Hilbert Spaces Contained in Quotients of Kreĭn Spaces, with Applications to Passive State/Signal Realization Theory.

Damir Z. Arov
Division of Mathematical Analysis
Institute of Physics and Mathematics
South-Ukrainian Pedagogical University
65020 Odessa, Ukraine

Olof J. Staffans
Åbo Akademi University
Department of Mathematics
FIN-20500 Åbo, Finland
http://www.abo.fi/~staffans/

Abstract

Let $\mathcal{Z}$ be a maximal nonnegative subspace of a Kreĭn space $\mathfrak{K}$ with (indefinite) inner product $[\cdot,\cdot]_{\mathfrak{K}}$, let $\mathcal{Z}^{[\perp]}$ be the orthogonal companion to $\mathcal{Z}$ in $\mathfrak{K}$, and let $\mathcal{Z}_0 = \mathcal{Z} \cap \mathcal{Z}^{[\perp]}$ be the maximal neutral subspace of $\mathcal{Z}$. Then $[\cdot,\cdot]_{\mathfrak{K}}$ induces a positive inner product in the quotient space $\mathcal{Z}/\mathcal{Z}_0$, and $-[\cdot,\cdot]_{\mathfrak{K}}$ induces a positive inner product in the quotient space $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$. These two inner product spaces way are not, in general, complete. We show that the completions of $\mathcal{Z}/\mathcal{Z}_0$ and $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ can be identified in a natural way with certain subspaces of the quotient spaces $\mathfrak{K}/\mathcal{Z}^{[\perp]}$ and $\mathfrak{K}/\mathcal{Z}$, respectively. The construction of these subspaces is similar to the deBrange–Rovnyak construction used to realize an operator-valued Schur function in the unit disk $\mathbb{D}$ as the characteristic function of a discrete time input/state/output system. More precisely, the completion of $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ can be identified with the following subspace $\mathcal{X}[\mathcal{Z}]$ of $\mathfrak{K}/\mathcal{Z}$. For each $k \in \mathfrak{K}$ we denote the equivalence class in $\mathfrak{K}/\mathcal{Z}$ to which k belongs by $[k] = k + \mathcal{Z}$. Then

$$\mathcal{X}[\mathcal{Z}] = \bigl\{[k] \in \mathfrak{K}/\mathcal{Z} \bigm| \|[k]\|_{\mathcal{X}[\mathcal{Z}]} < \infty\bigr\}, \tag{1}$$

where the norm $\|\cdot\|_{\mathcal{X}[\mathcal{Z}]}$ in $\mathcal{X}[\mathcal{Z}]$ is given by

$$\bigl\|[k]\bigr\|_{\mathcal{X}[\mathcal{Z}]} = \sqrt{\sup_{z\in\mathcal{Z}}(-[k-z,k-z]_{\mathfrak{K}})}. \tag{2}$$

The subspace $\mathcal{X}[\mathcal{Z}^{[\perp]}]$ of $\mathfrak{K}/\mathcal{Z}^{[\perp]}$ is defined in an analogous fashion.

We apply the technique described above to construct three canonical passive state/signal realizations of a given passive behavior $\mathfrak{W}$, namely a) a controllable forward conservative, b) an observable backward conservative, and c) a simple conservative state/signal realization. All of these are determined unique by $\mathfrak{W}$ up to unitary similarity. The passive behavior $\mathfrak{W}$ is roughly the time-domain counterpart of a shift-invariant maximal nonnegative subspace $\mathcal{Z}$ of the Kreĭn space $\mathfrak{K} := H^2(\mathbb{D};\mathcal{W})$, where $\mathcal{W}$ is a Kreĭn space. By decomposing $\mathcal{W}$ in different ways into the direct sum of an input space and an output space and interpreting $\mathcal{Z}$ as the graph of a shift-invariant operator we get the standard input/state/output realizations of Schur functions, Charathéodory functions, and Potapov functions in the unit disk.

References

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