# Affine Input/State/Output Representations of State/Signal Systems

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## Summary

- Discrete time-invariant i/s/o systems
- State/signal systems
- Affine representations of state/signal systems
- Generalized transfer functions
- Coprime Representations
- Extensions

# Discrete Time-Invariant I/S/O Systems

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Linear discrete-time-invariant systems are typically modeled as i/s/o (input/state/output) systems of the type

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0, \\ y(n) &= Cx(n) + Du(n), \qquad n \in \mathbb{Z}^+. \end{aligned} \tag{1}$$

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By a trajectory of this system we mean a triple of sequences $(u, x, y)$ satisfying (1).

We denote this system by $\Sigma_{i/s/o} = \left( \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$.

# Frequency Domain Interpretation

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The $Z$-transform of a sequence $\{x(n)\}_{n=0}^{\infty}$ is given by $\hat{x}(z) = \sum_{n=0}^{\infty} x(n) z^n$. Taking $Z$-transforms in (1) and solving for $\hat{x}(z)$ we get the frequency domain i/s/o equtions

$$\begin{aligned} \hat{x}(z) &= \mathfrak{A}(z) x_0 + \mathfrak{B}(z) \hat{u}(z), \\ \hat{y}(z) &= \mathfrak{C}(z) x_0 + \mathfrak{D}(z) \hat{u}(z), \qquad \text{for small } |z|. \end{aligned} \tag{2}$$

where

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} (1_{\mathcal{X}} - zA)^{-1} & z(1_{\mathcal{X}} - zA)^{-1} B \\ C(1_{\mathcal{X}} - zA)^{-1} & zC(1_{\mathcal{X}} - zA)^{-1} B + D \end{bmatrix} \tag{3}$$

is the four block transfer function of $\Sigma$ corresponding to the i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. The bottom right block $\mathfrak{D}(z) = zC(1_{\mathcal{X}} - zA)^{-1} B + D$ is the i/o transfer function of $\Sigma$ corresponding to this i/o decomposition.

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The natural domain of definition is the set $\Lambda_A$ consisting of those $z \in \mathbb{C}$ for which $1_{\mathcal{X}} - zA$ has a bounded inverse (including $z = \infty$ if $A$ has a bounded inverse).

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One way to avoid this problem is to ignore the distinction between an input and an output, and to replace the i/s/o model by a state/signal model.

# State/Signal Systems

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A linear discrete time-invariant s/s system $\Sigma$ is modelled by a system of equations

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Here $F$ is a bounded linear operator with a closed domain $\mathcal{D}(F) \subset \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ ($\mathbb{Z}^+ = 0, 1, 2, \ldots$) and a certain additional propertiy.

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By a trajectory of this system we mean a pair of sequences $(x, w)$ satisfying (4).

In the case of an i/s/o system we take $w = \left[\begin{smallmatrix}y\\ u\end{smallmatrix}\right]$, $F\left[\begin{smallmatrix}x\\ u\\ y\end{smallmatrix}\right] = Ax + Bu$, and
$\mathcal{D}(F) = \left\{\left[\begin{smallmatrix}x\\ u\\ y\end{smallmatrix}\right] \,\middle|\, y = Cx + Du\right\}$.

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(iii) The trajectory $(x, w)$ depends continuously on the intial state $x_0$ and the signal part $w$.

# Admissible I/O Decompositions

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Let $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ be a direct sum decomposition of the signal space $\mathcal{W}$. We call this decomposition an admissible i/o decomposition of $\mathcal{W}$ for the s/s system $\Sigma$ (with $\mathcal{U}$ as input space and $\mathcal{Y}$ as output space) if the s/s equation

$$x(n+1) = F \begin{bmatrix} x(n) \\ w(n) \end{bmatrix}, \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0, \tag{4}$$

can be written in i/s/o form (for some bounded linear operators $A$, $B$, $C$, $D$)

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where $u(n)$ and $y(n)$ are the projections of $w(n)$ onto $\mathcal{U}$ and $\mathcal{Y}$, respectively.

We then call $\Sigma_{i/s/o} = \left( \left[ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ the i/s/o representation of $\Sigma$ corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$.

# Affine Representations of State/Signal Systems

# Affine Representations of $\Sigma$

Not every i/o decomposition of $\mathcal{W}$ is admissible.

To be able to treat also the nonadmissible case we introduce right and left affine (= fractional) generalizations of the notions of i/s/o representations and their transfer functions.

These are defined for arbitrary i/o decompositions $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ (also nonadmissible ones).

# Right Affine Representations

By a right affine i/s/o representation of $\Sigma$ we mean an i/s/o system $\Sigma^r_{i/s/o}$ generated by the system of equations (a driving variable representation)

$$\begin{aligned} x(n+1) &= A'x(n) + B'\ell(n), \\ y(n) &= C'_{\mathcal{Y}}x(n) + D'_{\mathcal{Y}}\ell(n), \\ u(n) &= C'_{\mathcal{U}}x(n) + D'_{\mathcal{U}}\ell(n), \qquad n \in \mathbb{Z}^+, \ \ell(n) \in \mathcal{L} \end{aligned}$$

(where the new input space $\mathcal{L}$ is an auxiliary Hilbert space) with the following two properties:

1) $D' = \begin{bmatrix} D'_{\mathcal{Y}} \\ D'_{\mathcal{U}} \end{bmatrix}$ has a bounded left-inverse,

2) $\left(x(\cdot), \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix}\right)$ is a trajectory of $\Sigma$ if and only if $\left(x(\cdot), \ell(\cdot), \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix}\right)$ is a trajectory of $\Sigma^r_{i/s/o}$ for some sequence $\ell(\cdot)$ with values in $\mathcal{L}$.

# Left Affine Representations

By a left affine i/s/o representation of $\Sigma$ we mean an i/s/o system $\Sigma^{l}_{i/s/o}$ generated by the system of equations (an output nulling representation)

$$\begin{aligned} x(n+1) &= A''x(n) + B''_{\mathcal{Y}}y(n) + B''_{\mathcal{U}}u(n), \\ e(n) &= C''x(n) + D''_{\mathcal{Y}}y(n) + D''_{\mathcal{U}}u(n) = 0, \qquad n \in \mathbb{Z}^{+} \end{aligned}$$

(where the new output space $\mathcal{K}$ is another auxiliary Hilbert space) with the following two properties:

1) $D'' = \begin{bmatrix} D''_{\mathcal{Y}} & D''_{\mathcal{U}} \end{bmatrix}$ has a bounded right-inverse,

2) $\left(x(\cdot), \left[\begin{smallmatrix} y(\cdot) \\ u(\cdot) \end{smallmatrix}\right]\right)$ is a trajectory of $\Sigma$ if and only if $\left(x(\cdot), \left[\begin{smallmatrix} y(\cdot) \\ u(\cdot) \end{smallmatrix}\right], 0\right)$ is a trajectory of $\Sigma^{l}_{i/s/o}$ (i.e., the output is identically zero in $\mathcal{K}$).

## Right and Left Affine Four Block Transfer Functions

The frequency domain versions of these representations are

$$\begin{aligned}
\hat{x}(z) &= \mathfrak{A}'(z)x_0 + \mathfrak{B}'(z)\hat{\ell}(z), \\
\hat{y}(z) &= \mathfrak{C}'_{\mathcal{Y}}(z)x_0 + \mathfrak{D}'_{\mathcal{Y}}(z)\hat{\ell}(z), \\
\hat{u}(z) &= \mathfrak{C}'_{\mathcal{U}}(z)x_0 + \mathfrak{D}'_{\mathcal{U}}(z)\hat{\ell}(z), \qquad \text{for small } |z|,\ \hat{\ell}(z) \text{ is free.}
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The corresponding transfer functions are called the right, respectively left affine transfer functions of $\Sigma$ corresponding to the (possibly non-admissible) i/o decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$. Note, in particular, that the right and left affine i/o transfer functions are now decomposed into $\mathfrak{D}' = \begin{bmatrix} \mathfrak{D}'_{\mathcal{Y}} \\ \mathfrak{D}'_{\mathcal{U}} \end{bmatrix}$ and $\mathfrak{D}'' = \begin{bmatrix} \mathfrak{D}''_{\mathcal{Y}} & \mathfrak{D}''_{\mathcal{U}} \end{bmatrix}$.

# Generalized Transfer Functions

# Generalized Right Transfer Function

Solving (5) for $\hat{x}(z)$ and $\hat{y}(z)$ we get the following generalized right four block transfer function with input space $\mathcal{U}$ and output space $\mathcal{Y}$

$$\begin{bmatrix} \mathfrak{A}_r(z) & \mathfrak{B}_r(z) \\ \mathfrak{C}_r(z) & \mathfrak{D}_r(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}'(z) & \mathfrak{B}'(z) \\ \mathfrak{C}'_{\mathcal{Y}}(z) & \mathfrak{D}'_{\mathcal{Y}}(z) \end{bmatrix} \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \mathfrak{C}'_{\mathcal{U}}(z) & \mathfrak{D}'_{\mathcal{U}}(z) \end{bmatrix}^{-1}, \tag{7}$$

defined for all $z$ in the set

$$\Omega(\Sigma^r_{i/s/o}) := \{z \in \Lambda_{A'} \mid \mathfrak{D}'_{\mathcal{U}}(z) \text{ has a bounded inverse}\}.$$

In particular, the generalized right i/o transfer function is given by

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By varying the representation we can thus define $\left[\begin{smallmatrix} \mathfrak{A}_r(z) & \mathfrak{B}_r(z) \\ \mathfrak{C}_r(z) & \mathfrak{D}_r(z) \end{smallmatrix}\right]$ for all $z$ in the set

$$\Omega^r(\Sigma;\mathcal{U},\mathcal{Y}) := \text{the union of the above sets } \Omega(\Sigma^r_{i/s/o}).$$

# Generalized Left Transfer Function

Solving (6) for $\hat{x}(z)$ and $\hat{y}(z)$ we get the following generalized left four block transfer function with input space $\mathcal{U}$ and output space $\mathcal{Y}$

$$\begin{bmatrix} \mathfrak{A}_l(z) & \mathfrak{B}_l(z) \\ \mathfrak{C}_l(z) & \mathfrak{D}_l(z) \end{bmatrix} = \begin{bmatrix} 1_{\mathcal{X}} & -\mathfrak{B}''_{\mathcal{Y}}(z) \\ 0 & -\mathfrak{D}''_{\mathcal{Y}}(z) \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A}''(z) & \mathfrak{B}''_{\mathcal{U}}(z) \\ \mathfrak{C}''(z) & \mathfrak{D}''_{\mathcal{U}}(z) \end{bmatrix}. \tag{8}$$

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$$\Omega^l(\Sigma;\mathcal{U},\mathcal{Y}) := \text{the union of the above sets } \Omega(\Sigma^l_{i/s/o}).$$

# Domains of Right and Left Generalized Transfer Functions

Thus,

- $z \in \Omega^r(\Sigma;\mathcal{U},\mathcal{Y})$ if there exists at least one right affine representation $\Sigma^r_{i/s/o}$ for which formula (7) defining $\left[\begin{smallmatrix} \mathfrak{A}_r(z) & \mathfrak{B}_r(z) \\ \mathfrak{C}_r(z) & \mathfrak{D}_r(z) \end{smallmatrix}\right]$ makes sense.

- $z \in \Omega^l(\Sigma;\mathcal{U},\mathcal{Y})$ if there exists at least one left affine representation $\Sigma^l_{i/s/o}$ for which formula (8) defining $\left[\begin{smallmatrix} \mathfrak{A}_l(z) & \mathfrak{B}_l(z) \\ \mathfrak{C}_l(z) & \mathfrak{D}_l(z) \end{smallmatrix}\right]$ makes sense.

# Right and Left Generalized Transfer Functions Are Well Defined

**Theorem 1.** The right and left generalized four block transfer functions defined by (7) and (8), respectively, do not depend on the choice of $\Sigma^r_{i/s/o}$ and $\Sigma^l_{i/s/o}$, as long as $z \in \Omega(\Sigma^r_{i/s/o})$ or $z \in \Omega(\Sigma^r_{i/s/o})$.

# Right and Left Generalized Transfer Functions Coincide

**Theorem 2.** The right and left generalized four block transfer functions defined by (7) and (8), respectively, coincide on

$$\Omega(\Sigma; \mathcal{U}, \mathcal{Y}) = \Omega^r(\Sigma; \mathcal{U}, \mathcal{Y}) \cap \Omega^l(\Sigma; \mathcal{U}, \mathcal{Y})$$

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The decomposition is admissible if and only if $0 \in \Omega(\Sigma;\mathcal{U},\mathcal{Y})$.

If the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is admissible, and if $A$ is the main operator of the corresponding i/s/o representation of $\Sigma$, then

$$\Omega^r(\Sigma;\mathcal{U},\mathcal{Y}) = \Omega^l(\Sigma;\mathcal{U},\mathcal{Y}) = \Lambda_A,$$

and the right and left generalized four block transfer functions coincide with the ordinary four block transfer function corresponding to the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$.

# Coprime Representations

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- An i/s/o system $\Sigma_{i/s/o} = ([\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ is stable if the trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of this system has the property that $x(\cdot) \in \ell^\infty(\mathcal{X})$ and $y(\cdot) \in \ell^2(\mathcal{Y})$ whenever $u(\cdot) \in \ell^2(\mathcal{U})$.

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- A right or left affine i/s/o representation is stable if it is stable when regarded as an i/s/o system.

- The main operator $A$ of a stable system has the property that $\mathbb{D} \subset \Lambda_A$ and that its i/o transfer function belongs to $H^\infty$ over the unit disk $\mathbb{D}$. (This applies also to right and left affine i/s/o representations.)

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In particular, every s/s system which is passive in the sense of [AS06a] is LFT-stabilizable.

## Generalized Transfer Functions In $H^\infty/H^\infty$

Recall: the right or left i/o transfer functions of stable affine representations are defined in the full unit disk $\mathbb{D}$, and they belong to $H^\infty$ over $\mathbb{D}$.

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Thus, if $\Sigma$ is stabilizable, then to every direct sum decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ of $\mathcal{W}$ (admissible or not) we obtain a generalized right i/o transfer function (from $\mathcal{U}$ to $\mathcal{Y}$) defined as a (formal) right fraction $\mathfrak{D}_r(z) = \mathfrak{D}'_{\mathcal{Y}}(z)\mathfrak{D}'_{\mathcal{U}}(z)^{-1} \in H^\infty(\mathbb{D})/H^\infty(\mathbb{D})$.

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If $\Sigma$ is detectable, then to every direct sum decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ of $\mathcal{W}$ we obtain a generalized left i/o transfer function defined as a (formal) left fraction $\mathfrak{D}_r(z) = \mathfrak{D}''_{\mathcal{Y}}(z)^{-1}\mathfrak{D}''_{\mathcal{U}}(z) \in H^\infty(\mathbb{D}) \setminus H^\infty(\mathbb{D})$.

# Generalized Transfer Functions In $H^\infty/H^\infty$

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If $\Sigma$ is LFT-stabilizable, then these generalized right and left affine i/o transfer functions are even right or left coprime in $H^\infty(\mathbb{D})$, respectively.

# Generalized Nevanlinna and Potapov Class Functions

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The corresponding right and left coprime affine i/o transfer functions will be generalized Potapov and Nevanlinna class functions (relations), respectively.

# Unbounded Impedance Representations

It is also possible to give an unbounded i/s/o impedance representation of a passive s/s system in the case where the impedance function is single-valued, but the values are unbounded maximal accretive operators.

In this representation the bounded block operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is replaced by an unbounded operator, and the theory resembles the continuous time system node theory presented in [Sta05].

The main difference is that the “inside” (the state space $\mathcal{X}$) and the “outside” (the common input and output space) have changed places.

# Details

For details, see [AS06b], [AS06c], and [Sta06].

## References


[AS05] Damir Z. Arov and Olof J. Staffans, State/signal linear time-invariant systems theory. Part I: Discrete time systems, The State Space Method, Generalizations and Applications (Basel Boston Berlin), Operator Theory: Advances and Applications, vol. 161, Birkhäuser-Verlag, 2005, pp. 115–177.

[AS06a] ______, State/signal linear time-invariant systems theory. Passive discrete time systems, Internat. J. Robust Nonlinear Control **16** (2006), 52 pages, Manuscript available at http://www.abo.fi/˜staffans/.

[AS06b] ______, State/signal linear time-invariant systems theory. Part III: Transmission and impedance representations of discrete time systems, Submitted in May, 2006.

[AS06c] ______, State/signal linear time-invariant systems theory. Part IV: Affine representations of discrete time systems, In preparation, 2006.

[Sta05] Olof J. Staffans, Well-posed linear systems, Cambridge University Press, Cambridge and New York, 2005.

[Sta06] ———, Passive linear discrete time-invariant systems, Proceedings of ICM2006, Madrid, 2006.