Passive Linear Discrete Time-Invariant Systems

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- Extensions

Discrete time-invariant i/s/o systems

Discrete Time-Invariant I/S/O System

Linear discrete-time-invariant systems are typically modeled as i/s/o (input/state/output) systems of the type

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \qquad k \in \mathbb{Z}^+, \qquad x(0) = x_0, \\ y(k) &= Cx(k) + Du(k), \qquad k \in \mathbb{Z}^+. \end{aligned} \tag{1}$$

Here $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$ and
A, B, C, D, are bounded operators.

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By a trajectory of this system we mean a triple of sequences (u, x, y) satisfying (1).

We denote this system by $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$.

Forward H-Passive I/S/O System

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The system (1) is forward H-passive if all trajectories satisfy the inequality

$$\|\sqrt{H}x(k+1)\|_{\mathcal{X}}^2 - \|\sqrt{H}x(k)\|_{\mathcal{X}}^2 \leq \left\langle \begin{bmatrix} y(k) \\ u(k) \end{bmatrix}, J \begin{bmatrix} y(k) \\ u(k) \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}, \ k \in \mathbb{Z}^+, \qquad (2)$$

where $H > 0$ and J is a given signature operator ($J = J^* = J^{-1}$).

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The positive quadratic form

$$E_H(x) = \|\sqrt{H}x\|_{\mathcal{X}}^2 = \langle x, Hx \rangle_{\mathcal{X}}$$

is called the storage function (Lyapunov function), and the indefinite bilinear form

$$j(u,y) = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, J \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle_{\mathcal{Y}\oplus\mathcal{U}}.$$

is called the supply rate.

Forward H-Conservative System

In terms of the storage function and the supply rate the forward H-passivity inequality (2) becomes

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Thus, forward H-conservative $\Rightarrow$ forward H-passive.

Adjoint I/S/O System

The corresponding backward notions refer to the adjoint (or dual) I/S/O system

$$\begin{aligned} x_*(k+1) &= A^* x_*(k) + C^* y_*(k), \\ u_*(k) &= B^* x_*(k) + D^* y_*(k), \qquad k \in \mathbb{Z}^+, \\ x_*(0) &= x_{*0}. \end{aligned} \tag{5}$$

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$$j_*(y_*, u_*) = \left\langle \begin{bmatrix} u_* \\ y_* \end{bmatrix}, J_* \begin{bmatrix} u_* \\ y_* \end{bmatrix} \right\rangle_{\mathcal{U}\oplus\mathcal{Y}}, \tag{6}$$

where

$$J_* = \begin{bmatrix} 0 & -1_{\mathcal{U}} \\ 1_{\mathcal{Y}} & 0 \end{bmatrix} J^{-1} \begin{bmatrix} 0 & -1_{\mathcal{Y}} \\ 1_{\mathcal{U}} & 0 \end{bmatrix}. \tag{7}$$

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(v) By passive or conservative (with or without the attributes "forward" or "backward") we mean $1_{\mathcal{X}}$-passive or $1_{\mathcal{X}}$-conservative, respectively.

The Three Most Common Supply Rates

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(i) The scattering supply rate $j_{\mathrm{sca}}(u,y) = \|u\|_{\mathcal{U}}^2 - \|y\|_{\mathcal{Y}}^2$ with signature operator $J_{\mathrm{sca}} = \begin{bmatrix} -1_{\mathcal{Y}} & 0 \\ 0 & 1_{\mathcal{U}} \end{bmatrix}$. The signature operator of the dual supply rate is $J_{\mathrm{sca}*} = \begin{bmatrix} -1_{\mathcal{U}} & 0 \\ 0 & 1_{\mathcal{Y}} \end{bmatrix}$.

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(ii) The impedance supply rate $j_{\mathrm{imp}}(u,y) = 2\Re\langle \Psi u, y\rangle_{\mathcal{Y}}$ with signature operator $J_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & \Psi \\ \Psi^* & 0 \end{smallmatrix}\right]$, where Ψ is a unitary operator $\mathcal{U} \to \mathcal{Y}$. The signature operator of the dual supply rate is $J_{\mathrm{imp}*} = \left[\begin{smallmatrix} 0 & \Psi^* \\ \Psi & 0 \end{smallmatrix}\right]$.

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(iii) The transmission supply rate $j_{\rm tra}(u,y) = \langle u, J_{\mathcal U} u\rangle_{\mathcal U} - \langle y, J_{\mathcal Y} y\rangle_{\mathcal Y}$ with signature operator $J_{\rm tra} = \begin{bmatrix} -J_{\mathcal Y} & 0 \\ 0 & J_{\mathcal U} \end{bmatrix}$, where $J_{\mathcal Y}$ and $J_{\mathcal U}$ are signature operators in $\mathcal Y$ and $\mathcal U$, respectively. The signature operator of the dual supply rate is $J_{\rm tra*} = \begin{bmatrix} -J_{\mathcal U} & 0 \\ 0 & J_{\mathcal Y} \end{bmatrix}$.

The KYP Inequality

Easy: $\Sigma_{i/s/o}$ is forward H-passive if and only if $H > 0$ is a solution of the (forward) generalized i/s/o KYP (Kalman–Yakubovich–Popov) inequality[1]

$$E_H(Ax + Bu) - E_H(x) \leq j(u, Cx + Du), \quad x \in \mathcal{D}(\sqrt{H}), \quad u \in \mathcal{U}, \tag{8}$$

[1] In particular, in order for the first term in this inequality to be well-defined we require A to map $\mathcal{D}(\sqrt{H})$ into itself and B to map $\mathcal{U}$ into $\mathcal{D}(\sqrt{H})$.

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Named after Kalman [Kal63], Yakubovich [Yak62], and Popov [Pop61] (the finite-dimensional case with scattering or impedance supply rate).

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- Unbounded H and H^{-1}: [AKP06].

Scattering Systems

$$j_{\rm sca}(u,y) = \|u\|^2_{\mathcal{U}} - \|y\|^2_{\mathcal{Y}}.$$

$$j_{\rm sca*}(y_*,u_*) = \|y_*\|^2_{\mathcal{Y}} - \|u_*\|^2_{\mathcal{U}}.$$

$$\|\sqrt{H}(Ax+Bu)\|^2_{\mathcal{X}} - \|\sqrt{H}x\|^2_{\mathcal{X}} \le \|u\|^2_{\mathcal{U}} - \|Cx+Du\|^2_{\mathcal{Y}}.$$

Forward Scattering H-passive $\Leftrightarrow$ Backward H-passive

A scattering system is forward H-passive $\Leftrightarrow$ backward H-passive.[2] Proof:

[2] This, together with the correspodning impedance result, is why Kalman, Popov and Yakubovich never mention backward H-passivity.

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Case $H = 1_{\mathcal{X}}$:

- $\Sigma_{i/s/o}$ is forward passive $\Leftrightarrow \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is a contraction
- $\Leftrightarrow \left[\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right]$ is a contraction $\Leftrightarrow \Sigma_{i/s/o}^*$ is forward passive
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Forward scattering H-conservative $\not\Rightarrow$ backward H-conservative (not every isometric operator is unitary).

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The Transfer Function

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The answer is related to the transfer function or characteristic function $\mathfrak{D}$ of this system. It is given by

$$\mathfrak{D}(z) = zC(1_{\mathcal{X}} - zA)^{-1}B + D, \qquad z \in \Lambda(A),$$

where $\Lambda(A)$ is the set of points $z \in \mathbb{C}$ for which $1_{\mathcal{X}} - zA$ has a bounded inverse, plus the point at infinity if A has a bounded inverse.

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Roughly:

The KYP-inequlity has a nonnegative solution $\approx$ $\mathfrak{D}$ is a Schur function.

The Restricted Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \Omega)$

The Schur class $\mathcal{S}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$ is the unit ball in $H^\infty(\mathcal{U}, \mathcal{Y}, \mathbb{D})$, i.e.,

$\theta \in \mathcal{S}(\mathcal{U}, \mathcal{Y}; \Omega) \Leftrightarrow \theta$ is a $\mathcal{B}(\mathcal{U}; \mathcal{Y})$-valued analytic function in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ satisfying $\|\theta(z)\| \leq 1$ for all $z \in \mathbb{D}$.

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However, the transfer function $\mathfrak{D}$ of $\Sigma_{i/s/o}$ need not be defined in the full unit disk $\mathbb{D}$.

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(In our case Ω is open, the set of data points is infinite, and the solution is unique.)

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$$K_{\mathrm{sca}}^{\theta *}(z,\zeta) = \frac{1_{\mathcal{U}} - \theta(\zeta)^*\theta(z)}{1 - \overline{\zeta}z}, \quad z,\ \zeta \in \Omega,$$

is nonnegative definite on $\Omega \times \Omega$ (see [RR82]).

Controllable, Observable, Minimal

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- $\Sigma_{i/s/o}$ is controllable if the sets of all states $x(n)$, $n \geq 1$, which appear in some trajectory (u, x, y) of $\Sigma_{i/s/o}$ with $x_0 = 0$ (i.e., an externally generated trajectory) is dense in $\mathcal{X}$.

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- $\Sigma_{i/s/o}$ is minimal if $\Sigma_{i/s/o}$ is both controllable and observable.

The "Bounded Real Lemma"

Theorem 1. *Let* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\text{sca}}\right)$ *be an i/s/o system with scattering supply rate and transfer function* $\mathfrak{D}$*, and let* $\Lambda_0(A)$ *be the connected component of* $\Lambda(A) \cap \mathbb{D}$ *which contains the origin.*

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Impedance Systems

$$j_{\mathrm{imp}}(u, y) = 2\Re\langle \Psi u, y\rangle_{\mathcal{Y}}.$$

$$j_{\mathrm{imp}*}(y_*, u_*) = 2\Re\langle \Psi^* y_*, u_*\rangle_{\mathcal{U}}.$$

$$\|\sqrt{H}(Ax + Bu)\|_{\mathcal{X}}^2 - \|\sqrt{H}x\|_{\mathcal{X}}^2 \leq \langle \Psi u, Cx + Dy\rangle_{\mathcal{Y}} + \langle Cx + Dy, \Psi u\rangle_{\mathcal{Y}}.$$

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- The above transformation has been designed so that $j_{\rm imp}(u, y) = j_{\rm sca}(y^\times, u^\times)$. Thus, the resulting system $\Sigma^\times_{i/s/o}$ is forward scattering H-passive.

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- This implies that $\Sigma_{i/s/o}$ itself is backward impedance H-passive.

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All the results about scattering systems can be converted into results for impedance systems by means of the external Cayley transform.

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This is true if and only if the Carathéodory kernel

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Theorem 2. *Let* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]; \mathcal{U}, \mathcal{X}, \mathcal{U}; j_{\mathrm{imp}} \right)$ *be an i/s/o system with impedance supply rate, signature operator* $J_{\mathrm{imp}} = \left[\begin{smallmatrix} 0 & 1_{\mathcal{U}} \\ 1_{\mathcal{U}} & 0 \end{smallmatrix}\right]$*, and transfer function* $\mathfrak{D}$*. Let* $\Lambda_0(A)$ *be the connected component of* $\Lambda(A) \cap \mathbb{D}$ *which contains the origin.*

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Which one is the better reference case: Impedance or scattering?

Impedance or Scattering as the Reference Case?

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There exist scattering systems which have no impedance counterpart (even if we take $\mathcal{Y} = \mathcal{U}$).

The external Cayley transform maps the class of impedance systems into but not onto the class of scattering systems:

For a given scattering system there need not exist any operator Ψ such that $\Psi + D$ is invertible, hence the external Cayley transform cannot be defined for every scattering system (even if $\mathcal{Y} = \mathcal{U}$).

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Solution: State/signal systems!

Transmission Systems

$$j_{\mathrm{tra}}(u, y) = \langle u, J_{\mathcal{U}} u \rangle_{\mathcal{U}} - \langle y, J_{\mathcal{Y}} y \rangle_{\mathcal{Y}}$$

$$j_{\mathrm{tra}*}(y_*, u_*) = \langle y_*, J_{\mathcal{Y}} y_* \rangle_{\mathcal{Y}} - \langle u_*, J_{\mathcal{U}} u_* \rangle_{\mathcal{U}}.$$

$$\|\sqrt{H}(Ax + Bu)\|_{\mathcal{X}}^2 - \|\sqrt{H}x\|_{\mathcal{X}}^2 \leq \langle u, J_{\mathcal{U}} u \rangle_{\mathcal{U}} - \langle Cx + Dy, J_{\mathcal{Y}}(Cx + Dy) \rangle_{\mathcal{Y}}.$$

Forward Transmission H-passive $\not\Rightarrow$ Backward H-passive

Recall: Forward impedance H-passive $\Rightarrow$ backward H-passive. The proof is based on the fact that the impedance case can be reduced to the scattering case by means of the external Cayley transform.

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Does there exist a counterpart to the external Cayley transform which maps transmission into scattering?

Yes: The Potapov–Ginzburg (or chain scattering) transform.

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Recall: Forward impedance H-passive $\Rightarrow$ backward H-passive. The proof is based on the fact that the impedance case can be reduced to the scattering case by means of the external Cayley transform.

Does there exist a counterpart to the external Cayley transform which maps transmission into scattering?

Yes: The Potapov–Ginzburg (or chain scattering) transform.

(Unfortunately, is is not always defined!)

The Potapov–Ginzburg Transform

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- Split both $\mathcal{Y}$ and $\mathcal{U}$ into a positive and a negative subspace, which are orthogonal to each other: $\mathcal{Y} = -\mathcal{Y}_-[\dot{+}]\mathcal{Y}_+$ and $\mathcal{U} = -\mathcal{U}_-[\dot{+}]\mathcal{U}_+$ (fundamental decompositions).

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- Split the feed-through operator D accordingly into $D = \left[\begin{smallmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{smallmatrix}\right]$. Note that D_{11} maps the negative part of $\mathcal{U}$ into the negative part of $\mathcal{Y}$.

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- Split the output y and the input u into $y = \left[\begin{smallmatrix} y_- \\ y_+ \end{smallmatrix}\right]$ and $u = \left[\begin{smallmatrix} u_- \\ u_+ \end{smallmatrix}\right]$.

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Thus, forward transmission H-passive $\Rightarrow$ the Potapov–Ginzburg transfrom is well defined if and only if D_{11} is surjective.

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The Potapov–Ginzburg transform has been designed so that $j_{\rm tra}(u,y) = j_{\rm sca}(y^{\curvearrowleft}, u^{\curvearrowleft})$. Thus, the resulting system $\Sigma^{\curvearrowleft}_{i/s/o}$ is forward scattering H-passive whenever $\Sigma_{i/s/o}$ is forward transmission H-passive.

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Many results about scattering systems can be converted into results for transmission systems by means of the Potapov–Ginzburg transform.

The Restricted Potapov class $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \Omega)$

The transfer of an transmission H-passive system belongs to the restricted Potapov class $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \Omega)$.

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Solution: We start by first defining the restricted Potapov class $\mathcal{P}(\mathcal{U},\mathcal{Y};\Omega)$.

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- We interpret $\mathcal{U}$ and $\mathcal{Y}$ as Kreĭn spaces, i.e., we replace the original Hilbert space inner products in $\mathcal{Y}$ and $\mathcal{U}$ by the Kreĭn space inner products

$$[y, y']_{\mathcal{Y}} = \langle y, J_{\mathcal{Y}} y' \rangle_{\mathcal{Y}}, \qquad [u, u']_{\mathcal{U}} = \langle u, J_{\mathcal{U}} u' \rangle_{\mathcal{U}}.$$

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- Let $\Omega \subset \mathbb{D}$. A function $\varphi \colon \Omega \to \mathcal{B}(\mathcal{U}; \mathcal{Y})$ belongs to $\mathcal{P}(\mathcal{U}, \mathcal{Y}; \Omega)$ if both the kernels

$$\begin{aligned} K^{\varphi}_{\mathrm{tra}}(z, \zeta) &= \frac{1_{\mathcal{Y}} - \varphi(z)\varphi(\zeta)^*}{1 - z\overline{\zeta}}, \quad z,\ \zeta \in \Omega, \\ K^{\varphi*}_{\mathrm{tra}}(z, \zeta) &= \frac{1_{\mathcal{U}} - \varphi^*(\zeta)\varphi(z)}{1 - \overline{\zeta}z}, \quad z,\ \zeta \in \Omega, \end{aligned} \tag{10}$$

are nonnegative definite on $\Omega \times \Omega$.

The "Potapov Real Lemma"

Theorem 3. *Let* $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right]; \mathcal{U}, \mathcal{X}, \mathcal{Y}; j_{\mathrm{tra}} \right)$ *be an i/s/o system with transmission supply rate, signature operator* $J_{\mathrm{tra}} = \begin{bmatrix} J_{\mathcal{Y}} & 0 \\ 0 & J_{\mathcal{U}} \end{bmatrix}$*, and transfer function* $\mathfrak{D}$*. Let* $\Lambda_0(A)$ *be the connected component of* $\Lambda(A) \cap \mathbb{D}$ *which contains the origin.*

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(i) *If* $\Sigma_{i/s/o}$ *is* H*-passive for some* $H > 0$*, then* $\mathfrak{D}|_{\Lambda_0(A)} \in \mathcal{P}(\mathcal{U}, \mathcal{Y}; \Lambda_0(A))$*.*

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As shown in [AS06b], if $\varphi \in \mathcal{P}(\mathcal{U}, \mathcal{Y}; \mathbb{D})$, then φ does not have an analytic extension to any boundary point of its domain contained in the open unit disk $\mathbb{D}$.

Generalized Potapov Class

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Thus, the Potapov class of functions should be replaced by the Potapov class of relations!

Combine the Scattering, Impedance, and Transmission Cases into One Master Case!

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Yes: Use a state/signal system!

State/Signal Systems

The Signal Space

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We start by combining the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ into one signal space $\mathcal{W} = \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]$. This signal space has a natural Kreĭn space inner product obtained from the signature operator J in the supply rate j, namely

$$\left[\begin{bmatrix}y\\u\end{bmatrix}, \begin{bmatrix}y'\\u'\end{bmatrix}\right]_{\mathcal{W}} = \left\langle \begin{bmatrix}y\\u\end{bmatrix}, J\begin{bmatrix}y'\\u'\end{bmatrix}\right\rangle_{\mathcal{Y}\oplus\mathcal{U}}.$$

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The (forward) H-passivity-inequality (2) now becomes (with $w(k) = \left[\begin{smallmatrix} y(k) \\ u(k) \end{smallmatrix}\right]$)

$$\|\sqrt{H}x(k+1)\|^2_{\mathcal{X}} - \|\sqrt{H}x(k)\|^2_{\mathcal{X}} \le [w(k), w(k)]_{\mathcal{W}}, \qquad k \in \mathbb{Z}^+.$$

The Node Space and the Generating Subspace

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When we combine the input sequence u and the output sequence y into one signal sequence $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, then the basic i/s/o relation (1) can be rewritten in the form

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}, \qquad x(0) = x_0, \tag{11}$$

where the generating subspace V is the subspace of the node space $\mathfrak{K} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ given by (in this case)

$$V = \left\{ \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \,\middle|\, \begin{matrix} z = Ax + Bu, \\ y = Cx + Du, \end{matrix} \; w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \; x \in \mathcal{X}, \; u \in \mathcal{U} \right\}. \tag{12}$$

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By a trajectory of this system we mean a pair of sequences (x, w) satisfying (11).

Properties of the Generating Subspace

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Easy: The generating subspace V has the following properties:

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(i) V is closed in $\mathfrak{K}$;

(ii) For every $x \in \mathcal{X}$ there is some $\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ \mathcal{W} \end{smallmatrix}\right]$ such that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$;

(iii) If $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$, then $z = 0$;

(iv) The set $\left\{ \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right] \,\middle|\, \left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \text{ for some } z \in \mathcal{X} \right\}$ is closed in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

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(i) & (iv) The trajectory (x, w) depends continuously on the intial state x_0 and the signal part w.

State/Signal System: Definition

Definition 4. A triple $\Sigma = (V; \mathcal{X}, \mathcal{W})$, where the (internal) state space $\mathcal{X}$ is a Hilbert space and the (external) signal space $\mathcal{W}$ is a Kreĭn space and V is a subspace of the product space $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ is called a s/s (state/signal) node if it has properties (i)–(iv) listed above. We interpret $\mathfrak{K}$ as a Kreĭn space with the inner product

$$\left[\begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z' \\ x' \\ w' \end{bmatrix} \right]_{\mathfrak{K}} = -\langle z, z' \rangle_{\mathcal{X}} + \langle x, x' \rangle_{\mathcal{X}} + [w, w']_{\mathcal{W}}, \quad \begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z' \\ x' \\ w' \end{bmatrix} \in \mathfrak{K}, \qquad (13)$$

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By a trajectory of Σ we mean a pair of sequences (x, w) satisfying (11). We call x the state component and w the signal component of this trajectory. By the s/s system Σ we mean the s/s node Σ together with all its trajectories.

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We say that a (general) behavior if forward passive if (14) holds for all $w \in \mathfrak{W}$. It is backward passive if the adjoint behavior[3] $\mathfrak{W}_*$ is forward passive. It is passive if it is realizable[4] and both forward and backward passive.

[3] The adjoint behavior is the intersection of the null spaces of the convolution operators $w*$ where $w \in \mathfrak{W}$.

[4] $\mathfrak{W}$ is realizable if it is induced by some s/s system.

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(iv) *If Σ is minimal and $\mathfrak{W}$ is passive, then Σ is H-passive for some $H > 0$.*

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Thus, the state/signal setting contains all the other settings!

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- Left and right coprime representations of s/s systems (of, e.g., impedance or transmission type).
- Generalized input/state/output representations of impedance systems where the bounded operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has been replaced by a closed unbounded system operator.

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- Extension of the s/s theory to continuous time systems.

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