mathcal{X}$ and a signal space $\mathcal{W}$, where the state space is used to represent internal properties of the system, and the signal space describes interactions with the surrounding world. It resembles an input/state/output system apart from the fact that inputs and outputs are not separated from each other. By decomposing the signal space $\mathcal{W}$ into a direct sum of an input space $\mathcal{U}$ and an output space $\mathcal{Y}$ one gets a standard input/state/output system, provided the decomposition is *admissible*. Here we discuss systems which are passive with respect to a quadratic storage function in the state space, represented by a positive self-adjoint operator H which may be unbounded and have an unbounded inverse. The quadratic supply rate, which describes the energy flow between the system and the surroundings, imposes a Kreĭn space structure on the signal space, but the state space is a Hilbert space. Our main results relate the existence of an operator $H > 0$ such that the system is H-passive to the existence of a solution of a generalized Kalman–Yakubovich–Popov inequality, and also to the passivity properties of the behavior induced by the system.**

The evolution of a *linear discrete time-invariant s/s (= state/signal) system* Σ with a Hilbert *state space* $\mathcal{X}$ and a Kreĭn *signal space* $\mathcal{W}$ is described by the system of equations

$$x(n+1) = F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \quad n \in \mathbb{Z}^+, \quad x(0) = x_0, \qquad (1)$$

where the initial state $x_0 \in \mathcal{X}$ may be arbitrary and F is a bounded linear operator with a closed domain $\mathcal{D}(F) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ ($\mathbb{Z}^+ = 0, 1, 2, \ldots$). By a *trajectory* $(x(\cdot), w(\cdot))$ of this system we mean pairs of sequences $x(\cdot) \in \mathcal{X}$ and $w(\cdot) \in \mathcal{W}$ satisfying (1). Each s/s system Σ has an *adjoint s/s system* Σ_* with the same state space $\mathcal{X}$ and the Kreĭn signal space $\mathcal{W}_* = -\mathcal{W}$. This system is determined by the fact that $(x_*(\cdot), w_*(\cdot))$ is a trajectory of Σ_* if and only if

$$\begin{aligned} &-(x(n+1), x_*(0))_{\mathcal{X}} + (x(0), x_*(n+1))_{\mathcal{X}} \\ &\quad + \sum_{k=0}^{n} [w(k), w_*(n-k)]_{\mathcal{W}} = 0, \quad n \in \mathbb{Z}^+, \end{aligned}$$

for all trajectories $(x(\cdot), w(\cdot))$ of Σ. The adjoint of Σ_* is the original system Σ. A s/s system Σ is *controllable* if the sets of all states $x(n)$, $n \geq 1$, which appear in some trajectory $(x(\cdot), w(\cdot))$ of Σ with $x(0) = 0$ (i.e., an *externally generated trajectory*) is dense in $\mathcal{X}$. The system Σ is *observable* if there do not exist any nontrivial trajectories $(x(\cdot), w(\cdot))$ where the signal component $w(\cdot)$ is identically zero. Equivalently, Σ is observable if and only Σ_* is controllable. Finally, Σ is *minimal* if Σ is both controllable and observable.

Definition 1. Let H be a positive self-adjoint operator in the Hilbert space $\mathcal{X}$.[1] A s/s system Σ is

(i) *forward H-passive* if $x(n) \in \mathcal{D}(\sqrt{H})$ and
$$\begin{aligned} &\|\sqrt{H} x(n+1)\|_{\mathcal{X}}^2 - \|\sqrt{H} x(n)\|_{\mathcal{X}}^2 \\ &\qquad \leq [w(n), w(n)]_{\mathcal{W}}, \quad n \in \mathbb{Z}^+, \end{aligned}$$
for every trajectory (x, w) of Σ with $x(0) \in \mathcal{D}(\sqrt{H})$,

(ii) *backward H-passive* if Σ_* is forward H^{-1}-passive,

(iii) *H-passive* if it is both forward H-passive and backward H-passive.

(iv) *passive* if it is $1_{\mathcal{X}}$-passive ($1_{\mathcal{X}}$ is the identity operator in $\mathcal{X}$).

It is not difficult to see that a s/s system Σ whose trajectories are defined by (1) is forward H-passive if and only if $H > 0$ is a solution of the generalized s/s KYP (Kalman–Yakubovich–Popov) inequality[2]

$$\begin{aligned} &\|H^{1/2} F \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]\|_{\mathcal{X}}^2 - \|H^{1/2} x\|_{\mathcal{X}}^2 \leq [w, w]_{\mathcal{W}}, \\ &\qquad \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F), \quad x \in \mathcal{D}(H^{1/2}). \end{aligned} \qquad (2)$$

There is a rich literature on the finite-dimensional i/s/o (= input/state/output) version of this inequality and the corresponding equality; see, e.g., [PAJ91], [IW93], and [LR95], and the references mentioned there. This inequality is named after Kalman [Kal63], Popov [Pop73], and Yakubovich [Yak62]. In the seventies the classical results on the KYP inequalities were extended to systems with $\dim \mathcal{X} = \infty$ by V. A. Yakubovich and his students and collaborators (see [Yak74], [Yak75], [LY76] and the references listed there). There is now also a rich literature on this subject; see, e.g., the discussion in [Pan99] and the references cited there. The i/s/o version of our notion of a generalized solution of (2) was introduced and studied in [AKP05].

[1] Note that neither H itself nor H^{-1} is required to be bounded. In [AS06a] an example is given based on the heat equation where *all* solutions of the continuous time version of the generalized KYP inequality are unbounded and have an unbounded inverse.

[2] In particular, in order for the first term in this inequality to be well-defined we require F to map $\{\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{D}(F) \mid x \in \mathcal{D}(H^{1/2})\}$ into $\mathcal{D}(H^{1/2})$.

The notion of H-passivity of a s/s system Σ involves both the state component and the signal component of the trajectories of Σ. There is another weaker version of passivity which involves only the signal components of the externally generated trajectories of Σ.

By a *behavior*[3] *on the signal space* $\mathcal{W}$ we mean a closed right-shift invariant subspace of the Fréchet space $\mathcal{W}^{\mathbb{Z}^+}$. Thus, in particular, the set $\mathfrak{W}$ of all sequences w that are the signal parts of externally generated trajectories (x, w) of a s/s system Σ is a behavior. We call this the *behavior induced by* Σ, and refer to Σ as a *s/s realization of* $\mathfrak{W}$, or, in the case where Σ is minimal, as a *minimal s/s realization of* $\mathfrak{W}$. A behavior is *realizable* if it has a s/s realization.

Two s/s systems $\Sigma = (V; \mathcal{X}, \mathcal{W})$ and $\Sigma_1 = (V_1; \mathcal{X}_1, \mathcal{W})$ are called *pseudo-similar* if there exists an injective densely defined closed linear operator $R\colon \mathcal{X} \to \mathcal{X}_1$ with dense range such that the following conditions hold:

> If $(x(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$ with $x(0) \in \mathcal{D}(R)$, then $x(n) \in \mathcal{D}(R)$ for all $n \in \mathbb{Z}^+$ and $(Rx(\cdot), w(\cdot))$ is a trajectory of Σ_1 on $\mathbb{Z}^+$, and conversely, if $(x_1(\cdot), w(\cdot))$ is a trajectory of Σ_1 on $\mathbb{Z}^+$ with $x_1(0) \in \mathcal{R}(R)$, then $x_1(n) \in \mathcal{R}(R)$ for all $n \in \mathbb{Z}^+$ and $(R^{-1}x_1(\cdot), w(\cdot))$ is a trajectory of Σ on $\mathbb{Z}^+$.

Two s/s systems Σ_1 and Σ_2 with the same signal space are *externally equivalent* if they induce the same behavior. In particular, if Σ_1 and Σ_2 are pseudo-similar, then they are externally equivalent. Conversely, if Σ_1 and Σ_2 are minimal and externally equivalent, then they are necessarily pseudo-similar. Moreover, a realizable behavior $\mathfrak{W}$ on the signal space $\mathcal{W}$ has a minimal s/s realization, which is determined by $\mathfrak{W}$ up to pseudo-similarity. (See [AS05, Section 7] for details.)

The *adjoint* of the behavior $\mathfrak{W}$ on $\mathcal{W}$ is a behavior $\mathfrak{W}_*$ on $\mathcal{W}_*$ defined as the set of sequences w_* satisfying

$$\sum_{k=0}^{n} [w(k), w_*(n-k)]_{\mathcal{W}} = 0, \qquad n \in \mathbb{Z}^+,$$

for all $w \in \mathfrak{W}$. If $\mathfrak{W}$ is induced by Σ, then $\mathfrak{W}_*$ is (realizable and) induced by Σ_*, and the adjoint of $\mathfrak{W}_*$ is the original behavior $\mathfrak{W}$.

Definition 2. A behavior $\mathfrak{W}$ on $\mathcal{W}$ is

(i) *forward passive* if

$$\sum_{k=0}^{n} [w(k), w(k)]_{\mathcal{W}} \geq 0, \quad w \in \mathfrak{W}, \quad n \in \mathbb{Z}^+,$$

(ii) *backward passive* if $\mathfrak{W}_*$ is forward passive,

(iii) *passive* if it is realizable[4] and both forward and backward passive.

[3]Our behaviors are what Polderman and Willems call *linear time-invariant mainfest behaviors* in [PW98, Definitions 1.3.4, 1.4.1, and 1.4.2]. We refer the reader to this book for further details on behaviors induced by systems with a finite-dimensional state space and for an account of the extensive literatur on this subject.

[4]We do not know if the realizability assumption is redundant or not.

Proposition 3. *Let $\mathfrak{W}$ be the behavior induced by a s/s system Σ.*

(i) *If Σ is forward H-passive for some $H > 0$, then $\mathfrak{W}$ is forward passive.*

(ii) *If Σ is backward H-passive for some $H > 0$, then $\mathfrak{W}$ is backward passive.*

(iii) *If Σ is forward H_1-passive for some $H_1 > 0$ and backward H_2-passive for some $H_2 > 0$, then Σ is both H_1-passive and H_2-passive, and $\mathfrak{W}$ is passive.*

Theorem 4. *Let $\mathfrak{W}$ be a passive behavior on $\mathcal{W}$. Then*

(i) *$\mathfrak{W}$ has a minimal passive s/s realization.*

(ii) *Every H-passive realization Σ of $\mathfrak{W}$ is pseudo-similar to a passive realization Σ_H with pseudo-similarity operator $\sqrt{H}$. The system Σ_H is determined uniquely by Σ and H.*

(iii) *Every minimal realization of $\mathfrak{W}$ is H-passive for some $H > 0$, and it is possible to choose H in such a way that the system Σ_H in (ii) is minimal.*

Assertion (ii) can be interpreted in the following way: we can always convert an H-passive s/s system into a passive one by simply replacing the original norm $\|\cdot\|_{\mathcal{X}}$ in the state space by the new norm $\|x\|_H = \|\sqrt{H}x\|_{\mathcal{X}}$, which is finite for all $x \in \mathcal{D}(\sqrt{H})$, and then completing $\mathcal{D}(\sqrt{H})$ with respect to this new norm.

Our final theorem says that a suitable subclass of all operators $H > 0$ for which a s/s system Σ is H-passive can be partially ordered. Here we use the following partial ordering of nonnegative self-adjoint operators on $\mathcal{X}$: if H_1 and H_2 are two nonnegative self-adjoint operators on the Hilbert space $\mathcal{X}$, then we write $H_1 \preceq H_2$ whenever $\mathcal{D}(H_2^{1/2}) \subset \mathcal{D}(H_1^{1/2})$ and $\|H_1^{1/2}x\| \leq \|H_2^{1/2}x\|$ for all $x \in \mathcal{D}(H_2^{1/2})$. For *bounded* nonnegative operators H_1 and H_2 with $\mathcal{D}(H_2) = \mathcal{D}(H_1) = \mathcal{X}$ this ordering coincides with the standard ordering of bounded self-adjoint operators.

For each s/s system Σ we denote the set of operators $H > 0$ for which Σ is H-passive by M_Σ, and we let $M_\Sigma^{\min}$ be the set of $H \in M_\Sigma$ for which the system Σ_H in assertion (ii) of Theorem 4 is minimal.

Theorem 5. *Let Σ be a minimal s/s system with a passive behavior. Then $M_\Sigma^{\min} \neq \emptyset$ and $M_\Sigma^{\min}$ contains a minimal element $H_\circ$ and a maximal element $H_\bullet$, i.e., $H_\circ \preceq H \preceq H_\bullet$ for every $H \in M_\Sigma^{\min}$.*

The two extremal storage functions $E_{H_\circ}$ and $E_{H_\bullet}$ correspond to Willems' [Wil72a], [Wil72b] *available storage* and *required supply*, respectively (there presented in an i/s/o setting). In the terminology of Arov [Aro79], [Aro95], [Aro99] (likewise in an i/s/o setting), $\Sigma_{H_\circ}$ is the *optimal* and $\Sigma_{H_\bullet}$ is the $*$-*optimal* realization of $\mathfrak{W}$.

The results presented above were obtained by reducing the problem to the corresponding problems concerning the existence of generalized positive solutions of a KYP inequality for an i/s/o linear discrete time invariant system $\Sigma_{i/s/o}$ with scattering supply rate solved in [AKP05].

This reduction is based on the existence of *admissible decompositions* $\mathcal{W} = -\mathcal{Y} \dotplus \mathcal{U}$ of the Kreĭn signal space $\mathcal{W}$ of Σ. By this we mean that that there exists a (unique) i/s/o system $\Sigma_{i/s/o}$ with the same state space $\mathcal{X}$ is Σ, with input space $\mathcal{U}$ and output space $\mathcal{Y}$, and with trajectories $(x(\cdot), u(\cdot), y(\cdot))$ given by a system of equations

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \\ y(n) &= Cx(n) + Dx(n), \qquad n \in \mathbb{Z}^+, \\ x(0) &= x_0, \end{aligned} \tag{3}$$

where $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right])$, with the property that $(x(\cdot), u(\cdot), y(\cdot))$ is a trajectory of $\Sigma_{i/s/o}$ if and only if $(x(\cdot), w(\cdot))$ with $w(\cdot) = u(\cdot) + y(\cdot)$ is a trajectory of Σ. We show that

(i) a forward H-passive s/s system Σ is H-passive if and only if at least one fundamental decomposition $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ of the Kreĭn signal space $\mathcal{W}$ of Σ is a admissible,

(ii) if Σ is H-passive, then every fundamental decomposition $\mathcal{W} = -\mathcal{Y} [\dotplus] \mathcal{U}$ is admissible,

(iii) if the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is admissible for Σ, then the set of generalized positive solutions H of the KYP inequality for Σ coincides with the set of generalized positive solutions H of the KYP inequality for $\Sigma_{i/s/o}$ with the supply rate on $\mathcal{Y} \times \mathcal{U}$ inherited from the inner product $[\cdot, \cdot]_{\mathcal{W}}$.

Further details and proofs will be given in [AS06b] and [Sta06]. Different i/s/o representations and affine representations of s/s systems will be discussed in [AS06c] and [AS06d].

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