# On the Past Time Final State and Initial State Future Time Optimal Control Problems

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# The Continuous Time Linear System

This talk is about optimal control of a linear time-invariant i/s/o (input/state/output) systems whose dynamics is described by an equation of the type

$$\Sigma : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \quad x(0) = x_0. \tag{1}$$

It has a
input space $\mathcal{U}$ (a Hilbert space),
state space $\mathcal{X}$ (a Hilbert space),
output space $\mathcal{Y}$ (a Hilbert space).
The operator $S$ is supposed to be a system node (or more generally, an operator node). It need not be well-posed.
A system node $S$ has a main operator $A$, a control operator $B$, an observation operator $C$, and a transfer function $\widehat{\mathfrak{D}}$ defined on $\rho(A)$.

# Two Cost Minimization Problems

- In the initial state future time cost minimization problem we fix an initial state $x_0 \in \mathcal{X}$ and minimize the future cost

$$J_{\rm fut}(x_0, u, y) = \int_0^\infty \left(\|u(t)\|_{\mathcal{U}}^2 + \|y(t)\|_{\mathcal{Y}}^2\right) dt, \qquad (2)$$

over a suitable set of generalized stable future trajectories $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ of $\Sigma$ with the given initial state $x(0) = x_0$. (This cost may be zero, or finite and nonzero, or $+\infty$.)

- In the past time final state cost minimization problem we fix a final state $x_0 \in \mathcal{X}$ and minimize the past cost

$$J_{\rm past}(x_0, u, y) = \int_{-\infty}^0 \left(\|u(t)\|_{\mathcal{U}}^2 + \|y(t)\|_{\mathcal{Y}}^2\right) dt, \qquad (3)$$

over a suitable set of generalized stable past trajectories $\begin{bmatrix} x \\ u \\ y \end{bmatrix}$ of $\Sigma$ with the given final state $x(0) = x_0$. (This cost may be zero, or finite and nonzero, or $+\infty$.)

# Classical Stable Trajectories of (1)

Recall the original equation:

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+. \tag{1}$$

**Definition**

Let $S\colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \supset \mathrm{Dom}\,(S) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ be a closed operator.

1. A triple $(x, u, y)$ is called a classical solution of (1) on the interval interval $I$ (where $I = \mathbb{R}^+$ or $I = \mathbb{R}^-$) if $x \in C^1(I; \mathcal{X})$, $u \in C(I; \mathcal{U})$, $y \in C(I; \mathcal{Y})$, and (1) holds.
2. This trajectory is (externally) stable if, in addition, $u \in L^2(I; \mathcal{U})$ and $y \in L^2(I; \mathcal{Y})$.

- Note that we do not require the state $x(t)$ to be bounded (because this is irrelevant at the moment).

# Generalized Future Trajectories (Motivation)

- By taking Laplace transforms in the equation

$$\begin{bmatrix}\dot{x}(t)\\ y(t)\end{bmatrix} = S\begin{bmatrix}x(t)\\ u(t)\end{bmatrix}, \quad t\in\mathbb{R}^+, \quad x(0)=x_0, \qquad (1)$$

we find that the Laplace transform of a classical stable future trajectory $(x, u, y)$ satisfies

$$\begin{bmatrix}\lambda\hat{x}(\lambda)-x_0\\ \hat{y}(\lambda)\end{bmatrix} = S\begin{bmatrix}\hat{x}(\lambda)\\ \hat{u}(\lambda)\end{bmatrix}, \qquad \Re\lambda > 0.$$

- For each $\lambda\in\rho(A)\cap\mathbb{C}^+$ we can solve for $\hat{x}(\lambda)$ and $\hat{y}(\lambda)$ in terms of $x_0$ and $\hat{u}(\lambda)$ to get

$$\hat{x}(\lambda) = (\lambda - A)^{-1}x_0 + (\lambda - A_{|\mathcal{X}})^{-1}B\hat{u}(\lambda), \qquad (4)$$

$$\hat{y}(\lambda) = C(\lambda - A)^{-1}x_0 + \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda). \qquad (5)$$

- In the sequel we ignore (4) but use (5) as a definition of a generalized stable future trajectory of $\Sigma$. (The cost $J_{\mathrm{fut}}(x_0, u, y)$ depends only on $x_0$, $u$, and $y$.)

# Fixing the Transfer Function

Throughout the rest of this talk I fix some (connected) component $\Omega$ of $\rho(A) \cap \mathbb{C}^+$. (If $\rho(A) \cap \mathbb{C}^+$ is connected, then $\Omega = \rho(A) \cap \mathbb{C}^+$.)

For example, $\Omega = \Omega_\infty =$ the component of $\rho(A)$ which contains some right half-plane.

# Generalized Future Trajectories (Definition)

**Definition**

By the set of generalized stable future trajectories of $\Sigma$ we mean the set of all triples $\begin{bmatrix} x_0 \\ u \\ y \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{bmatrix}$ which satisfy

$$\hat{y}(\lambda) = \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda) + C(\lambda - A)^{-1}x_0, \qquad \lambda \in \Omega, \tag{6}$$

where $\hat{u}$ and $\hat{y}$ are the Laplace transforms of $u$ and $y$, respectively.

- We call $x_0$ the initial state, $u$ the input component, and $y$ the output component.
- Note that we here do not actually define the state component $x(t)$ of the trajectory for $t > 0$, but only for $t = 0$.
- However, the input $u$ and output $y$ are almost everywhere defined $L^2$-functions.

# Solution of Initial State Future Time Cost Minimization Problem

Because of the way in which I have defined the notion of a "generalized stable future trajectory of $\Sigma$" (and the assumption that $S$ is a system node), the following result is true:

- The minimum of the future cost function

$$J_{\text{fut}}(x_0, u, y) = \int_0^\infty \left(\|u(t)\|_{\mathcal{U}}^2 + \|y(t)\|_{\mathcal{Y}}^2\right) dt, \qquad (2)$$

  is always achieved at some generalized stable future trajectory $(x_0, u, y)$ of $\Sigma$.
- Thus, I can compute the future cost $\|x_0\|_{\text{fut}}^2$ of every possible initial state $x_0$.
- This cost may be $+\infty$, or it may be zero, or it may be finite and nonzero.

# Generalized Past Trajectories

We define the notion of a general stable past trajectory in a slightly different way, by taking the closure in $\begin{bmatrix} \mathcal{X} \\ L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{bmatrix}$ of the span $\mathfrak{V}_-$ of all classical exponential past trajectories:

$$\mathfrak{V}_- := \operatorname{span} \left\{ \begin{bmatrix} x_0 \\ u \\ y \end{bmatrix} = \begin{bmatrix} (\lambda - A_{|\mathcal{X}})^{-1} B u_0 \\ \mathrm{e}_\lambda u_0 \\ \mathrm{e}_\lambda \widehat{\mathfrak{D}}(\lambda) u_0 \end{bmatrix} \;\middle|\; \begin{matrix} \lambda \in \Omega, \\ u_0 \in \mathcal{U} \end{matrix} \right\}. \tag{7}$$

Here $\mathrm{e}_\lambda$ is the function

$$\mathrm{e}_\lambda(t) = \mathrm{e}^{\lambda t}, \qquad t \in \mathbb{R}^-.$$

# Solution of Past Time Final State Cost Minimization Problem

Because of the way in which I have defined the notion of a "generalized stable past trajectory of $\Sigma$" (and the assumption that $S$ is a system node), the following result is true:

- The minimum of the past time cost function

$$J_{\mathrm{past}}(x_0, u, y) = \int_{-\infty}^{0} \left(\|u(t)\|_{\mathcal{U}}^2 + \|y(t)\|_{\mathcal{Y}}^2\right) dt, \qquad (3)$$

  is always achieved at some generalized stable past trajectory $(x_0, u, y)$ of $\Sigma$.
- Thus, I can compute the past cost $\|x_0\|_{\mathrm{past}}^2$ of every possible final state $x_0$.
- This cost may be $+\infty$, or it may be zero, or it may be finite and nonzero.

# States with Finite or Nonzero Costs

We now encounter the following two crucial questions:

- Which initial states $x_0 \in \mathcal{X}$ have a finite future cost $\|x_0\|^2_{\mathrm{fut}}$?
- Which final states $x_0 \in \mathcal{X}$ have a nonzero past cost $\|x_0\|^2_{\mathrm{past}}$?

It turns out that the answer to these questions are related to the following three questions:

- Does the transfer function $\widehat{\mathfrak{D}}$ have a right $H^\infty$ factorization over $\mathbb{C}^+$?
- Does the transfer function $\widehat{\mathfrak{D}}$ have a left $H^\infty$ factorization over $\mathbb{C}^+$?
- Does the transfer function $\widehat{\mathfrak{D}}$ have a doubly coprime $H^\infty$ factorization over $\mathbb{C}^+$?

(Skip next three slides!)

# Right $H^\infty$ Factorizations

## Definition

1. $\widehat{\mathfrak{D}}$ has a right $H^\infty(\mathbb{C}^+)$ factorization valid in $\Omega$ if there exist two functions $\mathsf{M} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}))$ and $\mathsf{N} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$ such that $\mathsf{M}(\lambda)$ has a bounded inverse for all $\lambda \in \Omega$ and $\widehat{\mathfrak{D}}(\lambda) = \mathsf{N}(\lambda)\mathsf{M}(\lambda)^{-1}$ for all $\lambda \in \Omega$.
2. The factorization in (i) is normalized if the function $\begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix}$ is inner.
3. The factorization in (i) is weakly coprime if N and M have no common right $H^\infty$ factors.

# Left $H^\infty$ Factorizations

**Definition**

1. $\widehat{\mathfrak{D}}$ has a left $H^\infty(\mathbb{C}^+)$ factorization valid in $\Omega$ if there exist two functions $\widetilde{\mathrm{M}} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{Y}))$ and $\widetilde{\mathrm{N}} \in H^\infty(\mathbb{C}^+; \mathcal{L}(\mathcal{U}; \mathcal{Y}))$ such that $\widetilde{\mathrm{M}}(\lambda)$ has a bounded inverse for all $\lambda \in \Omega$ and $\widehat{\mathfrak{D}}(\lambda) = \widetilde{\mathrm{M}}(\lambda)^{-1}\widetilde{\mathrm{N}}(\lambda)$ for all $\lambda \in \Omega$.
2. The factorization in (i) is normalized if the operator function $\begin{bmatrix} \widetilde{\mathrm{N}} & \widetilde{\mathrm{M}} \end{bmatrix}$ is co-inner.
3. The factorization in (i) is weakly coprime if $\widetilde{\mathrm{N}}$ and $\widetilde{\mathrm{M}}$ have no common left $H^\infty$ factors.

**Definition**

$\widehat{\mathfrak{D}}$ has a doubly coprime (Bezout) $H^\infty(\mathbb{C}^+)$ factorization valid in $\Omega$ if there exist functions $\mathrm{M} \in H^\infty(\mathbb{C}^+;\mathcal{L}(\mathcal{U}))$, $\mathrm{N} \in H^\infty(\mathbb{C}^+;\mathcal{L}(\mathcal{U};\mathcal{Y}))$, $\widetilde{\mathrm{X}} \in H^\infty(\mathbb{C}^+;\mathcal{L}(\mathcal{U}))$, $\widetilde{\mathrm{Y}} \in H^\infty(\mathbb{C}^+;\mathcal{L}(\mathcal{Y};\mathcal{U}))$, $\widetilde{\mathrm{M}} \in H^\infty(\mathbb{C}^+;\mathcal{L}(\mathcal{Y}))$, $\widetilde{\mathrm{N}} \in H^\infty(\mathbb{C}^+;\mathcal{L}(\mathcal{U};\mathcal{Y}))$, $\mathrm{X} \in H^\infty(\mathbb{C}^+;\mathcal{L}(\mathcal{Y}))$ and $\mathrm{Y} \in H^\infty(\mathbb{C}^+;\mathcal{L}(\mathcal{U};\mathcal{Y}))$ such that $\left[\begin{smallmatrix}\mathrm{M}\\ \mathrm{N}\end{smallmatrix}\right]$ is a right $H^\infty(\mathbb{C}^+)$ factorization valid in $\Omega$, $[\widetilde{\mathrm{M}},\widetilde{\mathrm{N}}]$ is a left $H^\infty(\mathbb{C}^+)$ factorization valid in $\Omega$ and

$$\begin{bmatrix}\mathrm{M} & \mathrm{Y}\\ \mathrm{N} & \mathrm{X}\end{bmatrix}\begin{bmatrix}\tilde{\mathrm{X}} & -\tilde{\mathrm{Y}}\\ -\tilde{\mathrm{N}} & \tilde{\mathrm{M}}\end{bmatrix} = \begin{bmatrix}\tilde{\mathrm{X}} & -\tilde{\mathrm{Y}}\\ -\tilde{\mathrm{N}} & \tilde{\mathrm{M}}\end{bmatrix}\begin{bmatrix}\mathrm{M} & \mathrm{Y}\\ \mathrm{N} & \mathrm{X}\end{bmatrix} = \begin{bmatrix}1_{\mathcal{Y}} & 0\\ 0 & 1_{\mathcal{U}}\end{bmatrix}$$

on $\mathbb{C}^+$.

# The Input Finite Future Cost Condition

## Definition

1. The system $\Sigma$ satisfies the input finite future cost condition at the point $\alpha \in \Omega$ if $(\alpha - A|_{\mathcal{X}})^{-1} B u_0$ has a finite future cost for every $u_0 \in \mathcal{U}$.
2. The system $\Sigma$ satisfies the finite future cost condition if every initial state in $\mathcal{X}$ has a finite future cost.

Note that $(\alpha - A|_{\mathcal{X}})^{-1} B u_0$ is the state at time zero of the stable classical past exponential trajectory corresponding to the intput function $u(t) = \mathrm{e}^{\alpha t} u_0$.

- Out of these the finite future cost condition (= "optimizability") is the standard assumption in papers dealing with the initial state future time cost minimization problem.
- However, the important condition is not the finite future cost condition but the input finite future cost condition.

# Input Finite Future Cost $\Longleftrightarrow$ Right $H^\infty$ Factorization

**Theorem**

*The following conditions are equivalent for the system $\Sigma$:*

1. *$\Sigma$ satisfies the input finite future cost condition at some point (or equivalently, at every point) $\alpha \in \Omega$.*
2. *The control Riccati equation for $\Sigma$ has an $\alpha$-normalized nonnegative solution for some (or equivalently, for all) $\alpha \in \Omega$.*
3. *The transfer function $\widehat{\mathfrak{D}}$ of $\Sigma$ has a right $H^\infty$-factorization valid in some open subset of $\Omega$.*
4. *The transfer function $\widehat{\mathfrak{D}}$ of $\Sigma$ has a normalized weakly coprime right $H^\infty$-factorization valid in $\Omega$.*

*When these equivalent conditions hold, then the optimal future cost is equal to the minimal $\alpha$-normalized nonnegative solution of the continuous time control Riccati equation for all $\alpha \in \Omega$.*

# The Control Riccati Equation

## Definition

Let $S = \begin{bmatrix} S_0 \\ S_1 \end{bmatrix}$ be a system node with main operator $A$, and control operator $B$, and let $\alpha \in \rho(A) \cap \mathbb{C}^+$. By an $\alpha$-normalized solution of the (generalized) continuous time control Riccati equation induced by $S$ we mean a closed nonnegative sesquilinear symmetric form $q$ on $\mathcal{X}$ with domain $\mathcal{Z}$ satisfying the following conditions:

1. $(\alpha - A)^{-1}\mathcal{Z} \subset \mathcal{Z}$ and $(\alpha - A|_{|\mathcal{X}})^{-1}B\mathcal{U} \subset \mathcal{Z}$;
2. $q$ satisfies the "natural" $\alpha$-normalized control Riccati equation (see next two pages).

# Interpretation of Control Riccati Equation

- The classical interpretation of the control Riccati equation is that we are looking for a feedback pair $S_2\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]=Kx-u$ and a nonnegative Riccati operator $Q$ such that the solution of the equation

$$\begin{bmatrix}\dot{x}(t)\\y(t)\\v(t)\end{bmatrix}=\begin{bmatrix}S_0\\S_1\\S_2\end{bmatrix}\begin{bmatrix}x(t)\\u(t)\end{bmatrix}$$

  satisfies the energy balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|Q^{1/2}x(t)\right\|_{\mathcal{X}}^2+\|y(t)\|_{\mathcal{Y}}^2+\|u(t)\|_{\mathcal{U}}^2=\|v(t)\|_{\mathcal{U}}^2,$$

- The optimal solution to the forward cost minimization with initial state $x(0)=x_0$ is obtained for the input $u$ for which $v(t)\equiv 0$, i.e., $u(t)=Kx(t)$.
- Thus, the minimizing input $u(t)$ is of feedback type $u(t)=Kx(t)$.

# The Control Riccati Equation (continues)

Formally: the statement that the "natural" $\alpha$-normalized control Riccati euation holds" means that

2. There exists an operator $S_2 \colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \mathcal{U}$ with

$$\mathrm{Dom}\,(S_2) = \{\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right] \in \mathrm{Dom}\,(S_0) \mid x_0 \in \mathcal{Z} \text{ and } S_0\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right] \in \mathcal{Z}\} \tag{8}$$

such that for all $\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right] \in \mathrm{Dom}\,(S_2)$,

$$2\Re q\left[S_0\begin{bmatrix}x_0\\ u_0\end{bmatrix}, x_0\right]_{\mathcal{X}} + \left\|S_1\begin{bmatrix}x_0\\ u_0\end{bmatrix}\right\|_{\mathcal{Y}}^2 + \|u_0\|_{\mathcal{U}}^2 = \left\|S_2\begin{bmatrix}x_0\\ u_0\end{bmatrix}\right\|_{\mathcal{U}}^2, \tag{9}$$

and

$$S_2\begin{bmatrix}(\alpha - A|_{|\mathcal{X}})^{-1}B\\ 1_{\mathcal{U}}\end{bmatrix} \text{ has a bounded inverse in } \mathcal{L}(\mathcal{U}). \tag{10}$$

# The Output Coercive Past Cost Condition

## Definition

The system $\Sigma$ satisfies the output coercive past cost condition at the point $\alpha \in \Omega$ if there exists a constant $M > 0$ such that

$$\|C(\alpha - A)^{-1}x_0\|^2_{\mathcal{Y}} \leq M\big(\|u\|^2_{L^2(\mathbb{R}^-;\mathcal{U})} + \|y\|^2_{L^2(\mathbb{R}^-;\mathcal{Y})}\big) \qquad (11)$$

for every generalized stable past trajectory $\begin{bmatrix} x_0 \\ u \\ y \end{bmatrix}$ of $\Sigma$.

This is the dual of the input finite future cost condition:

- the system $\Sigma$ satisfies the output coercive past cost condition at some point $\alpha \in \Omega$ if and only if the dual system $\Sigma_*$ (with system node $S^*$) satisfies the input finite future cost condition at the point $\overline{\alpha} \in \overline{\Omega}$.

It is a weaker condition than "estimatability" = finite cost condition for the dual system.

# Output Coercive Past Cost $\Longleftrightarrow$ Left $H^\infty$ Factorization

**Theorem**

*The following conditions are equivalent for the system $\Sigma$:*

1. *$\Sigma$ satisfies the output coercive past cost condition at some point (or equivalently, at every point) $\alpha \in \Omega$.*
2. *The filter Riccati equation for $\Sigma$ has an $\alpha$-normalized nonnegative solution for some $\alpha \in \Omega$.*
3. *The transfer function $\widehat{\mathfrak{D}}$ of $\Sigma$ has a left $H^\infty$-factorization valid in some open subset of $\Omega$.*
4. *The transfer function $\widehat{\mathfrak{D}}$ of $\Sigma$ has a weakly coprime left $H^\infty$-factorization valid in $\Omega$.*

*When these equivalent conditions hold, then the optimal past cost is the inverse of the minimal $\alpha$-normalized nonnegative solution of the continuous time filter Riccati equation for all $\alpha \in \Omega$.*

Filter Riccati equation for $S$ = control Riccati equation for $S^*$.

# The Past Cost Dominance Condition

## Definition

The system $\Sigma$ satisfies the past cost dominance condition (with respect to $\Omega$) if the optimal future cost $\|\cdot\|^2_{\mathrm{fut}}$ is dominated by the optimal past cost $\|\cdot\|^2_{\mathrm{past}}$, i.e., there is a finite constant $M$ such that $\|x\|^2_{\mathrm{fut}} \leq M\|x\|^2_{\mathrm{past}}$ for every $x \in \mathcal{X}$.

- The past cost dominance condition implies both the input finite future cost condition and the output coercive past cost condition.
- In particular, the past cost dominance condition implies that both the control Riccati equation and the filter Riccati equation for $\Sigma$ have nonnegative solutions $p$ and $q$.
- The converse is not true.

# Past Cost Dominance $\Longleftrightarrow$ Doubly Coprime $H^\infty$ Factorization

**Theorem**

*The following conditions are equivalent for the system $\Sigma$:*

1. *$\Sigma$ satisfies the past cost dominance condition (with respect to $\Omega$).*
2. *For some (or equivalently, for all) $\alpha \in \Omega$ the control Riccati equation for $\Sigma$ has an $\alpha$-normalized nonnegative solution $q$ and the filter Riccati equation for $\Sigma$ has an $\alpha$-normalized nonnegative solution $p$, and $q$ is dominated by the inverse of $p$.*
3. *The transfer function $\widehat{\mathfrak{D}}$ of $\Sigma$ has a doubly coprime $H^\infty$-factorization valid in $\Omega$ (or equivalently, in some open subset of $\Omega$).*

# Technical Progress **0**: Removal of Irrelevant Assumptions

(Almost) all of the existing literature makes at least the following two additional assumptions:

- The transfer function $\widehat{\mathfrak{D}}$ is defined in some right-half plane, and usually it is even bounded in this right-half plane.
- The main operator $A$ generates a strongly continuous semigroup (i.e., $S$ is a system node).

However, our removal of these two conditions is not so significant.

- The reason why we do not use either of the above assumptions is that they would not simplify any of the proofs (on the contrary, they just add irrelevant additional structure which obscures the basic simplicity of the solution).

# Conceptual Advance 1: Unbounded Riccati Operator

A much more significant fact is that we allow the Riccati operator $Q$ (or the quadratic form $q$) to be unbounded. This makes it possible to prove simple necessary and sufficient conditions for the existence of a coprime factorization.

The literature says:

- The function $\widehat{\mathfrak{D}}$ has a right $H^\infty$ factorization if and only if $\widehat{\mathfrak{D}}$ has a (minimal) realization which satisfies the finite future cost condition.
- Suppose that $\widehat{\mathfrak{D}}$ is the transfer function of some (maybe even well-posed) system $\Sigma$ which does not satisfy the finite cost condition. Then the above result tells us absolutely nothing.
- However, our new result does applies also in this case, and it says that
  $\widehat{\mathfrak{D}}$ has a right $H^\infty$ factorizatation
  $\iff$ $\Sigma$ satisfies the input finite future cost condition
  $\iff$ the control Riccati equation has a (unbouded) solution.

# Conceptual Advance 2: Non-Densly Defined Riccati Operator with Nontrivial Kernel

Due to the fact that we allow the Riccati operators $Q$ and $P$ (or the quadratic forms $q$ and $p$) to have a non-dense domain and a nontrivial kernel we can

- remove all controllability and observability assumptions on the underlying system $\Sigma$

# Conceptual Advance **3**: The Past Time Final State Cost Minimization Problem

- There seems to be virtually nothing written about the infinite-dimensional past time final state cost minimization problem in the literature.
- We show that the solution of the past time final state cost minimization problem is the inverse of the initial state future time cost minimization problem for the dual system $\Sigma_*$.

# Conceptual Advance 4: The Coupling Condition for Existence of Doubly Coprime Factorization

- We have shown: $\widehat{\mathfrak{D}}$ has a doubly coprime $H$ factorization $\Longleftrightarrow$ the future cost of $\Sigma$ is dominated by the past cost.
- This can be interpreted as (previously unknown) coupling condition between the solutions of the control and filter Riccati equations:
  $\Longleftrightarrow$ the product of the two Riccati Operators $Q$ and $P$ is bounded
  (although $Q$ and/or $P$ may be separately unbounded).
- This is the natural condition that one obtains from the $H^\infty$ minimization problem by letting the norm parameter $\gamma \to \infty$.
- This is in sharp contrast to the prevailing theory, which says that there is no coupling between the $H^2$-optimal control and the $H^2$-optimal filter. Indeed, there is formally no coupling, but as a matter of fact, $P$ and $Q$ are coupled in the above sense if and only if $\widehat{\mathfrak{D}}$ has a doubly coprime factorization.

# Conceptual Advance **5**: Construction of Stabilizable and Detectable Realization

The following result is more or less true (work in progress):

How to construct a stabilizable and detectable realization

- A necessary condition for $\widehat{\mathfrak{D}}$ to have a stabilizable and detectable realization is that $\widehat{\mathfrak{D}}$ has a doubly coprime factorization.
- Suppose that $\widehat{\mathfrak{D}}$ has a doubly coprime factorization, and that $\widehat{\mathfrak{D}}$ is bounded in some right half-plane (i.e., "well-posed").
  - Choose an arbitrary system (or operator) node relization $\Sigma$ of $\widehat{\mathfrak{D}}$.
  - Restrict $\Sigma$ to the reachable subspace, and factor out the unobservable subspace.
  - Replace the original norm by the half way intepolation of $\|\cdot\|_{\mathrm{fut}}$ and $\|\cdot\|_{\mathrm{past}}$, and complete the space with respect to this norm (the resulting realization will be minimal and LQG balanced).
- Then the resulting system is well-posed, stabilizable and detectable (and unique up to unitary similarity)

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