How to Complete a Maximal Nonnegative Subspace of a Kreĭn Space?

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Outline

PART I: Completion of a Maximal Nonnegative Subspace

- Maximal Nonnegative subspaces of Kreĭn spaces
- The Hilbert spaces $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z}^{[\perp]})$

PART II: Correspondence to the de Branges Complementary Space

PART III: Passive State/Signal Systems

- The co-isometric de Branges–Rovnyak i/s/o model
- The co-isometric state/signal model

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More precisely, there exist a Hilbert space inner product $(\cdot,\cdot)_{\mathfrak{K}}$ in $\mathfrak{K}$ and an operator $J \in \mathcal{B}(\mathfrak{K})$, $J = J^* = J^{-1}$ (i.e., J is both self-adjoint and unitary), such that

$$[k_1, k_2]_{\mathfrak{K}} = (k_1, Jk_2)_{\mathfrak{K}}, \quad k_1,\ k_2 \in \mathfrak{K}.$$

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The orthogonal companion $\mathcal{Z}^{[\perp]}$ to a subspace $\mathcal{Z} \subset \mathfrak{K}$ is given by

$$\mathcal{Z}^{[\perp]} = \{k \in \mathfrak{K} \mid [k, z]_{\mathfrak{K}} = 0\ \forall z \in \mathcal{Z}\}.$$

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Let $\mathcal{Z}$ be maximal nonnegative. The maximal neutral subspace $\mathcal{Z}_0$ of $\mathcal{Z}$ is given by $\mathcal{Z}_0 = \mathcal{Z} \cap \mathcal{Z}^{[\perp]}$. This is the largest neutral subspace in $\mathcal{Z}$, and also the largest neutral subspace in $\mathcal{Z}^{[\perp]}$.

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Special case: we take $\mathcal{Z}$ to be either maximal nonnegative or maximal nonpositive. Such a subspace is automatically closed (with respect to the standard quotient topology).

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This implies that $\langle \cdot, \cdot \rangle_{\mathcal{Z}}$ induces a positive (nondegenerate) inner product on the quotient space $\mathcal{Z}/\mathcal{Z}_0$. We denote this inner product by $(\cdot, \cdot)_{\mathcal{Z}/\mathcal{Z}_0}$. Thus,

$$([z_1], [z_2])_{\mathcal{Z}/\mathcal{Z}_0} := \langle z_1, z_2 \rangle_{\mathcal{Z}} = [z_1, z_2]_{\mathfrak{K}},$$

where $[z_1]$ and $[z_2]$ stand for the equivalence classes $[z_i] := z_i + \mathcal{Z}_0$, $i = 1, 2$. With this inner product $\mathcal{Z}/\mathcal{Z}_0$ becomes a pre-Hilbert space (not necessary complete).

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What does the completion of $\mathcal{Z}/\mathcal{Z}_0$ look like?

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The Completion of $\mathcal{Z}/\mathcal{Z}_0$

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(ii) the completion of the pre-Hilbert space $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ with the inner product inherited from $-\mathfrak{K}$ can be identified in a natural way with a certain subspace $\mathcal{H}(\mathcal{Z})$ of $-\mathfrak{K}/\mathcal{Z}$.

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The construction of the Hilbert spaces $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z}^{[\perp]})$ is an abstract version of the functional construction by Louis de Branges and James Rovnyak in [dBR66b].

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$$\mathcal{X}[\mathcal{Z}] = \left\{k + \mathcal{Z} \in \mathfrak{K}/\mathcal{Z} \;\middle|\; \|k + \mathcal{Z}\|_{\mathcal{X}[\mathcal{Z}]} < \infty\right\}, \tag{1}$$

where the (Hilbert space) norm $\|k+\mathcal{Z}\|_{\mathcal{X}[\mathcal{Z}]}$ of the equivalence class $k+\mathcal{Z} \in \mathfrak{K}/\mathcal{Z}$ is the squre root of

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Thus, the norm in $\mathcal{X}[Z]$ is simply the 'Kreĭn space version' of the quotient norm in $\mathfrak{K}/\mathcal{Z}$ when $\mathcal{Z}$ is maximal nonnegative!

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- Answer to the original question: The completion of $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ is the subspace of those vectors in the quotient $\mathfrak{K}/\mathcal{Z}$ whose 'natural quotient norm' is finite.

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- All our proofs are direct and elementary, and require no knowledge of reproducing kernel Hilbert spaces or de Branges–Rovnyak spaces.

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Alternatively, $\mathcal{H}(S) = \mathcal{M}((1 - SS^*)^{1/2})$.

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Moreover, $\mathfrak{K} = -\mathcal{Y} \dot{+} \mathcal{Z}$, i.e., $-\mathcal{Y}$ is a direct complement to $\mathcal{Z}$ in $\mathcal{W}$. This implies that to each equivalence class $x \in \mathfrak{K}/\mathcal{Z}$ there corresponds a unique $y \in \mathcal{Y}$ such that $x = y + \mathcal{Z}$.

The mapping $T : x \to y$, where $y \in \mathcal{Y}$ and $y + \mathcal{Z} = x$, is a continuous bijection $\mathfrak{K}/\mathcal{Z} \to \mathcal{Y}$, and

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Under this mapping the image of the dense subspace $\mathcal{Z}^{[\perp]}/\mathcal{Z}_0$ of $\mathcal{H}(\mathcal{Z})$ is mapped onto $\mathcal{R}(1 - SS^*)$ (recall that $\mathcal{H}(S)$ is the range space of $(1 - SS^*)^{1/2}$).

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Input/State/Output System

The i/s/o (input/state/output) system

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n) \\ y(n) &= Cx(n) + Du(n), \quad n \in \mathbb{Z}^+; \quad x(0) = x_0, \end{aligned} \tag{6}$$

is called a passive realization of the Schur function $\mathfrak{D}$ if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is contractive and $\mathfrak{D}(z) = zC(1 - zA)^{-1}B + D$, $z \in \mathbb{D}$.

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The i/s/o (input/state/output) system

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n) \\ y(n) &= Cx(n) + Du(n), \quad n \in \mathbb{Z}^+; \quad x(0) = x_0, \end{aligned} \tag{6}$$

is called a passive realization of the Schur function $\mathfrak{D}$ if $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ is contractive and $\mathfrak{D}(z) = zC(1 - zA)^{-1}B + D$, $z \in \mathbb{D}$.

If $u \in \ell^2_+(\mathcal{U})$, then the Z-transforms of x, u, and y satisfy

$$\begin{bmatrix} \frac{\hat{x}(z) - x_0}{z} \\ \hat{y}(z) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x}(z) \\ \hat{u}(z) \end{bmatrix}, \quad z \in \mathbb{D}. \tag{7}$$

From this equation we can solve $\hat{x}(z)$ and $\hat{y}(z)$ in terms of x_0 and $\hat{u}(z)$ (over):

Input/State/Output System (continues)

$$\begin{bmatrix} \hat{x}(z) \\ \hat{y}(z) \end{bmatrix} = \begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} \begin{bmatrix} x_0 \\ \hat{u}(z) \end{bmatrix}, \quad z \in \mathbb{D}, \tag{8}$$

where

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} z(1 - zA)^{-1} & z(1 - zA)^{-1}B \\ C(1 - zA)^{-1} & zC(1 - A)^{-1}B + D \end{bmatrix}, \qquad z \in \mathbb{D}. \tag{9}$$

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where

$$\begin{bmatrix} \mathfrak{A}(z) & \mathfrak{B}(z) \\ \mathfrak{C}(z) & \mathfrak{D}(z) \end{bmatrix} = \begin{bmatrix} z(1-zA)^{-1} & z(1-zA)^{-1}B \\ C(1-zA)^{-1} & zC(1-A)^{-1}B + D \end{bmatrix}, \qquad z \in \mathbb{D}. \tag{9}$$

The 2×2 block operator in (9) is the input-state/state-output transfer function, and the bottom right corner (the input/output transfer function) is required to be equal to the given Schur function.

Co-Isometric Observable I/S/O Realization

In the co-isometric observable passive realization we interpret $\mathfrak{D}$ as a contractive multiplication operator $S\colon H^2(\mathcal{U}) \to H^2(\mathcal{Y})$, and the state space is $\mathcal{X} = \mathcal{H}(S) \subset H^2(\mathcal{Y})$.

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The coefficient matrix is given by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x}(z) \\ u_0 \end{bmatrix} = \begin{bmatrix} \frac{\hat{x}(z)-\hat{x}(0)}{z} & \frac{\mathfrak{D}(z)-\mathfrak{D}(0)}{z} u_0 \\ \hat{x}(0) & \mathfrak{D}(0)u_0 \end{bmatrix}, \quad \left[\begin{smallmatrix} \hat{x} \\ u_0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right], \quad z \in \mathbb{D}.$$

This realization was discovered by de Branges and Rovnyak [dBR66a, dBR66b].

Outline

PART I: Completion of a Maximal Nonnegative Subspace

- Maximal Nonnegative subspaces of Kreĭn spaces
- The Hilbert spaces $\mathcal{H}(\mathcal{Z})$ and $\mathcal{H}(\mathcal{Z}^{[\perp]})$

PART II: Correspondence to the de Branges Complementary Space

PART III: Passive State/Signal Systems

- The co-isometric de Branges–Rovnyak i/s/o model
- The co-isometric state/signal model

State/Signal Systems

We get the corresponding s/s (state/signal) system by replacing $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ by its graph:

$$\begin{bmatrix} x(n+1) \\ x(n) \\ w(n) \end{bmatrix} \in V, \qquad n \in \mathbb{Z}^+, \quad x(0) = x_0, \tag{10}$$

where $w(n) = \begin{bmatrix} y(n) \\ u(n) \end{bmatrix}$ and

$$V = \left\{ \begin{bmatrix} Ax_0 + Bu_0 \\ x_0 \\ Cx_0 + Du_0 \\ u_0 \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{Y} \\ \mathcal{U} \end{bmatrix} \;\middle|\; \left[\begin{smallmatrix} x_0 \\ w_0 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \right\}, \qquad \mathcal{W} = \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right].$$

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The corresponding graph representation of (7) is given by

$$\begin{bmatrix} \frac{1}{z}(\hat{x}(z) - x_0) \\ \hat{x}(z) \\ \hat{w}(z) \end{bmatrix} \in V, \qquad z \in \mathbb{D}. \tag{11}$$

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The graph $\mathcal{Z} := \left\{ \left[\begin{smallmatrix} \mathfrak{D}\hat{u} \\ \hat{u} \end{smallmatrix}\right] \;\middle|\; \hat{u} \in H^2(\mathcal{U}) \right\}$ of $\mathfrak{D}$ is equal to the set of all $\hat{w} \in H^2(\mathcal{W})$ for which there exists some $\hat{x}$ such that (11) holds with $x_0 = 0$.

Co-Isometric Observable S/S Realization

In [AS08a, AS08b] we present a co-isometric observable passive s/s realization of the graph of $\mathfrak{D}$. It is unitarily similar to the s/s realization that one gets by interpreting the de Branges–Rovnyak i/s/o realization as a s/s system.

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The underlying Kreĭn space is $\mathcal{K} = K^2(\mathcal{W})$, which is the Hardy space over $\mathbb{D}$ with values in $\mathcal{W}$ and with the indefinite inner product inherited from the Kreĭn space $\mathcal{W} = -\mathcal{Y}\,[\dot{+}]\,\mathcal{U}$.

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The generating subspace has the image representation

$$V = \left\{ \begin{bmatrix} \frac{\hat{w}(z)-\hat{w}(0)}{z} + \mathcal{Z} \\ \hat{w}(z) + \mathcal{Z} \\ \hat{w}(0) \end{bmatrix} \,\middle|\, \hat{w}(\cdot) \in K^2(\mathcal{W}),\ \hat{w}(\cdot) + \mathcal{Z} \in \mathcal{X} \right\}.$$

Isometric and Conservative S/S Realizations

In [AS08b] we also present an isometric controllable passive s/s realization of the graph of $\mathfrak{D}$. It is unitarily similar to the s/s realization that one gets by interpreting the isometric controllable de Branges–Rovnyak i/s/o realization as a s/s system.

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There also exists a simple unitary s/s realization. This one is unitarily similar to the s/s realization that one gets by interpreting the simple unitary de Branges–Rovnyak i/s/o realization as a s/s system.

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There also exists a simple unitary s/s realization. This one is unitarily similar to the s/s realization that one gets by interpreting the simple unitary de Branges–Rovnyak i/s/o realization as a s/s system.

Open problem: Do there exist s/s versions of the Sz.-Nagy–Foias and Pavlov models?

Nevanlinna and Potapov Functions

The above construction is based entirely on the maximal nonnegative subspace $\mathcal{Z}$ of $K^2(\mathcal{W})$. The graph representation $\mathcal{Z} = \left\{ \left[\begin{smallmatrix} \mathfrak{D}\hat{u} \\ \hat{u} \end{smallmatrix} \right] \;\middle|\; \hat{u} \in H^2(\mathcal{U}) \right\}$ of $\mathcal{Z}$ is irrelevant.

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The same construction can be used to give a passive s/s realization of an arbitrary maximal nonnegative shift-invariant subspace of $K^2(\mathcal{W})$.

Thus, instead of realizing the graph of a Schur function we can use the same method to realize the graph of a Nevanlinna function (or relation) or of a Potapov function (or relation).

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Thus, instead of realizing the graph of a Schur function we can use the same method to realize the graph of a Nevanlinna function (or relation) or of a Potapov function (or relation).

All the corresponding i/s/o realizations can be regarded as i/s/o representations of the s/s realization of $\mathcal{Z}$ that one obtains by decomposing the signal space $\mathcal{W}$ in different ways:

Nevanlinna and Potapov Functions

- If the decomposition $\mathcal{W} = \mathcal{Y} \dot{+} \mathcal{U}$ is fudamental (i.e., $\mathcal{U}$ is uniformly positive and $\mathcal{Y} = \mathcal{U}^{[\perp]}$), then we get a scattering passive i/s/o system.

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- By taking both $\mathcal{U}$ and $\mathcal{Y}$ to be neutral we get an impedance passive i/s/o system.

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This gives us

Nevanlinna and Potapov Functions

- If the decomposition $\mathcal{W} = \mathcal{Y} \dotplus \mathcal{U}$ is fudamental (i.e., $\mathcal{U}$ is uniformly positive and $\mathcal{Y} = \mathcal{U}^{[\perp]}$), then we get a scattering passive i/s/o system.

- By taking both $\mathcal{U}$ and $\mathcal{Y}$ to be neutral we get an impedance passive i/s/o system.

- By taking $\mathcal{U}$ to be a Kreĭn subspace of $\mathcal{W}$ and $\mathcal{Y} = \mathcal{U}^{[\perp]}$ we get a transmission passive i/s/o system.

This gives us

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Nevanlinna and Potapov Functions

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This gives us

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This gives us

- scattering passive i/s/o realizations of given Schur function,

- impedance passive i/s/o realizations of a given Nevanlinna function,

- transmission passive i/s/o realizations of a given Potapov function.

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