Passive and Conservative Discrete Time State/Signal Systems

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Outline

- Discrete time-invariant i/s/o systems
- State/signal systems
- Passive state/signal systems
- Representations of state/signal systems
- Realization theory
- I/s/o invariant properties of state/signal systems
- Advantages of state/signal systems
- Applications: LQ optimal control, Kalman filter, etc.
- Continuous time?

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Discrete Time-Invariant I/S/O System

Linear discrete-time-invariant i/s/o (input/state/output) system

$$\Sigma_{i/s/o} : \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \qquad x(0) = x_0, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+. \end{cases} \tag{1}$$

A, B, C, D, are bounded linear operators and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

the **input** $u(n) \in \mathcal{U}$ = the input space,
the **state** $x(n) \in \mathcal{X}$ = the state space,
the **output** $y(n) \in \mathcal{Y}$ = the output space (all Hilbert spaces).

A **trajectory** = a triple of sequences (u, x, y) satisfying (1).

Forward Passive and Conservative I/S/O System

$\Sigma_{i/s/o}$ is **forward passive** if all trajectories satisfy

$$\|x(n+1)\|_{\mathcal{X}}^2 \leq \|x(n)\|_{\mathcal{X}}^2 + \left\langle \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, J \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}, \quad n \in \mathbb{Z}^+.$$

Here

$$j(u,y) = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}.$$

is a **supply rate** induced by the **signature operator** $J = J^* = J^{-1}$.

$\Sigma_{i/s/o}$ is **forward conservative** if we have equality

$$\|x(n+1)\|_{\mathcal{X}}^2 = \|x(n)\|_{\mathcal{X}}^2 + \left\langle \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, J \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}, \quad n \in \mathbb{Z}^+.$$

The Three Most Common Supply Rates

(i) The **scattering** supply rate $j_{\rm sca}(u,y) = -\|y\|^2_{\mathcal{Y}} + \|u\|^2_{\mathcal{U}}$ with signature operator $J_{\rm sca} = \begin{bmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}$.

(ii) The **impedance** supply rate $j_{\rm imp}(u,y) = 2\Re\langle y, \Psi u\rangle_{\mathcal{U}}$ with signature operator $J_{\rm imp} = \begin{bmatrix} 0 & \Psi \\ \Psi^* & 0 \end{bmatrix}$, where Ψ is a unitary operator $\mathcal{U} \to \mathcal{Y}$.

(iii) The **transmission** supply rate $j_{\rm tra}(u,y) = -\langle y, J_{\mathcal{Y}} y\rangle_{\mathcal{Y}} + \langle u, J_{\mathcal{U}} u\rangle_{\mathcal{U}}$ with signature operator $J_{\rm tra} = \begin{bmatrix} -J_{\mathcal{Y}} & 0 \\ 0 & J_{\mathcal{U}} \end{bmatrix}$, where $J_{\mathcal{Y}}$ and $J_{\mathcal{U}}$ are signature operators in $\mathcal{Y}$ and $\mathcal{U}$, respectively.

It is possible to **combine all these cases** into one single setting, called the **s/s (state/signal)** setting. The idea is to introduce a class of systems which **does not distinguish between inputs and outputs**.

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The Signal Space

We start by combining the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ into one **signal space** $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. This signal space has a **natural Kreĭn space inner product** obtained from the signature operator J in the supply rate j, namely

$$\left[\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y' \\ u' \end{bmatrix}\right]_{\mathcal{W}} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y' \\ u' \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}.$$

The **forward passivity inequality** now becomes (with $w(n) = \left[\begin{smallmatrix} y(n) \\ u(n) \end{smallmatrix}\right]$)

$$\|x(n+1)\|^2_{\mathcal{X}} \le \|x(n)\|^2_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}}, \qquad n \in \mathbb{Z}^+.$$

The **forward conservativity equality** becomes

$$\|x(n+1)\|^2_{\mathcal{X}} = \|x(n)\|^2_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}}, \qquad n \in \mathbb{Z}^+.$$

The Node Space and the Generating Subspace

After combining the input and output sequences u and y into one **signal sequence** $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ we can rewrite the basic i/s/o relation

$$\Sigma_{i/s/o} : \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \quad x(0) = x_0, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+. \end{cases} \tag{1}$$

in graph form

$$\Sigma : \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0, \tag{2}$$

where

$$V = \left\{ \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \,\middle|\, \begin{matrix} z = Ax + Bu, \\ y = Cx + Du, \end{matrix} \; w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \; x \in \mathcal{X}, \; u \in \mathcal{U} \right\}.$$

The Node Space and the Generating Subspace (continues)

Repetition:

$$\Sigma : \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \qquad x(0) = x_0. \tag{2}$$

where V is a certain subspace of $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

We call V the **generating subspace** and $\mathfrak{K}$ the **node space** of the **state/system** Σ.

By a **trajectory** of Σ we mean a pair of sequences (x, w) satisfying (2).

We call x the **state component** and w the **signal component** of the trajectory.

Properties of the Generating Subspace

Easy: The generating subspace V has the following properties:

(i) V is closed in $\mathfrak{K}$;

(ii) For every $x \in \mathcal{X}$ there is some $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$;

(iii) If $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V$, then $z = 0$;

(iv) The set $\left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \text{ for some } z \in \mathcal{X} \right\}$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

Interpretation of (i)–(iv)

(ii) For every $x \in \mathcal{X}$ there is some $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \Leftrightarrow$

(ii)' For **every initial state** $x_0 \in \mathcal{X}$ there is **some trajectory** (x, w) satisfying $x(0) = x_0$.

(iii) If $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$, then $z = 0 \Leftrightarrow$

(iii)' A **trajectory** (x, w) **is uniquely determined** by the initial state x_0 and the signal part w.

(i)&(iv) V is closed and $\left\{\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \;\middle|\; \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \text{ for some } z \in \mathcal{X}\right\}$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \Leftrightarrow$

(iv)' The **trajectory** (x, w) **depends continuously** on the intial state x_0 and the signal part w.

State/Signal System: Definition

Definition 1. A triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$, where the **(internal) state space** $\mathcal{X}$ and the **(external) signal space** $\mathcal{W}$ are Kreĭn (or Hilbert) spaces and V is a subspace of the **node space** $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ is called a **s/s (state/signal) node** if V has properties (i)–(iv) listed above.

Note: Different type of state and signal spaces in different applications:

- Passive and conservative systems: $\mathcal{X}$ is a **Hilbert space** and $\mathcal{W}$ is a **Kreĭn space**.
- Suboptimal Nehari (Nehari–Takagari) problem: $\mathcal{X}$ is a **Pontryagin space** (Kreĭn space with finite negative dimension) and $\mathcal{W}$ is a **Kreĭn space**.
- LQ optimal control problem: both $\mathcal{X}$ and $\mathcal{W}$ are **Hilbert spaces**.

The Node Space $\mathfrak{K}$ is Always a Kreĭn Space

The node space $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ inherits a natural inner product from its components:

$$\left[\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix}\right]_{\mathfrak{K}} = -[z_1, z_2]_{\mathcal{X}} + [x_1, x_2]_{\mathcal{X}} + [w_1, w_2]_{\mathcal{W}},$$

$$\begin{bmatrix} z_1 \\ x_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} z_2 \\ x_2 \\ w_2 \end{bmatrix} \in \mathfrak{K}.$$

Thus $\mathfrak{K} = \mathfrak{X} [\dot{+}] \mathcal{W}$, where $\mathfrak{X} := \begin{bmatrix} -\mathcal{X} \\ \mathcal{X} \end{bmatrix}$. Note that the 'future time' component $-[z_1, z_2]_{\mathcal{X}}$ and the 'present time' component $[x_1, x_2]_{\mathcal{X}}$ have opposite signs in $\mathfrak{X}$.

In particular, since $\mathfrak{X}$ has the same positive and negative dimensions, $\mathfrak{K}$ **is always a Kreĭn space** if $\mathcal{X}$ is infinite-dimensional (**not a Pontryagin space**).

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Forward Passive S/S Systems

Recall the **forward passivity inequality and conservativity equality**

$$[x(n+1), x(n+1)]^2_{\mathcal{X}} \le [x(n), x(n)]^2_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}}, \qquad n \in \mathbb{Z}^+, \text{ or}$$
$$[x(n+1), x(n+1)]^2_{\mathcal{X}} = [x(n), x(n)]^2_{\mathcal{X}} + [w(n), w(n)]_{\mathcal{W}}, \qquad n \in \mathbb{Z}^+.$$

Rewrite this in the form

$$\left[\left[\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix}, \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \right]\right]_{\mathfrak{K}} \ge 0 \text{ (or } = 0).$$

True for all trajectories $\Leftrightarrow$ true for all $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] = \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V$. Thus,

- Σ is **forward passive** $\Leftrightarrow$ V is a **nonnegative** subspace of the node space $\mathfrak{K}$,
- Σ is **forward conservative** $\Leftrightarrow$ V is a **neutral** subspace of the node space $\mathfrak{K}$.

Passive S/S Systems

Definition 2. A state/signal system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is

(i) **forward passive** if V is a nonnegative subspace of $\mathfrak{K}$,

(ii) **backward passive** if $V^{[\perp]}$ is a nonpositive subspace of $\mathfrak{K}$,

(iii) **passive** if V is a maximal nonnegative subspace of $\mathfrak{K}$,

(iv) **forward conservative** if V is a neutral subspace of $\mathfrak{K}$ ($V \subset V^{[\perp]}$),

(v) **backward conservative** if $V^{[\perp]}$ is a neutral subspace of $\mathfrak{K}$ ($V^{[\perp]} \subset V$),

(vi) **conservative** if V is a Lagrangian subspace of $\mathfrak{K}$ ($V = V^{[\perp]}$).

The Adjoint System Σ_*

If V is the generating subspace of a s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ with Kreĭn state and signal spaces, then $V^{[\perp]}$ is the generating subspace of another (anti-causal) s/s system that evolves backwards in time.

From this system we get the **adjoint s/s system** Σ_* by reflecting the time direction (to make the system causal) and replacing the signal space $\mathcal{W}$ by $-\mathcal{W}$ (to compensate for the change of sign in the balance equation caused by the change of time direction).

- Σ is **backward** passive or conservative $\Leftrightarrow$ Σ_* is **forward** passive or conservative.
- Σ is **passive** $\Leftrightarrow$ Σ is **both forward and backward** passive.
- Σ is **conservative** $\Leftrightarrow$ Σ is **both forward and backward** conservative.

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I/S/O Repesentations of S/S Systems

Recall: Trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of the i/s/o system $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ satisfy

$$\Sigma_{i/s/o} : \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+, \end{cases} \tag{1}$$

and trajectories $(x(\cdot), w(\cdot))$ of the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ satisfy

$$\Sigma : \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+. \tag{2}$$

A direct sum decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is called an **admissible i/o decomposition of $\mathcal{W}$ for Σ with the corresponding i/s/o representation**[1] $\Sigma_{i/s/o}$ if there is an one-to-one correspondence between the trajectories $(x(\cdot), u(\cdot), y(\cdot))$ of $\Sigma_{i/s/o}$ and the trajectories $(x(\cdot), w(\cdot))$ of Σ (with $w(\cdot) = y(\cdot) + u(\cdot)$, $y(n) = P_{\mathcal{Y}}^{\mathcal{U}} w(n)$, $u(n) = P_{\mathcal{U}}^{\mathcal{Y}} w(n)$).

[1] $\Sigma_{i/s/o}$ is unique as soon as $\mathcal{U}$ and $\mathcal{Y}$ have been fixed.

By splitting $\mathcal{W}$ in different ways we recover 'standard' passivity and conservativity results for different supply rates:

- A **fundamental** decomposition $\mathcal{W} = -\mathcal{Y} \, [\dot{+}] \, \mathcal{U}$ (where $-\mathcal{Y}$ is negative and $\mathcal{U}$ is positive) gives a **scattering representation**,
- A **Lagrangean** decomposition $\mathcal{W} = \mathcal{F} \dot{+} \mathcal{E}$ (where $\mathcal{F} = \mathcal{F}^{[\perp]}$ and $\mathcal{E} = \mathcal{E}^{[\perp]}$) gives an **impedance representation**,
- A **regular** (orthogonal) decomposition $\mathcal{W} = -\mathcal{Y} \, [\dot{+}] \, \mathcal{U}$ (where $\mathcal{Y}$ and $\mathcal{U}$ have the same negative dimension) gives a **transmission representation**.

Thus, all the above i/s/o systems can be seen as 'i/s/o projections' of s/s systems. From a state/signal point of view, they **all represent the same s/s system**. For example, the **Potapov–Ginzburg transform** can be interpreted as a formula which simply describes the **connection between a scattering and a transmission representation** of one and the same s/s system, and the **external Cayley transform** describes the **connection between a scattering and an impedance representation**

Driving Variable Repesentations of S/S Systems

A **driving variable representation** of the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is an i/s/o system[2] $\Sigma_{dv} = \left(\left[\begin{smallmatrix} A' & B' \\ C' & D' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{L}, \mathcal{W} \right)$ with the property that $(x(\cdot), w(\cdot))$ is a trajectory of

$$\Sigma\colon \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \tag{2}$$

if and only if there exists some $\ell(\cdot)$ such that $(x(\cdot), \ell(\cdot), w(\cdot))$ is a trajectory of

$$\Sigma_{dv} : \begin{cases} x(n+1) = A'x(n) + B'\ell(n), & n \in \mathbb{Z}^+, \\ w(n) = C'x(n) + D'\ell(n), & n \in \mathbb{Z}^+. \end{cases} \tag{3}$$

In addition we require D' to have a left-inverse (so that $\ell(\cdot)$ is uniquely determined by and depends continuously on $x(\cdot)$ and $w(\cdot)$).

[2]Note that Σ_{dv} has the same state space as Σ, and that the output space of Σ_{dv} is the signal space of Σ.

Output Nulling Repesentations of S/S Systems

An **output nulling representation** of the s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is an i/s/o system[3] $\Sigma_{on} = \left(\left[\begin{smallmatrix} A'' & B'' \\ C'' & D'' \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}, \mathcal{K} \right)$ with the property that $(x(\cdot), w(\cdot))$ is a trajectory of

$$\Sigma\colon \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \tag{2}$$

if and only if

$$\Sigma_{dv} : \begin{cases} x(n+1) = A''x(n) + B''w(n), & n \in \mathbb{Z}^+, \\ \qquad\quad 0 = C''x(n) + D''w(n), & n \in \mathbb{Z}^+. \end{cases} \tag{4}$$

In addition we require D'' to be surjective (so that the error space $\mathcal{K}$ ($=$ the output space of Σ_{on}) is as small as possible).

[3] Note that Σ_{on} has the same state space as Σ, and that the input space of Σ_{on} is the signal space of Σ.

Every I/S/O Repesentation is a Driving Variable Representation

We can rewrite the standard i/s/o system $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ of Σ in the form

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), && n \in \mathbb{Z}^+, \\ w(n) &= \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} = \begin{bmatrix} C \\ 0 \end{bmatrix} x(n) + \begin{bmatrix} D \\ 1_{\mathcal{U}} \end{bmatrix} u(n), && n \in \mathbb{Z}^+. \end{aligned} \tag{1}$$

This has the form of a **driving variable** representation of Σ, with driving variable u ($=$ the input variable of $\Sigma_{i/s/o}$), and

$$A' = A, \qquad B' = B,$$

$$C' = \begin{bmatrix} C \\ 0 \end{bmatrix}, \qquad D' = \begin{bmatrix} D \\ 1_{\mathcal{U}} \end{bmatrix}.$$

Every I/S/O Repesentation is an Output Nulling Representation

We can rewrite the standard i/s/o system $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ of Σ in the form

$$\begin{aligned} x(n+1) &= Ax(n) + \begin{bmatrix} 0 & B \end{bmatrix} \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, \qquad n \in \mathbb{Z}^+, \\ 0 &= Cx(n) + \begin{bmatrix} -1_{\mathcal{Y}} & D \end{bmatrix} \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, \qquad n \in \mathbb{Z}^+. \end{aligned} \tag{1}$$

This has the form of an **output nulling** representation of Σ, with error variable y (= the output variable of $\Sigma_{i/s/o}$), and

$$\begin{aligned} A'' &= A, \qquad & B'' &= \begin{bmatrix} 0 & B \end{bmatrix}, \\ C'' &= C, \qquad & D'' &= \begin{bmatrix} -1_{\mathcal{Y}} & D \end{bmatrix}. \end{aligned}$$

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Transfer Functions and Behaviors

A **transfer function** describes the **relation between the input and the output**.

In a state/signal system we do not specify which part of the signal space is the input, and which part is the output. **What is the transfer function of a s/s system**?

The i/o transfer function of an i/s/o representation of Σ depends on how we choose the i/o decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$, but **the graph of the transfer function is a s/s invariant** (it does not depend on the i/o decomposition).

Thus, we must replace the notion of "transfer function" by the notion of **the graph of the transfer function**.

By mapping this graph back into the time-domain then we get the notion of a **signal behavior** (= the inverse Laplace transform of the graph of the transfer function).

Below I restrict myself to the **passive** case (so that the behavior $\subset \ell^2(\mathbb{Z}^+)$).

The Behavior of a S/S System

Let $\mathcal{W}$ be a Kreĭn space.

An ℓ^2 **signal behavior** on $\mathcal{W}$ = a **closed right-shift invariant subspace** of $\ell^2(\mathbb{Z}^+;\mathcal{W})$.

Recall: If $(x(\cdot), w(\cdot))$ is a trajectory of a s/s system Σ, then w is called the **signal component** of $(x(\cdot), w(\cdot))$.

A trajectory is **externally generated** if $x(0) = 0$. Such a trajectory is determined uniquely by its signal component w.

The ℓ^2**-behavior** $\mathfrak{W}$ **induced by a s/s system** Σ = the set of all **signal components in** $\ell^2(\mathbb{Z}^+;\mathcal{W})$ **of all externally generated trajectories**. (Easy to see that this is a closed right-shift invariant subspace of $\ell^2(\mathbb{Z}^+;\mathcal{W})$.)

The s/s system Σ is a **realization of the** ℓ^2 **signal behavior** $\mathfrak{W}$ $\Leftrightarrow$ $\mathfrak{W}$ is the ℓ^2-behavior induced by Σ.

The Behavior of a Passive S/S System

Supppose that $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is **forward passive**. Then

$$[x(k+1), x(k+1)]_{\mathcal{X}}^2 \leq [x(k), x(k)]_{\mathcal{X}}^2 + [w(k), w(k)]_{\mathcal{W}}, \qquad k \in \mathbb{Z}^+.$$

Take $x(0) = 0$ and sum over $k = 0, 1, \ldots n$ to get

$$\sum_{k=0}^{n} [w(k), w(k)]_{\mathcal{W}} \geq [x(n+1), x(n+1)]_{\mathcal{X}}^2.$$

In particular, if $w(\cdot) \in \ell^2(\mathbb{Z}^+; \mathcal{W})$ (i.e., $w(\cdot)$ belongs to the ℓ^2-behavior $\mathfrak{W}$ indcued by Σ), then

$$\sum_{k=0}^{\infty} [w(k), w(k)]_{\mathcal{W}} \geq 0.$$

Thus, Σ **forward passive** $\Rightarrow$ **the behavior** $\mathfrak{W}$ **is a nonnegative subspace of** $\ell^2(\mathbb{Z}^+; \mathcal{W})$.

Passive Behaviors

An ℓ^2-behavior $\mathfrak{W}$ on a Kreĭn space $\mathcal{W}$ is **passive** if

(i) $\mathfrak{W}$ is a **nonnegative** subspace of $\ell^2(\mathbb{Z}^+;\mathcal{W})$.

(ii) The **zero section** $\mathfrak{W}(0) = \{w(0) \mid w \in \mathfrak{W}\}$ is a **maximal nonnegative** subspace of $\mathcal{W}$.

This implies, in particular, that $\mathfrak{W}$ is a **maximal nonnegative** subspace of $\ell^2(\mathbb{Z}^+;\mathcal{W})$.

Realizations of Passive Behaviors

It is easy to see:

The behavior indcued by a passive s/s system is passive! (Use a scattering representation to show that also condition (ii) above holds.)

The converse is more interesting:

Does every passive behavior have a passive s/s realization?

YES! There is a **complete passive s/s realization theory** that contains (as projections) the corresponding i/s/o realization theories for

- Schur functions
- Nevanlinna functions
- Potapov functions

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I/S/O Invariant Properties of S/S Systems

There are may properties of s/s system which are i/s/o invariant in the sense that **if one i/s/o representation** of a s/s system Σ has this property, **then every other i/s/o representation of Σ has the same property**. This includes

- **Controllability**, observability, simplicity ([AS05]).
- **Similarity** and pseudo-similarity ([AS05]).
- **Dilations** of s/s systems correspond to dilations of i/s/o representations ([AS05]).
- **Duality** of s/s systems correspond to duality of i/s/o representations ([AS06]).
- **Passivity** (with respect to the supply rate induced by Σ), forward passivity, backward passivity ([AS06]).
- **Conservativity**, forward conservativity, backward conservativity ([AS06]).
- **Optimality**, $*$-optimality ([AS07c]).
- **Losslessness** (transfer function is J-inner) ([AS07c]);

I/S/O Invariant Properties of S/S Systems (continues)

Some other properties are common for all **scattering representations** of passive s/s system (those that correspond to a fundamental decomposition $\mathcal{W} = -\mathcal{Y} \,[\dot{+}]\, \mathcal{U}$ of the signal space), such as

- **Stability** ([AS06]);
- **Strong (forward or backward or both) stability** ([AS07c]).

These stability properties can also be **characterized directly in terms of the underlying s/s system or its behavior** (without any explicit reference to any i/s/o representation) and they are also reflected in the behavior of non-scattering representations of the system.

Losslessness

A **passive behavior** $\mathfrak{W}$ on the signal space $\mathcal{W}$ is

- **forward lossless** if $\mathfrak{W}$ is a neutral subspace of $\ell^2(\mathbb{Z}^+;\mathcal{W})$ ($\mathfrak{W} \subset \mathfrak{W}^{[\perp]}$),
- **backward lossless** if $\mathfrak{W}^{[\perp]}$ is a neutral subspace of $\ell^2(\mathbb{Z}^+;\mathcal{W})$ ($\mathfrak{W}^{[\perp]} \subset \mathfrak{W}$),
- **lossless** if $\mathfrak{W}$ is a Lagrangean subspace of $\ell^2(\mathbb{Z}^+;\mathcal{W})$ ($\mathfrak{W} = \mathfrak{W}^{[\perp]}$).

A **state/signal system** Σ is forward lossless, or backward lossless, or lossless it the behavior $\mathfrak{W}$ induced by Σ has this property.

Note: The transfer function of a scattering representation of Σ is **inner if Σ is forward lossless, co-inner if Σ is backward lossless, and bi-inner if Σ is lossless**. The converse is also true.

Stable I/S/O Systems

We call the i/s/o system

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n), \qquad n \in \mathbb{Z}^+, \\ y(n) &= Cx(n) + Du(n), \qquad n \in \mathbb{Z}^+. \end{aligned} \tag{1}$$

stable if the trajectories of $\Sigma_{i/s/o}$ have the following property:

If $u(\cdot) \in \ell^2(\mathbb{Z}^+;\mathcal{U})$**, then** $x(\cdot) \in \ell^\infty(\mathbb{Z}^+;\mathcal{X})$ **and** $y(\cdot) \in \ell^2(\mathbb{Z}^+;\mathcal{Y})$ (for all possible initial states $x_0 \in \mathcal{X}$).

Σ is **forward strongly stable**, if, in addition $x(n) \to 0$ in $\mathcal{X}$ as $n \to \infty$.
Σ is **backward strongly stable** if Σ_* is forward strongly stable.

A **driving variable** representation Σ_{dv} and an **output nulling** representation Σ_{on} of a s/s system Σ is (strongly) stable if it is (strongly) stable **when interpreted as an i/s/o system**.

Strong Stabilizability $\leftrightarrow$ Losslessness

A **minimal passive** s/s system $\Sigma = (V; \mathcal{X}, \mathcal{W})$ is

- **forward lossless** if and only if Σ **is forward conservative and has a forward strongly stable driving variable representation** (i.e., Σ is forward strongly stabilizable),
- **backward lossless** if and only if Σ **is backward conservative and has a backward strongly stable output nulling representation** (i.e., Σ is backward strongly detectable),
- **lossless** if and only if Σ is **conservative and has an i/s/o representation which is both forward and backward strongly stable** (i.e., Σ is both forward and backward strongly LFT-stabilizable).

In each of the cases described above Σ is **determined uniquely by its behavior** $\mathfrak{W}$ (up to a unitary similarity transformation in the state space).

Outline

- Discrete time-invariant i/s/o systems
- State/signal systems
- Passive state/signal systems
- Representations of state/signal systems
- Realization theory
- I/s/o invariant properties of state/signal systems
- **Advantages of state/signal systems**
- Applications: LQ optimal control, Kalman filter, etc.
- Continuous time?

Advantages of State/Signal Systems

- When one uses the s/s system formulation it is **enough to prove a result for one supply rate** (scattering, impedance, or transmission), and the corresponding results for the other supply rates come **almost for free** (maybe 90% of the proofs are common for all cases and can be carried out in a s/s setting).
- State/signal systems have many different representations (**i/s/o** representations, **driving variable** representations, **output nulling** representations). The appropriate choice of representation simplifies the argument significantly. (Use stable driving variable representations to get **right factorizations** of the transfer function, stable output nulling representations to get **left factorizations**, and stable i/s/o representations to get **coprime factorizations**.)
- Many problem, although typically stated in i/s/o form, are **inherently of state/signal nature**. In this case the **s/s signal setting is even more natural** than the i/s/o setting. **This leads to a better (intiutive) understanding of the problem, and simplifies the formulation of the essential results.**

State/Signal Systems Have Been Used

to study (among others)

- realizations of (passive) behaviors ([AS06]),
- connections between scattering, impedance, and transmission systems ([AS07a]),
- i/s/o invariant tests for controllability and observability ([AS07b]),
- right and left affine representations of transfer functions ([AS07b]),
- right and left coprime representations of transfer functions ([AS07b]),
- realizations of generalized transfer functions (for example of Potapov type) which may have a singularity at the origin ([AS07b]),
- the maximal domain of a Potapov function ([AS07b]).

See **[AS05, AS06, AS07a, AS07b, AS07c, Sta06]** for details.

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Example: LQ Optimal Control

LQ Optimal I/S/O Control Problem: For each given initial state x_0, **find the input sequence $u(\cdot)$ which minimizes the cost function**

$$J(x_0, u) = \sum_{k=0}^{\infty} (\|y(k)\|^2_{\mathcal{Y}} + \|u(k)\|^2_{\mathcal{U}}),$$

where $y(\cdot)$ is the output of the i/s/o system

$$\Sigma_{i/s/o} : \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \quad x(0) = x_0, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+. \end{cases}$$

This is a state/signal problem: It does not matter which part of the signal $w(\cdot) = \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix}$ we regard to be the input!
If, for example, D is invertible, then we can rewrite the equation so that $y(\cdot)$ becomes the input and $u(\cdot)$ the output, **but $J(x_0, w(\cdot))$ stays the same!**

State/Signal LQ Control

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system where $\mathcal{X}$ **and** $\mathcal{W}$ **are Hilbert spaces**. **LQ Optimal S/S Control Problem**: For each given initial state x_0, **find the trajectory** $(x(\cdot), w(\cdot))$ **of the s/s system**

$$\Sigma : \left\{ \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^+, \quad x(0) = x_0. \right.$$

for which $\|w(\cdot)\|_{\ell^2(\mathbb{Z}^+;\mathcal{W})}$ **is minimal.**
As in the LQ optimal i/s/o control problem it turns out that the optimal signal $w(\cdot)$ is of **state feedback type**.

The solution of this problem leads to a **strongly stable forward conservative driving variable representation** of the behavior induced by Σ, and it can be used to construct **right normalized weakly coprime factorizations** of all the transfer functions of all the different i/s/o representations of Σ (work in progress with **Mark Opmeer**).

Example: Deterministic Kalman Filter

Deterministic I/S/O Kalman Filter: For each given final state x_0 which can be reached in a finite number of steps, **find the input sequence** $u(\cdot)$ **which minimizes the cost function**

$$J(x_0, u) = \sum_{k=-\infty}^{-1} (\|y(k)\|_{\mathcal{Y}}^2 + \|u(k)\|_{\mathcal{U}}^2),$$

under the condition $x(0) = x_0$, where $(x(\cdot), u(\cdot), y(\cdot))$ is a trajectory of the i/s/o system

$$\Sigma_{i/s/o} : \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^-, \quad x(-\infty) = 0, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^-. \end{cases}$$

This is a state/signal problem: It does not matter which part of the signal $w(\cdot) = \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix}$ we regard to be the input!

State/Signal Deterministic Kalman Filter

Let $\Sigma = (V; \mathcal{X}, \mathcal{W})$ be a state/signal system where $\mathcal{X}$ **and** $\mathcal{W}$ **are Hilbert spaces**.

S/S Deterministic Kalman Filter: For each given final state x_0 which can be reached in a finite number of steps, **find the trajectory** $(x(\cdot), w(\cdot))$ **of the s/s system**

$$\Sigma : \left\{ \begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \quad n \in \mathbb{Z}^- = \{-1, -2, \ldots\}, \quad x(-\infty) = 0. \right.$$

satisfying $x(0) = x_0$ **for which** $\|w(\cdot)\|_{\ell^2(\mathbb{Z}^-;\mathcal{W})}$ **is minimal.**

As in the deterministic i/s/o Kalman filter it turns out that the optimal signal $w(\cdot)$ is of **signal injection type**.

The solution of this problem leads to a **strongly** $*$**-stable backward conservative output nulling representation** of the behavior induced by Σ, and it can be used to construct **left normalized weakly coprime factorizations** of all the transfer functions of all the different i/s/o representations of Σ (work in progress with **Mark Opmeer**).

Example: Available Storage (Optimal Passive Realization)

I/S/O Available Storage: For each given initial state x_0, **find the input sequence $u(\cdot)$ which maximizes the cost function**

$$J(x_0, u) = -\sum_{k=0}^{\infty} \left\langle \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, J \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right\rangle_{\mathcal{Y} \oplus \mathcal{U}},$$

where $y(\cdot)$ is the output of the i/s/o system

$$\Sigma_{i/s/o} : \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^+, \quad x(0) = x_0, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^+. \end{cases}$$

This is a state/signal problem: It does not matter which part of the signal $w(\cdot) = \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix}$ we regard to be the input! (Use the Kreĭn space inner product in the signal space $\mathcal{W} = \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$ induced by the signature operator J.) (See [AS07c].)

Example: Required Supply ($*$-Optimal Passive Realization)

I/S/O Required Supply: For each given final state x_0 which can be reached in a finite number of steps, **find the input sequence** $u(\cdot)$ **which minimizes the cost function**

$$J(x_0, u) = \sum_{k=-\infty}^{-1} \left\langle \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}, J \begin{bmatrix} y(n) \\ u(n) \end{bmatrix} \right\rangle_{\mathcal{Y} \oplus \mathcal{U}},$$

under the condition $x(0) = x_0$, where $(x(\cdot), u(\cdot), y(\cdot))$ is a trajectory of the i/s/o system

$$\Sigma_{i/s/o} : \begin{cases} x(n+1) = Ax(n) + Bu(n), & n \in \mathbb{Z}^-, \qquad x(-\infty) = 0, \\ y(n) = Cx(n) + Du(n), & n \in \mathbb{Z}^-. \end{cases}$$

This is a state/signal problem: It does not matter which part of the signal $w(\cdot) = \begin{bmatrix} y(\cdot) \\ u(\cdot) \end{bmatrix}$ we regard to be the input! (Use the Kreĭn space inner product in the signal space $\mathcal{W} = \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$ induced by the signature operator J.) (See [AS07c].)

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Continuous Time?

Recall that the node space $\mathfrak{K} := \mathfrak{X} \, [\dot{+}] \, \mathcal{W}$, where $\mathfrak{X} := \left[\begin{smallmatrix} -\mathcal{X} \\ \mathcal{X} \end{smallmatrix} \right]$.

In **discrete time** we throughout interpret the negative copy of $\mathcal{X}$ in $\mathfrak{X}$ as the **future state** $x(n+1)$ ($=$ output), and the positive copy of $\mathcal{X}$ in $\mathfrak{X}$ as the **present state** $x(n)$ ($=$ input).

Thus, the **discrete time theory is based on a fundamental decomposition of** $\mathfrak{X}$ ("internal scattering representation").

To derive the corresponding **continuous time** results one simply **replaces the fundamental decomposition of** $\mathfrak{X}$ **by a Lagrangean decomposition**: $\mathfrak{X} = \mathcal{F} \dot{+} \mathcal{E}$, where $\mathcal{E} := \mathcal{R}\left(\left[\begin{smallmatrix} 1_{\mathcal{X}} \\ 1_{\mathcal{X}} \end{smallmatrix} \right] \right)$ represents the present state $x(t)$ and $\mathcal{F} := \mathcal{R}\left(\left[\begin{smallmatrix} -1_{\mathcal{X}} \\ 1_{\mathcal{X}} \end{smallmatrix} \right] \right)$ represents the present velocity $\dot{x}(t)$ ("internal impedance representation").

Thus, **we pass from discrete to continuous time simply by making a 45° rotation in** $\mathfrak{X}$ ($=$ the state part of $\mathfrak{K}$) (work in progress with **Mikael Kurula**).

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