Time Versus Frequency Domain

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Lorentz@Snellius February 19, 2013

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Outline of Talk

- Time domain well-posed input/state/output systems
- State/signal systems and boundary relations
- Frequency domain well-posed input/state/output systems
- Intertwinement in time and frequency domain
- Compressions and dilations in time and frequency domain
- Controllability, observability, and minimality
- Work in progress

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“Classical” infinite-dimensional i/s/o system

One of the first serious attempts to do infinite-dimensional control theory was to study systems of the type

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (1)$$

$x(t) \in \mathcal{X}$ is the state,
$u(t) \in \mathcal{U}$ is the input,
$y(t) \in \mathcal{Y}$ is the output
$\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces.
The main operator A is the generator of a C_0 semigroup, but
the control operator B,
the observation operator C, and
the feed-through operator D are all bounded linear operators.
This class of systems is studied in the book (CZ95).
However, it is not really “good enough” to study boundary control systems.

"Regular" infinite-dimensional i/s/o systems

One gets a significantly more powerful theory by keeping the same set of equations

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (1)$$

but allowing also B and C to be unbounded:
A is the generator of a C_0 semigroup,
C maps $\mathrm{dom}\,(A) \to \mathcal{Y}$ (continuous w.r.t. graph norm of A),
B maps $\mathcal{U} \to \mathcal{X}_{-1}$, where $\mathcal{X}_{-1}$ is an "extrapolation space", which contains $\mathcal{X}$ as a dense subspace,
D maps $\mathcal{U} \to \mathcal{Y}$.
This class of systems has been studied in a sequence of papers by George Weiss (the first of these appeared in 1989). (See also (Sal87) and (Šmu86).)
After a small modification (replace "regular" by "compatible") this becomes a good class for the study of boundary control systems.

System Nodes

In the theory of "regular" and "compatible" systems the definition of the operator feed-through operator D causes some problems. One solution to this problem is to collapse the block matrix operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ into one operator, called the system node S, and to rewrite (1) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{2}$$

In the regular case the operators A, B, C, and D can be recovered from S, but (2) makes sense also without any "regularity" assumptions. Of course, we still need some assumptions on S. All the systems in (Sta05) are of this type (when they are not regular or compatible).

System Node (the "simplest" version from (Sta05))

Definition

By an operator node on a triple of Hilbert spaces $(\mathcal{X}, \mathcal{U}, \mathcal{Y})$ we mean a (possibly unbounded) linear operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ with the following properties. We denote $\mathrm{dom}\,(A) = \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right] \in \mathrm{dom}\,(S)\right\}$, define $A\colon \mathrm{dom}\,(A) \to \mathcal{X}$ by $Ax = P_{\mathcal{X}} S \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right]$, and require the following conditions to hold:

1. S is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ (with domain $\mathrm{dom}\,(S)$).
2. $P_{\mathcal{X}} S$ is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\mathcal{X}$ (with domain $\mathrm{dom}\,(S)$).
3. $\mathrm{dom}\,(A)$ is dense in $\mathcal{X}$ and $\rho(A) \neq \emptyset$.
4. For every $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ with $\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{dom}\,(S)$.

We call S a system node if, in addition, A generates a C_0 semigroup.

The operators A, B, C, and $\widehat{\mathfrak{D}}(\lambda)$ of system node

If S is an operator node in the above sense, then we define the main operator A of S as we did above, and the definition of the observation operator C is analogous:

$$\begin{aligned} \mathrm{dom}\,(A) = \mathrm{dom}\,(C) &= \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \mathrm{dom}\,(S)\right\}, \\ Ax &= P_{\mathcal{X}} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \qquad x \in \mathrm{dom}\,(A)\,, \\ Cx &= P_{\mathcal{Y}} S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right], \qquad x \in \mathrm{dom}\,(A)\,. \end{aligned}$$

The definition of the control operator B is more complicated (it maps $\mathcal{U}$ into an "extrapolation space" $\mathcal{X}_{-1}$). The transfer function $\widehat{\mathfrak{D}}$ is defined by

$$\widehat{\mathfrak{D}}(\lambda) = P_{\mathcal{Y}} S \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1} B \\ 1_{\mathcal{U}} \end{bmatrix}, \qquad \lambda \in \rho(A).$$

where $A|_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X}_{-1}$ stands for an extended version of A.

Time domain well-posed i/s/o systems

Definition

An i/s/o system $\Sigma = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$, where S is a "system node", is time-domain well-posed there exists a nonnegative function η such that all trajectories $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ of Σ on $\mathbb{R}^+$ satisfy

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(s)\|_{\mathcal{Y}}^2 \, ds \leq \eta(t)^2 \left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2 \, ds \right), \qquad t \in \mathbb{R}^+.$$

Every well-posed system has four characteristic operator:

- The evolution semi-group $\mathfrak{A}$,
- The input map $\mathfrak{B}$,
- The output map $\mathfrak{C}$,
- The input/output map $\mathfrak{D}$.

The characteristic time domain operators

- $\mathfrak{A}^t$ is the map from the initial state $x_0 \in \mathcal{X}$ at time $t = 0$ to the final state $x(t) \in \mathcal{X}$ at time $t \geq 0$ when the input is zero.
- $\mathfrak{B}$ is the map from an input $u \in L^2(\mathbb{R}^-; \mathcal{U})$ with compact support into the final state $x(0) \in \mathcal{X}$ at time zero, when we take the initial state to be zero for large negative time.
- $\mathfrak{C}$ is the map from the initial state $x_0 \in \mathcal{X}$ at time $t = 0$ to the output $y \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{Y})$ when the input is zero.
- $\mathfrak{D}$ is the map from an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{U})$ whose support is bounded to the left to the output $y \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{Y})$, when we take the initial state to be zero for large negative time.

With the help of these four operators (and some shifts, etc.) one can write a general formula for how to compute $x(t)$ and y from x_0 and u for trajectories on arbitrary intervals $[t_0, t_1]$, or $t_0, \infty)$, or $(-\infty, t_1]$ (see (Sta05) for details).

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We need a different look at the notion of "a dynamical system" when we want to model, e.g., an electrical circuit with distributed components. This circuit is connected to the outside world by a number of "terminals". The model should be "universal" in the sense that we do not specify in advance which of the terminals should be interpreted as "inputs", and which should be interpreted as "outputs".

This can be achieved by first rewriting the above equations in "graph form", and then combining the input and output signals into one. This leads to the notion of a state/signal system. This class of systems has up to now been studied primarily by Damir Arov, Mikael Kurula and myself.

Graph form of i/s/o system

We can rewrite the i/s/o equation

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \quad t\in\mathbb{R}^+, \quad x(0)=x_0, \qquad (2)$$

in graph form to get:

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\\ x(t)\\ u(t)\end{bmatrix} \in \mathrm{graph}\,(S), \qquad t\in\mathbb{R}^+, \qquad x(0)=x_0, \qquad (3)$$

$$\mathrm{graph}\,(S) := \left\{\begin{bmatrix}z\\ y\\ x\\ u\end{bmatrix} \,\middle|\, \left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{dom}\,(S),\ \left[\begin{smallmatrix}z\\ y\end{smallmatrix}\right] = S\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right]\right\}, \qquad (4)$$

This can be further simplified into

$$\begin{bmatrix}\dot{x}(t)\\ x(t)\\ w(t)\end{bmatrix} \in V, \qquad t\in\mathbb{R}^+, \qquad x(0)=x_0, \qquad (5)$$

by combining the input u and the output y into one signal $w = \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]$ (we get V from $\mathrm{graph}\,(S)$ by reordering the components).

Standard assumptions for s/s systems

In the study of the s/s system

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \tag{5}$$

the following (minimal) assumtions are usually used:

1. V is closed.
2. If $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$, then $z = 0$.
3. The set $V_{\mathcal{X}} = \left\{ x \in \mathcal{X} \;\middle|\; \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \text{ for some } z \in \mathcal{X},\ w \in \mathcal{W} \right\}$ is dense in $\mathcal{X}$.

Interpretation:

1. Weakest possible continuity assumption.
2. $\dot{x}(t)$ is uniquely determined by $x(t)$ and $w(t)$.
3. The set of "possible initial states" is dense in $\mathcal{X}$.

Connection to Boundary Relations

- In my talk here in 3 years ago I discussed the relationship between "conservative s/s systems" and "conservative boundary relations", and concluded that these two notions were "essentially the same", i.e., they were different points of view to the same problem. See (AKS12) for details.
- The "non-essential" difference was that in the theory of boundary relations conditions (2) and (3) did not appear "naturally", but they could be "imposed without loss of generality".

Why "non-essential?"

1. V is closed.
2. If $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$, then $z = 0$.
3. The set $V_{\mathcal{X}} = \left\{ x \in \mathcal{X} \;\middle|\; \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \text{ for some } z \in \mathcal{X},\; w \in \mathcal{W} \right\}$ is dense in $\mathcal{X}$.

- In the conservative (and also in the passive) case: Let

$$\mathcal{X}_0 = \left\{ z \in \mathcal{X} \;\middle|\; \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \right\}, \qquad \mathcal{X}_1 = \overline{V_{\mathcal{X}}},$$

  Then $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_0$.
- By replacing $\mathcal{X}$ by $\mathcal{X}_1$ we get a "compressed" conservative (or passive) system which satisfies conditions (1)–(3).
- The new system has the same "external characteristics" as the original (same "Weyl function" and same "Gamma field").

What can be done without conditions (2)–(3)?

1. V is closed.
2. If $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$, then $z = 0$.
3. The set $V_{\mathcal{X}} = \left\{ x \in \mathcal{X} \;\middle|\; \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \text{ for some } z \in \mathcal{X},\ w \in \mathcal{W} \right\}$ is dense in $\mathcal{X}$.

- Thus, at least in the conservative (and passive) case we can assume "without loss of generality" that (1)–(3) hold.
- What about the non-passive case?
- Is there a real need for a theory which does not use conditions (2) and (3)?
- Maybe yes? Such a theory can be developed fairly easily. Below I will discuss the i/s/o version of this theory.
- Recall that the theory of boundary relations is formulated in an i/s/o setting, not in a s/s setting.

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Back to graph version of the i/s/o setting

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \\ x(t) \\ u(t) \end{bmatrix} \in \operatorname{graph}(S), \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \qquad (3)$$

In this equation there is no particular reason why we could not allow S to be multi-valued, i.e., relation.

Notation: For each $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)$ we interpret $S\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ as a affine (= shifted) subspace of $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ and write

$$\begin{bmatrix} z \\ y \\ x \\ u \end{bmatrix} \in \operatorname{graph}(S) \Leftrightarrow \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S) \text{ and } \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] \in S\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right].$$

With this notation (3) becomes

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (6)$$

Can we say anything about equations of this type?

Multi-valued input/state/output system

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \\ x(t) \\ u(t) \end{bmatrix} \in \mathrm{graph}\,(S)\,, \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0. \qquad (3)$$

$\Leftrightarrow$

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \qquad (6)$$

For the moment I assume only that S is a closed relation (i.e., the graph of S is a closed subspace of $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{U} \\ \mathcal{Y} \end{bmatrix}$).

For this class of systems I shall not say anything about time domain well-posedness.

Instead I shall look at frequendy comain well-posedness.

Frequency domain well-posedness

By taking (formal) Laplace transforms in the equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \\ x(t) \\ u(t) \end{bmatrix} \in \operatorname{graph}(S), \qquad t \in \mathbb{R}^+, \qquad x(0) = x_0, \tag{3}$$

we get

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x_0 \\ \hat{y}(\lambda) \\ \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \operatorname{graph}(S). \tag{7}$$

Definition

The system (3) is frequency domain well-posed if there exists at least one $\lambda \in \mathbb{C}$ such that the equation (7) defines a bounded linear everywhere defined map from $\left[\begin{smallmatrix} x_0 \\ \hat{u}(\lambda) \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$.

(In the passive case we will typically require the above condition to be true for all $\lambda \in \mathbb{C}^+$.)

The node bundle

Clearly the condition

$$\begin{bmatrix} \lambda\hat{x}(\lambda)-x_0 \\ \hat{y}(\lambda) \\ \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \operatorname{graph}(S). \tag{7}$$

can be rewritten in the equivalent form

$$\begin{bmatrix} x_0 \\ \hat{y}(\lambda) \\ \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix} \in \mathfrak{E}(\lambda), \tag{8}$$

where

$$\mathfrak{E}(\lambda) = \begin{bmatrix} -1_{\mathcal{X}} & 0 & \lambda & 0 \\ 0 & 1_{\mathcal{Y}} & 0 & 0 \\ 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}} \end{bmatrix} \operatorname{graph}(S). \tag{9}$$

We call $\mathfrak{E}$ the node bundle of the system. It is a subspace-valued analytic function of the complex variable λ.

The graph representation

Lemma

The system (3) is frequency domain well-posed if and only if there exists at least one $\lambda \in \mathbb{C}$ *such that* $\mathfrak{E}(\lambda)$ *has the graph representation*

$$\mathfrak{E}(\lambda) = \operatorname{im}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) \tag{10}$$

for some bounded linear operator
$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.

The proof is trivial. Substituting this into (9) we get

$$\begin{bmatrix} -1_{\mathcal{X}} & 0 & \lambda & 0 \\ 0 & 1_{\mathcal{Y}} & 0 & 0 \\ 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}} \end{bmatrix} \operatorname{graph}(S) = \mathfrak{E}(\lambda) = \operatorname{im}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right).$$

Another graph representation

We can of course eliminate $\mathfrak{E}(\lambda)$ from the above formula:

Lemma

The system (3) *is frequency domain well-posed if and only if there exists at least one* $\lambda \in \mathbb{C}$ *such that* V *has the graph representation*

$$V = \operatorname{im}\left(\begin{bmatrix} \lambda\widehat{\mathfrak{A}}(\lambda) - 1_{\mathcal{X}} & \lambda\widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) \tag{11}$$

for some operator $\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \in \mathcal{L}\left(\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]\right)$.

The proof is still trivial.

The i/s/o resolvent matrix

Definition

1. The i/s/o resolvent set $\rho_{\rm iso}(S)$ of a closed relation $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ consists of those point $\lambda \in \mathbb{C}$ for which $\operatorname{graph}(S)$ has a representation of the type

$$\operatorname{graph}(S) = \operatorname{im}\left(\begin{bmatrix} \lambda\widehat{\mathfrak{A}}(\lambda)-1_{\mathcal{X}} & \lambda\widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix}\right) \tag{11}$$

for some operator $\widehat{\mathfrak{S}}(\lambda) = \left[\begin{smallmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda)\end{smallmatrix}\right] \in \mathcal{L}\left(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right];\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]\right)$.

2. The i/s/o resolvent matrix is the operator-valued function $\widehat{\mathfrak{S}}(\lambda)$ above defined for all $\lambda \in \operatorname{dom}\left(\widehat{\mathfrak{S}}(\lambda)\right) := \rho_{\rm iso}(S)$.

The i/s/o resolvent matrix

Definition

1. The i/s/o resolvent set $\rho_{\mathrm{iso}}(S)$ of a closed relation $S\colon \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ consists of those point $\lambda \in \mathbb{C}$ for which the following identity is valid

$$\begin{bmatrix} -1_{\mathcal{X}} & 0 & \lambda & 0 \\ 0 & 1_{\mathcal{Y}} & 0 & 0 \\ 0 & 0 & 1_{\mathcal{X}} & 0 \\ 0 & 0 & 0 & 1_{\mathcal{U}} \end{bmatrix} \operatorname{graph}(S) = \operatorname{im}\left(\begin{bmatrix} 1_{\mathcal{X}} & 0 \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \\ \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ 0 & 1_{\mathcal{U}} \end{bmatrix} \right), \qquad (11)$$

for some operator $\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} \in \mathcal{L}\left(\begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}\right)$.

2. The i/s/o resolvent matrix of S is the operator-valued function $\widehat{\mathfrak{S}}(\lambda)$ above defined for all $\lambda \in \operatorname{dom}\left(\widehat{\mathfrak{S}}(\lambda)\right) := \rho_{\mathrm{iso}}(S)$.

Terminology

We call:

- $\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix}$ is the i/s/o resolvent matrix,
- $\widehat{\mathfrak{A}}(\lambda)$ is the s/s resolvent function,
- $\widehat{\mathfrak{B}}(\lambda)$ is the i/s resolvent function (= the "Gamma field"),
- $\widehat{\mathfrak{C}}(\lambda)$ is the s/o resolvent function,
- $\widehat{\mathfrak{D}}(\lambda)$ is the i/o resolvent function (= the "Weyl function").

What type of propertiers do these functions have? Recall that they arise from a particular graph representation of a analytic bundle (= an analytic subspace-valued function). They should at least be analytic!

Theorem

The i/s/o resolvent matrix $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{smallmatrix}\right]$ *satisfies the following i/s/o resolvent identities for all* λ, $\mu \in \operatorname{dom}\left(\widehat{\mathfrak{S}}\right)$*:*

$$\widehat{\mathfrak{S}}(\lambda) = \widehat{\mathfrak{S}}(\mu) + (\mu - \lambda) \begin{bmatrix} \widehat{\mathfrak{A}}(\mu) \\ \widehat{\mathfrak{C}}(\mu) \end{bmatrix} \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \end{bmatrix}. \tag{12}$$

or equivalently,

$$\begin{aligned} \widehat{\mathfrak{A}}(\lambda) - \widehat{\mathfrak{A}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{A}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{A}}(\lambda)\widehat{\mathfrak{A}}(\mu), \\ \widehat{\mathfrak{B}}(\lambda) - \widehat{\mathfrak{B}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{B}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{A}}(\lambda)\widehat{\mathfrak{B}}(\mu), \\ \widehat{\mathfrak{C}}(\lambda) - \widehat{\mathfrak{C}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{C}}(\mu)\widehat{\mathfrak{A}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{C}}(\lambda)\widehat{\mathfrak{A}}(\mu), \\ \widehat{\mathfrak{D}}(\lambda) - \widehat{\mathfrak{D}}(\mu) &= (\mu - \lambda)\widehat{\mathfrak{C}}(\mu)\widehat{\mathfrak{B}}(\lambda) = (\mu - \lambda)\widehat{\mathfrak{C}}(\lambda)\widehat{\mathfrak{B}}(\mu). \end{aligned} \tag{13}$$

Mark Opmeer's "Resolvent Linear Systems

- In (Opm06) Mark Opmeer uses the above i/s/o resolvent identities to define what he calls a resolvent linear system. It consists of a quadruple of operator-valued functions $\widehat{\mathfrak{S}} = \begin{bmatrix} \widehat{\mathfrak{A}} & \widehat{\mathfrak{B}} \\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}} \end{bmatrix}$ which satisfy the i/s/o resolvent identities on some open connected subset Ω of the complex plane.
- By adding the condition that Ω contains some right half-plane and that the above functions are polynomially bounded on that half plane he gets a class of dynamical systems, which he calls integrated resolvent linear systems.
- He also defines a slightly larger class of dynamical systems that he calls distriutional resolvent linear systems.

Definition

1. A $\mathcal{L}\left(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right];\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]\right)$-valued function $\widehat{\mathfrak{S}} = \left[\begin{smallmatrix}\widehat{\mathfrak{A}} & \widehat{\mathfrak{B}}\\ \widehat{\mathfrak{C}} & \widehat{\mathfrak{D}}\end{smallmatrix}\right]$ defined on some open set $\Omega \in \mathbb{C}$ is called an i/s/o pseudo-resolvent matrix if it satisfies the i/s/o resolvent identity

$$\widehat{\mathfrak{S}}(\lambda) - \widehat{\mathfrak{S}}(\mu) = (\mu - \lambda)\begin{bmatrix}\widehat{\mathfrak{A}}(\mu)\\ \widehat{\mathfrak{C}}(\mu)\end{bmatrix}\begin{bmatrix}\widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda)\end{bmatrix} \tag{12}$$

for all λ, $\mu \in \Omega$.

2. A $\mathcal{L}(\mathcal{X})$-valued function $\widehat{\mathfrak{A}}$ defined on some open set $\Omega \in \mathbb{C}$ is called a pseudo-resolvent if it satisfies

$$\widehat{\mathfrak{A}}(\lambda) - \widehat{\mathfrak{A}}(\mu) = (\mu - \lambda)\widehat{\mathfrak{A}}(\mu)\widehat{\mathfrak{A}}(\lambda) \tag{14}$$

for all λ, $\mu \in \Omega$.

The (standard) resolvent of a relation

From our earlier definition of a i/s/o resolvent matrix we can extract that definition of the "standard" resolvent of a closed relation:

Definition

1. The resolvent set $\rho(A)$ of a closed relation $A\colon \mathcal{X} \to \mathcal{X}$ consists of those point $\lambda \in \mathbb{C}$ for which $\mathrm{graph}\,(\lambda - A) := \left\{ \left[\begin{smallmatrix} \lambda x - y \\ x \end{smallmatrix}\right] \,\middle|\, x \in \mathrm{dom}\,(A),\ y \in Ax \right\}$ has a representation of the type

$$\mathrm{graph}\ (\lambda - A) = \begin{bmatrix} -1_{\mathcal{X}} & \lambda \\ 0 & 1_{\mathcal{X}} \end{bmatrix} \mathrm{graph}\,(A) = \mathrm{im}\left(\begin{bmatrix} 1_{\mathcal{X}} \\ \widehat{\mathfrak{A}}(\lambda) \end{bmatrix} \right) \quad (15)$$

   for some operator $\widehat{\mathfrak{A}}(\lambda) \in \mathcal{L}\,(\mathcal{X})$.

2. The resolvent of A is the operator-valued function $\widehat{\mathfrak{S}}(\lambda)$ above defined for all $\lambda \in \mathrm{dom}\left(\widehat{\mathfrak{S}}(\lambda)\right) := \rho(A)$.

Pseudo-resolvents are resolvents!

Lemma

1. *If* $\widehat{\mathfrak{A}}$ *is the resolvent of a closed relation* $A\colon \mathcal{X} \to \mathcal{X}$, *then* $\widehat{\mathfrak{A}}$ *satisfies the resolvent identity* (14) *for all* $\lambda, \mu \in \rho(A)$.
2. *Conversely, if* $\widehat{\mathfrak{A}}$ *is a pseudo-resolvent defined on some open set* $\Omega \subset \mathbb{C}$, *then* $\widehat{\mathfrak{A}}$ *is the restriction to* Ω *of the resolvent of some closed relation* $A\colon \mathcal{X} \to \mathcal{X}$.
3. A *is single-valued if and only if* $\widehat{\mathfrak{A}}(\lambda)$ *is injective for some (and hence for all)* $\lambda \in \Omega$.
4. $\mathrm{dom}\,(A)$ *is dense in* $\mathcal{X}$ *if and only if* $\mathrm{im}\left(\widehat{\mathfrak{A}}(\lambda)\right)$ *is dense in* $\mathcal{X}$ *for some (and hence for all)* $\lambda \in \Omega$.
5. $\widehat{\mathfrak{A}}$ *is an analytic function of* λ *on* Ω.

I/s/o pseudo-resolvents are i/s/o resolvents!

Theorem

1. *Recall: If* $\widehat{\mathfrak{S}}$ *is the i/s/o resolvent of a closed relation* $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$*, then* $\widehat{\mathfrak{S}}$ *satisfies the resolvent identity* (12) *for all* λ*,* $\mu \in \rho_{\rm iso}(S)$*.*
2. *Conversely, if* $\widehat{\mathfrak{S}}$ *is an i/s/o pseudo-resolvent matrix defined on some open set* $\Omega \subset \mathbb{C}$*, then* $\widehat{\mathfrak{S}}$ *is the restriction to* Ω *of the i/s/o resolvent matrix of some closed relation* $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$*.*
3. S *is single-valued if and only if the s/s resolvent function* $\widehat{\mathfrak{A}}(\lambda)$ *is injective for some (and hence for all)* $\lambda \in \Omega$*.*
4. $\operatorname{dom}(S)$ *is dense in* $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ *if and only if* $\operatorname{im}\left(\widehat{\mathfrak{A}}(\lambda)\right)$ *is dense in* $\mathcal{X}$ *for some (and hence for all)* $\lambda \in \Omega$*.*
5. $\widehat{\mathfrak{S}}$ *is an analytic function of* λ *on* Ω*.*

How is all this related to "operator nodes"?

Definition

Recall: By an operator node on a triple of Hilbert spaces $(\mathcal{X},\mathcal{U},\mathcal{Y})$ we mean a (possibly unbounded) linear operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ with the following properties. We denote $\mathrm{dom}\,(A) = \left\{x \in \mathcal{X} \;\middle|\; \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right] \in \mathrm{dom}\,(S)\right\}$, define $A\colon \mathrm{dom}\,(A) \to \mathcal{X}$ by $Ax = P_{\mathcal{X}} S \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right]$, and require the following conditions to hold:

1. S is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ (with domain $\mathrm{dom}\,(S)$).
2. $P_{\mathcal{X}} S$ is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\mathcal{X}$ (with domain $\mathrm{dom}\,(S)$).
3. $\mathrm{dom}\,(A)$ is dense in $\mathcal{X}$ and $\rho(A) \neq \emptyset$.
4. For every $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ with $\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{dom}\,(S)$.

We call S a system node if, in addition, A generates a C_0 semigroup.

S operator node $\Leftrightarrow \rho_{\mathrm{iso}}(S) \neq \emptyset$

Theorem

A linear (single-valued) operator $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ is an operator node if and only if $\rho_{\mathrm{iso}}(S) \neq \emptyset$, i.e., the if and only if the system

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} \in S \begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{6}$$

is frequency domain well-posed.

- In particular, every time domain well-posed i/s/o system is automatically frequency domain well-posed. The converse is not true.
- The system (6) can be frequency domain well-posed even in the case where S is a relation.

Example: an integrator

Take $\mathcal{X} = \mathcal{U} = \mathcal{Y} = \mathbb{C}$,
$A = 0$, $B = 1$, $C = 1$, $D = 0$, $S = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$,

$$\Sigma : \begin{cases} \dot{x}(t) = u(t), \\ y(t) = x(t), \end{cases} \quad t \in \mathbb{R}^+, \quad x(0) = x_0.$$

This is a integrator: $y(t) = x_0 + \int_0^t u(s)\,ds$, $t \in \mathbb{R}^+$, and the i/s/o resolvent matrix of this system is

$$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} = \begin{bmatrix} 1/\lambda & 1/\lambda \\ 1/\lambda & 1/\lambda \end{bmatrix}.$$

Let us in this system change the meaning of u and y, so that y becomes the input, and u the output. This will be a system of the type

$$\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} \in S \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0, \tag{6}$$

for a suitable relation S.

Example: freq dom well-posed multi-valued system

It turns out that S is the purely multi-valued ralation whose graph is

$$\operatorname{graph}(S) = \operatorname{im}\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}\right).$$

Thus,

$$\operatorname{dom}(S) = \{\left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right] \mid x \in \mathbb{C}\}, \qquad \operatorname{mul}(S) = \operatorname{im}(S) = \{\left[\begin{smallmatrix} u \\ u \end{smallmatrix}\right] \mid u \in \mathbb{C}\}.$$

If $\begin{bmatrix} x(t) \\ y(t) \\ u(t) \end{bmatrix}$ is a trajectory of this system, then $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \in \operatorname{dom}(S)$, or equivalently, $x(t) = y(t)$, and $\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} \in \operatorname{im}(S)$, i.e., $\dot{x}(t) = u(t)$. Thus, $u(t) = \dot{y}(t)$. The i/s/o resolvent matrix of this system is

$$\widehat{\mathfrak{S}}(\lambda) = \begin{bmatrix} \widehat{\mathfrak{A}}(\lambda) & \widehat{\mathfrak{B}}(\lambda) \\ \widehat{\mathfrak{C}}(\lambda) & \widehat{\mathfrak{D}}(\lambda) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \lambda \end{bmatrix}.$$

If we reinterpret this as a s/s system, then all the three regularity condition (1)–(3) hold.

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- State/signal systems and boundary relations
- Frequency domain well-posed input/state/output systems
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- Compressions and dilations in time and frequency domain
- Controllability, observability, and minimality
- Work in progress

Intertwinement in time domain

Definition

Let $\Sigma_1 = (S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y})$ and $\Sigma_2 = (S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y})$ be two time domain well-posed i/s/o systems (with the same input and output spaces), and let R be a linear relation $\mathcal{X}_1 \to \mathcal{X}_2$. We say that Σ_1 and Σ_2 are intertwined by R if the following condition holds:
If $\begin{bmatrix} x_1 \\ y_1 \\ u \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ u \end{bmatrix}$ are trajectories of Σ_1 and Σ_2 on $\mathbb{R}^+$, respectively (with the same input function u), and if $x_2(0) \in Rx_1(0)$, then $y_1 = y_2$ and $x_2(t) \in Rx_1(t)$ for all $t \in \mathbb{R}^+$.

The characteristic time domain operators

Recall:

- $\mathfrak{A}^t$ is the map from the initial state $x_0 \in \mathcal{X}$ at time $t = 0$ to the final state $x(t) \in \mathcal{X}$ at time $t \geq 0$ when the input is zero.
- $\mathfrak{B}$ is the map from an input $u \in L^2(\mathbb{R}^-; \mathcal{U})$ with compact support into the final state $x(0) \in \mathcal{X}$ at time zero, when we take the initial state to be zero for large negative time.
- $\mathfrak{C}$ is the map from the initial state $x_0 \in \mathcal{X}$ at time $t = 0$ to the output $y \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{Y})$ when the input is zero.
- $\mathfrak{D}$ is the map from an input $u \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{U})$ whose support is bounded to the left to the output $y \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathcal{Y})$, when we take the initial state to be zero for large negative time.

Intertwinement in terms of characteristic operators

Theorem

The two time domain well-posed i/s/o systems Σ_1 and Σ_2 are intertwined by the closed relation R if and only if the characteristic time domain operators of these systems satisfy:

1. $\mathfrak{A}_2^t x_2 \in R\mathfrak{A}_1^t x_1$ *for all* $x_2 \subset Rx_1$ *and all* $t \in \mathbb{R}^+$.
2. *For all* $u \in L^2(\mathbb{R}^-;\mathcal{U})$ *with compact support we have* $\mathfrak{B}_2 u \in R\mathfrak{B}_1 u$.
3. $\mathfrak{C}_2 x_2 = \mathfrak{C}_1 x_1$ *for all* $x_2 \subset Rx_1$
4. $\mathfrak{D}_2 = \mathfrak{D}_1$.

Theorem

The two time domain well-posed i/s/o systems Σ_1 and Σ_2 are intertwined by some closed relation R if and only if they have the same i/o map.

Intertwinement in frequency domain

Theorem

Let Σ_1 and Σ_2 be two time domain well-posed linear systems, with growth rates ω_1 and ω_2, respectively, let $\omega = \max\{\omega_1, \omega_2\}$, and denote $\mathbb{C}^+_\omega = \{\lambda \in \mathbb{C} \mid \Re\lambda > \omega\}$. Then Σ_1 and Σ_2 are intertwined by the closed relation R if and only if the following frequency domain conditions hold:

1. $\widehat{\mathfrak{A}}_2(\lambda)x_2 \in R\widehat{\mathfrak{A}}_1(\lambda)x_1$ *for all* $x_2 \subset Rx_1$ *and all* $\lambda \in \mathbb{C}^+_\omega$.
2. $\widehat{\mathfrak{B}}_2(\lambda)u_0 \in R\widehat{\mathfrak{B}}_1(\lambda)u_0$ *for all* $u_0 \in \mathcal{U}$ *and* $\lambda \in \mathbb{C}^+_\omega$.
3. $\widehat{\mathfrak{C}}_2(\lambda)x_2 = \widehat{\mathfrak{C}}_1(\lambda)x_1$ *for all* $x_2 \subset Rx_1$ *and all* $\lambda \in \mathbb{C}^+_\omega$.
4. $\widehat{\mathfrak{D}}_2(\lambda) = \widehat{\mathfrak{D}}_1(\lambda)$ *for all* $\lambda \in \mathbb{C}^+_\omega$.

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Definition

Let $\mathcal{X}_1$ be a closed subspace of $\mathcal{X}_2$, and let $\Sigma_1 = \big(S_1; \mathcal{X}_1, \mathcal{U}, \mathcal{Y}\big)$ and $\Sigma_2 = \big(S_2; \mathcal{X}_2, \mathcal{U}, \mathcal{Y}\big)$ be two time domain well-posed i/s/s systems. We call Σ_1 the (orthogonal) compression of Σ_2 onto $\mathcal{X}_1$, and we call Σ_2 an (orthogonal) dilation of Σ_1, if the following condition holds:

- For each $x_0 \in \mathcal{X}$ and each $u \in L^2_{\mathrm{loc}}(\mathbb{R}^+; \mathcal{U})$, if we denote the (generalized) future trajectories of Σ_1 and Σ_2 with initial state x_0 and input function u by $\begin{bmatrix} x_1 \\ y_1 \\ u \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ u \end{bmatrix}$, respectively, then $y_1 = y_2$ and $x_1(t) = P_{\mathcal{X}_1} x_2(t)$ for all $t \in \mathbb{R}^+$.

Compressions in terms of characteristic operators

Theorem

The time domain well-posed i/s/o system Σ is the compression onto $\mathcal{X}$ of the time domain well-posed i/s/o system Σ_1 (i.e., Σ_1 is a dilation of Σ) if and only if the characteristic time domain operators of these systems satisfy:

1. $\mathfrak{A}_1^t = P_{\mathcal{X}_1}\mathfrak{A}_2^t|_{\mathcal{X}_1}$ *for all* $t \in \mathbb{R}^+$.
2. $\mathfrak{B}_1 = P_{\mathcal{X}_1}\mathfrak{B}_2$.
3. $\mathfrak{C}_1 = \mathfrak{C}_2|_{\mathcal{X}_1}$.
4. $\mathfrak{D}_1 = \mathfrak{D}_2$.

Theorem

Every dilation (and compression) can be interpreted as a special case of an intertwinement (for a suitable bounded single-valued intertwining operor R with closed domain).

Compressions in frequency domain

Theorem

Let Σ_1 and Σ_2 be two time domain well-posed linear systems, with growth rates ω_1 and ω_2, respectively, let $\omega = \max\{\omega_1, \omega_2\}$, and denote $\mathbb{C}^+_\omega = \{\lambda \in \mathbb{C} \mid \Re\lambda > \omega\}$. Then Σ_1 is the projection of Σ_2 onto $\mathcal{X}_1$ if and only if the following frequency domain conditions hold:

1. $\widehat{\mathfrak{A}}_1(\lambda) = P_{\mathcal{X}_1}\widehat{\mathfrak{A}}_2(\lambda)|_{\mathcal{X}_1}$ *for all* $\lambda \in \mathbb{C}^+_\omega$.
2. $\widehat{\mathfrak{B}}_1(\lambda) = \widehat{\mathfrak{B}}_2(\lambda)$ *for all* $\lambda \in \mathbb{C}^+_\omega$ *(in particular,* $\mathrm{im}\left(\widehat{\mathfrak{B}}_2(\lambda)\right) \subset \mathcal{X}_1$*).*
3. $\widehat{\mathfrak{C}}_1(\lambda) = \widehat{\mathfrak{C}}_2(\lambda)|_{\mathcal{X}_1}$ *for all* $\lambda \in \mathbb{C}^+_\omega$.
4. $\widehat{\mathfrak{D}}_2(\lambda) = \widehat{\mathfrak{D}}_1(\lambda)$ *for all* $\lambda \in \mathbb{C}^+_\omega$.

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Controllable and observable systems

Definition

Let $\Sigma = (X; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a time domain well-posed i/s/o system.

- Σ is controllable if $\operatorname{im}(\mathfrak{B})$ is dense in $\mathcal{X}$
- Σ is observable if $\ker(\mathfrak{C}) = \{0\}$.

Theorem

Let $\Sigma = (X; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ be a time domain well-posed i/s/o system with growth bound $\omega(\Sigma)$.

- *Σ is controllable if and only if $\bigvee_{\lambda \in \mathbb{C}^+_{\omega(\Sigma)}} \operatorname{im}\left(\widehat{\mathfrak{B}}(\lambda)\right) = \mathcal{X}$.*
- *Σ is observable if and only if $\bigcap_{\lambda \in \mathbb{C}^+_{\omega(\Sigma)}} \ker\left(\widehat{\mathfrak{C}}(\lambda)\right) = \{0\}$.*

Minimal Systems

Definition

A time domain well-posed i/s/o system Σ is minimal if it does not have any nontrivial compressions (i.e., it is not a nontrivial dilation of any other well-posed i/s/o system).

Theorem

A time domain well-posed i/s/o system Σ is minimal if and only if it is both controllable and observable.

Theorem

Every non-minimal time domain well-posed i/s/o system Σ can be compressed into a minimal time domain well-posed i/s/o system.

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Intertwinements, Dilations, Compressions for multi-valued i/s/o systems

- Above I defined the basic notions of intertwinements, dilations, compressions, controllability, observability, and minimality in the time domain, assuming time domain well-posedness, and then gave frequency domain interpretations of these notions.
- If a system is not time-domain well-posed, then the above time domain definitions are no longer valid.
- However, nothing prevents us from using the frequency domain characterizations of intertwinements, dilations, compressions, controllability, observability, and minimality as definitions of these notions. Such definitions make sense as soon as the system is frequency domain well-posed.
- This seems to work well even when the generating operator S is allowed to be multi-valued (as long as the system is frequency domain well-posed).

Observations

- In the frequency domain setting we start by choosing som open set $\Omega \subset \mathbb{C}$ where we require the i/s/o resolvent conditions to hold, i.e., we have start with a i/s/o pseudo-resolvent defined in Ω.
- In the above time-domain well-posed setting it turned out that frequency domain results were only valid if we choose Ω to be some right half-plane.
- Different choices of Ω give the the same result whenever $\rho_{\rm iso}(S)$ is connected.
- If $\rho_{\rm iso}(S)$ is not connected, then different choices of Ω can give different results.
- For example, by choosing Ω to be a left half-plane we can sometimers get "backward" results that are different from the corresponding "forward" results (this corresponds to a change of the direction of time).

More observations

- We seem to be able to prove more or less the same results in this frequency domain setting as in the standard time domain well-posed setting.
- So far we have encounterd only one major problem: We can still compress every nonminimal system into a minimal one, but we have not been able to prove that the compressed generating operator is always single-valued whenever the original generating operator S is single-valued.
- The situation is essentially the same in the state/signal setting: In the non-well-posed case it is easier to develop all the relevant notions in the frequency domain, but we still have problems with the "minimality conditions" (2)–(3).
- This is the main resason why we started to look at multi-valued generating operators S in the first place!
- We hope that this will give us an even better bridge to the theory of boundary relations!

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