

A PHYSICALLY MOTIVATED CLASS OF SCATTERING PASSIVE LINEAR SYSTEMS*

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Abstract. We introduce a class of scattering passive linear systems motivated by examples from mathematical physics. The state space of the system is $X = H \oplus E$, where H and E are Hilbert spaces. We also have a Hilbert space E_0 which is dense in E , with continuous embedding, and E'_0 is the dual of E_0 with respect to the pivot space E . The input space is the same as the output space, and it is denoted by U . The semigroup generator has the structure $A = \begin{bmatrix} 0 & -L \\ L^* & G - \frac{1}{2}K^*K \end{bmatrix}$, where $L \in \mathcal{L}(E_0, H)$ and $K \in \mathcal{L}(E_0, U)$ are such that $\begin{bmatrix} L \\ K \end{bmatrix}$, with domain E_0 , is closed as an unbounded operator from E to $H \oplus U$. The operator $G \in \mathcal{L}(E_0, E'_0)$ is such that $\operatorname{Re} \langle Gw_0, w_0 \rangle \leq 0$ for all $w_0 \in E_0$. The observation operator is $C = \begin{bmatrix} 0 & -K \end{bmatrix}$, the control operator is $B = -C^*$, and the output equation is $y = Cx + u = -Kw + u$, where u is the input function, $x = \begin{bmatrix} z \\ w \end{bmatrix}$ is the state trajectory, and y is the corresponding output function. We show that this system is scattering passive (hence, well-posed) and that classical solutions of the system equation $\dot{x} = Ax + Bu$ satisfy $\frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 + 2\operatorname{Re} \langle Gw, w \rangle$. Moreover, the dual system satisfies a similar power balance equation. Hence, this system is scattering conservative if and only if $\operatorname{Re} \langle Gw_0, w_0 \rangle = 0$ for all $w_0 \in E_0$. We give two examples involving the beam equation and one with Maxwell's equations.

Key words. scattering passive system, scattering conservative system, system node, Cayley transform, beam equation, Maxwell's equations

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1. The main results. Given four operators A, B, C, D on appropriate Hilbert spaces, a natural question is whether they determine a scattering passive or conservative (in particular, well-posed) linear system via the equations $\dot{x} = Ax + Bu$, $y = Cx + Du$. This was studied for the first time in Arov and Nudelman [4], using earlier results about discrete-time scattering passive systems and translating those results using the internal Cayley transform. More results about scattering passive systems were derived in Staffans and Weiss [35] (where they were called dissipative systems), and relatively simple necessary and sufficient conditions for a system node to be scattering conservative were provided in Malinen, Staffans, and Weiss [24]. A good overview of these results can be found in the book by Staffans [34], and the connection with impedance passive and conservative systems is studied in Staffans [31, 32, 33].

It is of interest to identify large classes of systems where the operators A, B, C, D have a special structure observed in models of mathematical physics, which implies that the system is scattering passive or conservative. Indeed, if we then find a system with this special structure, we do not have to take the trouble of checking the conditions for scattering passivity or conservativity given in the papers listed earlier. (This kind of checking need not be straightforward.) Such a special class of conservative

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systems (“from thin air”) has been introduced in Weiss and Tucsnak [46] and further studied in Tucsnak and Weiss [38] and in Staffans [33]. In this paper we give a larger special class, which includes the systems introduced in [46] and also others. We were led to introduce this class by our failure to fit Maxwell’s equations into the framework of [46]. We illustrate the new theory with two short examples based on the beam equation (only one of which falls within the class of systems treated in [46] and [38]), and we also outline an application to Maxwell’s equations. To keep the length of this paper within reasonable limits we have postponed a more complete treatment of Maxwell’s equations to the follow-up paper [44].

In this paper we consider a linear system Σ whose state space X can be decomposed as $X = H \oplus E$, where H and E are Hilbert spaces. The Hilbert space U is both the input space and the output space of Σ . We identify H , E , and U with their duals H' , E' , and U' . The Hilbert space E_0 is a dense subspace of E and the embedding $E_0 \hookrightarrow E$ is continuous. We denote by E'_0 the dual of E_0 with respect to the pivot space E so that

$$E_0 \subset E \subset E'_0$$

densely and with continuous embeddings. Such triples of Hilbert spaces are often encountered in the abstract treatment of partial differential equations. We denote $X_0 = H \oplus E_0$ so that $X'_0 = H \oplus E'_0$. We decompose the state of Σ as follows:

$$x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \quad z_0 \in H, \quad w_0 \in E.$$

We assume that we have three bounded operators

$$(1.1) \quad L \in \mathcal{L}(E_0, H), \quad K \in \mathcal{L}(E_0, U), \quad G \in \mathcal{L}(E_0, E'_0)$$

such that

$$(1.2) \quad \operatorname{Re} \langle Gw_0, w_0 \rangle_{E'_0, E_0} \leq 0 \quad \forall w_0 \in E_0,$$

and we define $\bar{A} \in \mathcal{L}(X_0, X'_0)$, $B \in \mathcal{L}(U, X'_0)$, and $\bar{C} \in \mathcal{L}(X_0, U)$ by

$$(1.3) \quad \bar{A} = \begin{bmatrix} 0 & -L \\ L^* & G - \frac{1}{2}K^*K \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ K^* \end{bmatrix}, \quad \bar{C} = [0 \quad -K].$$

The equations of the system are

$$(1.4) \quad \dot{x}(t) = \bar{A}x(t) + Bu(t), \quad y(t) = \bar{C}x(t) + u(t),$$

where x is the state trajectory, u is the input function, and y is the output function. Note that the differential equation above is an equation in X'_0 .

We define the domain $\mathcal{D}(A)$ by

$$(1.5) \quad \mathcal{D}(A) = \{x_0 \in X_0 \mid \bar{A}x_0 \in X\}$$

and we denote by A and C the restrictions of \bar{A} and \bar{C} to $\mathcal{D}(A)$. More explicitly,

$$(1.6) \quad \mathcal{D}(A) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in X_0 \mid L^*z_0 + \left(G - \frac{1}{2}K^*K\right)w_0 \in E \right\}.$$

Under the assumptions made so far, A is not necessarily closed. But for (1.4) to define a scattering passive system, we need A to be the generator of a strongly continuous semigroup of operators on X . One way to overcome this problem would be to assume that L is closed. This would indeed work, but it would be too restrictive: it would eliminate Maxwell’s equations, which we would like to fit into this abstract framework. A better alternative is to assume the following weaker condition:

$$(1.7) \quad \begin{bmatrix} L \\ K \end{bmatrix} \text{ (with domain } E_0 \text{) is closed as an unbounded operator } E \rightarrow H \oplus U.$$

As we shall see later (in Theorem 1.4), this assumption implies that A is m-dissipative (i.e., A is dissipative and $I + A$ has a bounded inverse) and hence it generates a semigroup of contractions.

Informal statement of the main result. *Under the assumptions (1.1)–(1.3) and (1.7), the equations of (1.4) determine a scattering passive system with state space X . This system is scattering conservative if and only if we have equality in (1.2).*

Scattering passive systems and scattering conservative systems will be formally defined in section 3. The above statement sounds simple and it contains the essence of what we are proving in this paper. However, a precise and formal statement is much more complicated. One reason for this is that we have to make it clear what we mean by the claim that certain equations determine a system. Another reason is that we want to give a precise description of the system in terms of its system node, semigroup generator, control and observation operators, and transfer function.

Before stating a more precise version of the above informal result, let us explain how systems of the above type arise in the modeling of physical systems. To do this we introduce $K_0 = \frac{1}{\sqrt{2}}K$ and remark that (1.7) holds if and only if it holds with K_0 in place of K . The formal dynamical system

$$(1.8) \quad \begin{aligned} \begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \end{bmatrix} &= \begin{bmatrix} 0 & -L \\ L^* & G \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K_0^* \end{bmatrix} e(t), & \quad t \geq 0, \\ f(t) &= \begin{bmatrix} 0 & K_0 \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix}, & \quad t \geq 0, \end{aligned}$$

often arises in physical modeling in the so-called impedance setting, where e stands for an effort variable and f stands for a flow variable. In Proposition 6.2 below we prove that if (1.1), (1.2), and (1.7) hold, then the above system is *impedance passive* in a weak sense, i.e., the operator

$$(1.9) \quad T = \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \\ 0 & -K_0 & 0 \end{bmatrix}$$

(with the appropriate domain) is m-dissipative. This implies that

$$(1.10) \quad \frac{d}{dt} \|x(t)\|^2 \leq 2\operatorname{Re} \langle e(t), f(t) \rangle$$

for all classical solutions $x(\cdot) = \begin{bmatrix} z(\cdot) \\ w(\cdot) \end{bmatrix}$ of (1.8). The physical interpretation of this inequality is that $\frac{1}{2}\|x(t)\|^2$ is the energy in the system, while $\operatorname{Re} \langle e(t), f(t) \rangle$ is the power entering the system. For an *impedance conservative* system the inequality in (1.10) holds as an equality. For closed $\begin{bmatrix} L \\ K_0 \end{bmatrix}$ the system will be impedance conservative

if and only if we have equality in (1.2). We refer to [2, 3, 5, 23, 31, 32, 33] for details on impedance passive and conservative systems.

Unfortunately, the system (1.8) is usually not well-posed. Well-posedness means for some (hence, for every) $t > 0$, the state $x(t)$ and the flow f (restricted to $[0, t]$) depend continuously on the initial state $x(0)$ and the effort e (restricted to $[0, t]$), and this does not follow from the above assumptions. In general, we do not even have a strongly continuous semigroup describing the evolution of the state $x(t)$ if $e = 0$. However, it is possible to reformulate the above problem into a *scattering* setting by choosing suitable combinations of e and f as input and output signals. If we define

$$(1.11) \quad u = \frac{1}{\sqrt{2}}(e + f), \quad y = \frac{1}{\sqrt{2}}(e - f),$$

then (1.8) becomes

$$\begin{aligned} \begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \end{bmatrix} &= \begin{bmatrix} 0 & -L \\ L^* & G - K_0^* K_0 \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \sqrt{2} K_0^* \end{bmatrix} u(t), & t \geq 0, \\ y(t) &= \begin{bmatrix} 0 & \sqrt{2} K_0 \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} + u(t), & t \geq 0, \end{aligned}$$

and instead of (1.10) we get the inequality that is characteristic of scattering passive systems:

$$(1.12) \quad \frac{d}{dt} \|x(t)\|^2 \leq \|u(t)\|^2 - \|y(t)\|^2.$$

After substituting $\sqrt{2} K_0 = K$ we arrive at the system (1.3)–(1.4).

The transformation (1.11) is called the *external Cayley transform*. It can be interpreted as a negative output feedback combined with a feed-forward term and a rescaling, as will be explained in more detail in section 5 (see Figure 1). As we prove in Theorem 5.2 and Remark 5.3, the system (1.3)–(1.4) is *scattering passive if and only if the operator T in (1.9) is m -dissipative*. Scattering passivity implies well-posedness. The system (1.3)–(1.4) is scattering conservative if the inequality in (1.2) holds as an equality. (Scattering conservative systems will be defined in section 3, and for them equality holds in (1.12).)

After this digression we now return to a more precise statement of our main results. The following three theorems use terminology that will be recalled in section 3.

THEOREM 1.1. *Let H, E, U, E_0 , and X_0 be as at the beginning of this section and let the operators L, K , and G be as (1.1), (1.2), and (1.7). Define the operator S_{sca} by*

$$S_{\text{sca}} = \begin{bmatrix} [A\&B]_{\text{sca}} \\ [C\&D]_{\text{sca}} \end{bmatrix},$$

where

$$(1.13) \quad [A\&B]_{\text{sca}} = \begin{bmatrix} 0 & -L & 0 \\ L^* & G - \frac{1}{2}K^*K & K^* \end{bmatrix}, \quad [C\&D]_{\text{sca}} = \begin{bmatrix} 0 & -K & I \end{bmatrix},$$

both have the domain

$$(1.14) \quad \mathcal{D}(S_{\text{sca}}) = \left\{ \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix} \in X_0 \times U \mid L^* z_0 + (G - \frac{1}{2}K^*K)w_0 + K^*u_0 \in E \right\}.$$

Then S_{sca} is a scattering passive (hence, well-posed) system node with input space U , state space $X = H \oplus E$ and output space U .

The following proposition is an easy consequence of the above theorem, using known general properties of well-posed system nodes. It tells us that the equation associated with the system node S_{sca} (see (1.17) below) has plenty of classical solutions. These coincide with the classical solutions of (1.4).

We denote by $\mathcal{H}_{\text{loc}}^1((0, \infty); U)$ the space of those functions on $(0, \infty)$ whose restriction to $(0, n)$ is in $\mathcal{H}^1((0, n); U)$ for every $n \in \mathbb{N}$.

PROPOSITION 1.2. *We use the notation and the assumptions of Theorem 1.1. The space $\mathcal{D}(S_{\text{sca}})$ is dense in $H \oplus E \oplus U$. It is a Hilbert space with the norm*

$$\left\| \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix} \right\|_{\mathcal{D}(S_{\text{sca}})}^2 = \|z_0\|^2 + \|w_0\|^2 + \|u_0\|^2 + \|Lw_0\|^2 + \|L^*z_0 + (G - \frac{1}{2}K^*K)w_0 + K^*u_0\|^2.$$

(On the right-hand side, we have used the norms of H, E , and U .)

If the input function u and the initial state $\begin{bmatrix} z(0) \\ w(0) \end{bmatrix}$ of S_{sca} satisfy

$$(1.15) \quad u \in \mathcal{H}_{\text{loc}}^1((0, \infty); U), \quad \begin{bmatrix} z(0) \\ w(0) \\ u(0) \end{bmatrix} \in \mathcal{D}(S_{\text{sca}}),$$

then the corresponding state trajectory $\begin{bmatrix} z \\ w \end{bmatrix}$ and output function y of S_{sca} satisfy

$$(1.16) \quad \begin{bmatrix} z \\ w \end{bmatrix} \in C^1([0, \infty); X), \quad \begin{bmatrix} z \\ w \\ u \end{bmatrix} \in C([0, \infty); \mathcal{D}(S_{\text{sca}})), \quad y \in \mathcal{H}_{\text{loc}}^1((0, \infty); Y),$$

and

$$(1.17) \quad \begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \\ y(t) \end{bmatrix} = S_{\text{sca}} \begin{bmatrix} z(t) \\ w(t) \\ u(t) \end{bmatrix} \quad \forall t \geq 0.$$

THEOREM 1.3. *We use the notation and the assumptions of Theorem 1.1. If the functions $u, x = \begin{bmatrix} z \\ w \end{bmatrix}$, and y are as in (1.15)–(1.17), then they satisfy the following power balance equation for every $t \geq 0$:*

$$(1.18) \quad \frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 + 2\text{Re} \langle Gw(t), w(t) \rangle.$$

The dual system node S_{sca}^* has the same structure but with L, K , and G replaced with $-L, -K$, and G^* . Hence, its classical solutions satisfy the same power balance equation (1.18). Therefore, S_{sca} is scattering conservative if and only if

$$(1.19) \quad \text{Re} \langle Gw_0, w_0 \rangle = 0 \quad \forall w_0 \in E_0.$$

THEOREM 1.4. *We use the notation and the assumptions of Theorem 1.1. The semigroup generator A of S_{sca} is \bar{A} from (1.3) restricted to $\mathcal{D}(A)$ from (1.6).*

Let $X_1 = \mathcal{D}(A)$ with the norm $\|x\|_1 = \|(I - A)x\|$, and let X_{-1} be the completion of X with respect to the norm $\|x\|_{-1} = \|(I - A)^{-1}x\|$. Then

$$(1.20) \quad X_1 \subset X_0 \subset X \subset X'_0 \subset X_{-1},$$

where all the embeddings are continuous and dense. A has a unique extension to an operator $A_{-1} \in \mathcal{L}(X, X_{-1})$, whose restriction to X_0 is \overline{A} from (1.3).

The control operator B of S_{sca} is as in (1.3) and the observation operator C of S_{sca} is \overline{C} from (1.3) restricted to $\mathcal{D}(A)$. The transfer function of S_{sca} is

$$\mathbf{G}(s) = I - K \left[sI + \frac{1}{2}K^*K - G + \frac{1}{s}L^*L \right]^{-1} K^*$$

for all s in the open right half-plane.

We mention that it follows immediately from the above theorem, combined with Theorem 1.3, that

$$A^* = \begin{bmatrix} 0 & L \\ -L^* & G^* - \frac{1}{2}K^*K \end{bmatrix},$$

$$\mathcal{D}(A^*) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in X_0 \mid -L^*z_0 + \left(G^* - \frac{1}{2}K^*K\right)w_0 \in E \right\}.$$

In section 2 we give several comments on the above results and we explain some connections with other classes of systems and with second order differential equations on Hilbert spaces. We give two examples based on the Euler–Bernoulli beam equation and outline an application to Maxwell’s equations.

To understand the theorems stated earlier, the reader should know the concepts of system node, well-posed system node, scattering passive system node, scattering conservative system node, classical solution, and well-posed system. The definitions of these concepts and some of their properties will be recalled in section 3. The concepts of semigroup generator, control operator, observation operator, and transfer function of a system node will also be recalled in section 3.

The internal and external Cayley transformations are discussed in sections 4 and 5. The proof of Theorem 1.1 is given in section 6. The proofs of Proposition 1.2 and Theorem 1.3 are also in section 6.

In section 7 we give a more explicit formula for the resolvent $(sI - \overline{A})^{-1}$ (as an operator matrix) and we prove Theorem 1.4.

A preliminary version of our results was presented in the conference paper [43].

2. Comments on the main results and examples.

Remark 2.1. If S_{sca} is a system node of the above structure, and moreover L is surjective, then for any $\begin{bmatrix} z \\ w \end{bmatrix}, u, y$ that satisfy (1.16) and (1.17) it is possible to identify a *hidden variable* q such that

$$z(t) = -Lq(t), \quad w(t) = \dot{q}(t).$$

Indeed, first we have to find an initial value for q such that $z(0) = -Lq(0)$ (this $q(0)$ may be nonunique). Then, for every $t \geq 0$ put $q(t) = q(0) + \int_0^t w(\sigma)d\sigma$. The hidden variable q satisfies the differential equation

$$\ddot{q}(t) + \left(\frac{1}{2}K^*K - G\right)\dot{q}(t) + L^*Lq(t) = K^*u(t),$$

where all the terms are in E'_0 , $q \in C^1([0, \infty); E_0)$, and $\dot{q} \in C^2([0, \infty); E)$. The difference between two different hidden variables associated to the same solution of (1.17) is a constant in $\text{Ker } L$. An alternative state for this system is (q, \dot{q}) , taking values in the new state space $E_0 \oplus E$ (this is often used in the literature). Note that

since L is not assumed to be closed, in general it is not possible to restrict L^*L to a positive (in particular, self-adjoint) operator from $\mathcal{D}(L^*L) \subset E$ to E . If L is closed, then this becomes possible; see Theorem 13.13 in Rudin [27].

Systems described by second order differential equations like the one above, with similar assumptions but with L closed, have been studied in Jacob and Morris [15, 16]. They have examined various properties of the associated transfer functions. For more material on second order linear differential equations with operator coefficients we refer to section VI.3 in Engel and Nagel [12] and to Fattorini [13].

As already mentioned, the expression $\frac{1}{2}\|x(t)\|^2 = \frac{1}{2}(\|z(t)\|^2 + \|w(t)\|^2)$ is interpreted as the energy of the system at time t . After introducing q , the second term $\frac{1}{2}\|w(t)\|^2 = \frac{1}{2}\|\dot{q}(t)\|^2$ can be interpreted as the *kinetic energy*, and the first term $\frac{1}{2}\|z(t)\|^2 = \frac{1}{2}\|Lq(t)\|^2$ can be interpreted as the *potential energy* stored in Σ .

Remark 2.2. An interesting generalization of the class of systems discussed here arises if instead of the quadratic expression for the potential energy $V(z(t)) = \frac{1}{2}\|z(t)\|^2$, we allow it to be a more general nonlinear C^1 function of $z(t)$, denoted by V . Here, we assume that all the Hilbert spaces are real, which is more realistic for nonlinear systems. The energy in the system is defined by $\mathcal{E}(t) = V(z(t)) + \frac{1}{2}\|w(t)\|^2$. The equations of the system are postulated to be

$$(2.1) \quad \begin{aligned} \frac{d}{dt} \begin{bmatrix} z(t) \\ w(t) \end{bmatrix} &= \begin{bmatrix} 0 & -L \\ L^* & G - \frac{1}{2}K^*K \end{bmatrix} \begin{bmatrix} (\nabla V)(z(t)) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K^* \end{bmatrix} u(t) \\ y(t) &= -Kw(t) + u(t), \end{aligned}$$

with the same assumptions on L, K, G as before. This is an instance of a scattering version of a port-Hamiltonian system, studied by van der Schaft and collaborators [7, 28], Villegas [40], and others. For every solution of the above equations we have the power balance equation

$$2\dot{\mathcal{E}}(t) = \|u(t)\|^2 - \|y(t)\|^2 + 2\langle Gw(t), w(t) \rangle \quad \forall t \geq 0.$$

For most systems in this class, it is challenging to prove the existence and uniqueness of suitable solutions of (2.1) for a dense set of initial states and input functions. We refer to Yao and Weiss [47] for a nonlinear scattering passive Rayleigh type beam equation that can be put into this framework.

Remark 2.3. The conservative systems “from thin air” considered in [38] and [46] are a subclass of the systems discussed here. Indeed, if we assume that L has an inverse $L^{-1} \in \mathcal{L}(H, E_0)$, $G = 0$ and if we denote $A_0 = L^*L$, $C_0 = K$, then we obtain the conservative systems from [38, 46]. From $L^{-1} \in \mathcal{L}(H, E_0)$ we see that L is closed (from E to H) so that the assumption (1.7) is satisfied. The operator $A_0 \in \mathcal{L}(E_0, E'_0)$ can be restricted to a positive operator on E with domain $\mathcal{D}(A_0)$ (see also Remark 2.1) and then $E_0 = \mathcal{D}(A_0^{\frac{1}{2}})$. The state vector used in [38, 46] is $\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} q \\ w \end{bmatrix}$, where q is the hidden variable introduced in Remark 2.1 ($q = -L^{-1}z$), and the corresponding state space is $E_0 \oplus E$. Due to this choice of state vector, there is no need to ever mention the space H or the operator L , the theory of systems “from thin air” uses only E , U , A_0 , and C_0 . In this case, our spaces E and E_0 correspond to what is denoted in [46] by H and $H_{\frac{1}{2}}$. The transformation from conservative systems “from thin air” to systems of the type discussed here can be found also in Schnaubelt and Weiss [29] (the proof of Theorem 5.1).

Example 2.4 (clamped Euler–Bernoulli beam). Consider $E = H = L^2[0, l]$, $E_0 = \mathcal{H}_0^2(0, l)$, and let $\alpha \in L^\infty[0, l]$ be positive and bounded from below, i.e., $\alpha(\xi) \geq \varepsilon > 0$

for almost every $\xi \in [0, l]$. Let $\kappa \in \mathcal{H}^{-2}(0, l)$ so that κ may contain several “Dirac pulses” and their derivatives placed in the interval $(0, l)$. We define $L \in \mathcal{L}(E_0, H)$ and $K \in \mathcal{L}(E_0, \mathbb{C})$ by

$$Lw = \alpha w'', \quad Kw = \langle w, \kappa \rangle_{\mathcal{H}_0^2, \mathcal{H}^{-2}}.$$

We consider $G = 0$. The operator L is the composition of a bounded and invertible operator (multiplication by α) with a closed operator (second derivative). Therefore, L is closed as an operator from E to H , so that (1.7) holds. Then according to Theorems 1.1 and 1.3, S_{sca} as in (1.13)–(1.14) is a scattering conservative system node with state space $L^2[0, l] \oplus L^2[0, l]$. Since L is invertible, this system fits into the class of conservative systems “from thin air” that were described in Remark 2.3. For every solution of (1.16)–(1.17) we can introduce the hidden variable $q = -L^{-1}z$, as explained in Remark 2.1, and we obtain that $q \in C^1([0, \infty); \mathcal{H}_0^2(0, l))$, $q \in C^2([0, \infty); L^2[0, l])$ and q satisfies the following Euler–Bernoulli beam equation with damping:

$$\ddot{q}(t) + \frac{\kappa}{2} \langle \dot{q}(t), \kappa \rangle_{\mathcal{H}_0^2, \mathcal{H}^{-2}} + (\alpha^2 q(t)'')'' = \kappa u(t)$$

with the (clamped) boundary conditions $q(0) = q(l) = 0$, $q'(0) = q'(l) = 0$. Here, the second derivative of $\alpha^2 q''$ is considered in the sense of distributions on $(0, l)$. The corresponding output signal is of course given by $y(t) = -\langle \dot{q}(t), \kappa \rangle + u(t)$.

Example 2.5 (free Euler–Bernoulli beam). This is a variation of the previous example, with noninvertible (but still closed) L . Consider again $E = H = L^2[0, l]$ but we take away the boundary conditions from E_0 : $E_0 = \mathcal{H}^2(0, l)$. Let α be as in Example 2.4. We denote by $\mathcal{H}^2(0, l)'$ the dual of $\mathcal{H}^2(0, l)$ with respect to the pivot space E . Let $\tilde{\kappa} \in \mathcal{H}^2(0, l)'$; then $\tilde{\kappa}$ has a unique decomposition of the form

$$(2.2) \quad \langle \varphi, \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} = a_0 \varphi(0) + b_0 \varphi'(0) + a_1 \varphi(l) + b_1 \varphi'(l) + \langle \varphi - \varphi_0, \kappa \rangle_{\mathcal{H}_0^2, \mathcal{H}^{-2}}$$

for all $\varphi \in \mathcal{H}^2(0, l)$, where $\kappa \in \mathcal{H}^{-2}(0, l)$. Here φ_0 is the polynomial of order at most three that satisfies the same (four) boundary conditions as φ so that $\varphi - \varphi_0 \in \mathcal{H}_0^2(0, l)$. We define the operators $L \in \mathcal{L}(E_0, H)$ and $K \in \mathcal{L}(E_0, \mathbb{C})$ by

$$Lw = \alpha w'', \quad Kw = \langle w, \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}}.$$

We consider $G = 0$. L is closed, for the same reason as in the previous example. According to Theorems 1.1 and 1.3, S_{sca} as in (1.13)–(1.14) is a scattering conservative system node with state space $L^2[0, l] \oplus L^2[0, l]$. Since L is not invertible, this system does not fit into the class of conservative systems “from thin air.” Nevertheless, L is surjective; hence for every classical solution of (1.17) we can introduce the hidden variable q , as explained in Remark 2.1, and we obtain that $q \in C^1([0, \infty); \mathcal{H}^2(0, l))$, $q \in C^2([0, \infty); L^2[0, l])$ and q satisfies the following equation:

$$(2.3) \quad \ddot{q}(t) + \frac{\tilde{\kappa}}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} + L^*(\alpha q(t)'') = \tilde{\kappa} u(t).$$

The output signal is $y(t) = -\langle \dot{q}(t), \tilde{\kappa} \rangle + u(t)$.

In general, we cannot say more about the abstract equation (2.3). However, we can get a better understanding of it by assuming a special structure for $\tilde{\kappa}$. In what follows, we assume that κ from (2.2) is in $L^2[0, l]$. Equivalently, $\tilde{\kappa}$ is given by

$$(2.4) \quad \langle \varphi, \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} = c_0 \varphi(0) + d_0 \varphi'(0) + c_1 \varphi(l) + d_1 \varphi'(l) + \langle \varphi, \kappa \rangle.$$

It is not a difficult computation to express the numbers c_0, d_0, c_1, d_1 in terms of the numbers a_0, b_0, a_1, b_1 and the function κ , but this is not needed here: we may take (2.4) as the definition of $\tilde{\kappa}$ with $\kappa \in L^2[0, l]$. To understand the meaning of (2.3) with $\tilde{\kappa}$ given by (2.4), we rewrite (2.3) in weak form,

$$\langle \varphi, \ddot{q}(t) \rangle + \langle \varphi, \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} \left[\frac{1}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} - u(t) \right] + \langle \alpha \varphi'', \alpha q(t)'' \rangle = 0,$$

for every $\varphi \in \mathcal{H}^2(0, l)$. Using (2.4) this becomes

$$(2.5) \quad \left[c_0 \varphi(0) + d_0 \varphi'(0) + c_1 \varphi(l) + d_1 \varphi'(l) + \langle \varphi, \kappa \rangle \right] \cdot \left[\frac{1}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} - u(t) \right] + \langle \varphi, \ddot{q}(t) \rangle + \langle \varphi'', \alpha^2 q(t)'' \rangle = 0$$

for every $t \geq 0$ and every $\varphi \in \mathcal{H}^2(0, l)$. In particular, for $\varphi \in \mathcal{H}_0^2(0, l)$ we obtain

$$\langle \varphi, \kappa \rangle \cdot \left[\frac{1}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} - u(t) \right] + \langle \varphi, \ddot{q}(t) \rangle + \langle \varphi'', \alpha^2 q(t)'' \rangle = 0.$$

This shows that (for each $t \geq 0$) the last term (which depends on φ'') can be extended continuously to all $\varphi \in L^2[0, l]$. Hence, $\alpha^2 q(t)''$ belongs to the domain of the adjoint of the second derivative operator with domain $\mathcal{H}_0^2(0, l)$, which is $\mathcal{H}^2(0, l)$:

$$\alpha^2 q(t)'' \in \mathcal{H}^2(0, l) \quad \forall t \geq 0.$$

Armed with this knowledge, we return to (2.5) and perform integration by parts twice. We obtain that for all $t \geq 0$ and for all $\varphi \in \mathcal{H}^2(0, l)$,

$$(2.6) \quad \left[c_0 \varphi(0) + d_0 \varphi'(0) + c_1 \varphi(l) + d_1 \varphi'(l) + \langle \varphi, \kappa \rangle \right] \cdot \left[\frac{1}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} - u(t) \right] + \langle \varphi, \ddot{q}(t) \rangle + \varphi' \alpha^2 \overline{q(t)''} \Big|_0^l - \varphi \left(\alpha^2 \overline{q(t)''} \right)' \Big|_0^l + \langle \varphi, (\alpha^2 q(t)'')'' \rangle = 0.$$

In particular, for $\varphi \in \mathcal{H}_0^2(0, l)$ we obtain

$$\langle \varphi, \kappa \rangle \cdot \left[\frac{1}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} - u(t) \right] + \langle \varphi, \ddot{q}(t) \rangle + \langle \varphi, (\alpha^2 q(t)'')'' \rangle = 0,$$

which is equivalent to

$$\ddot{q}(t) + \frac{\kappa}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} + (\alpha^2 q(t)'')'' = \kappa u(t).$$

The last equation is the (second order in time) partial differential equation that describes the behavior of our system in the open interval $(0, l)$. We recognize this as an Euler–Bernoulli beam equation with a nonlocal distributed damping term (which may depend also on the boundary values of \dot{q}). The physical interpretation of $q(t)$ is the vertical displacement and the input $u(t)$ is acting on the beam as a distributed vertical force with intensity proportional to κ .

We want to understand the boundary conditions that q satisfies. For this we have to go back to (2.6), where we take φ such that $\varphi(0) = 1$, $\varphi(l) = 0$, $\varphi'(0) = 0$, and $\varphi'(l) = 0$ and $\|\varphi\|_{L^2}$ is very small (it is an easy exercise to see that this is possible). Then, in the limit as $\|\varphi\|_{L^2} \rightarrow 0$, we obtain

$$c_0 \left[\frac{1}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} - u(t) \right] + \left(\alpha^2 \overline{q(t)}'' \right)'(0) = 0.$$

If it happens that $c_0 = 0$, then this equation means that there is no force acting on the beam at the left end (where $\xi = 0$). The system corresponding to a $c_0 \neq 0$ can be interpreted as being obtained from the system with $c_0 = 0$ by closing a boundary force feedback loop and adding a boundary force input for the beam, because the last term in the above equation is the force acting at $\xi = 0$.

Similarly, in (2.6) we now take φ such that $\varphi(0) = 0$, $\varphi(l) = 0$, $\varphi'(0) = 1$, and $\varphi'(l) = 0$ and $\|\varphi\|_{L^2}$ is very small. Then, in the limit as $\|\varphi\|_{L^2} \rightarrow 0$, we obtain

$$d_0 \left[\frac{1}{2} \langle \dot{q}(t), \tilde{\kappa} \rangle_{\mathcal{H}^2, \mathcal{H}^{2'}} - u(t) \right] - \left(\alpha^2 \overline{q(t)}'' \right)'(0) = 0.$$

For $d_0 = 0$ this equation means that there is no torque acting on the beam at $\xi = 0$. The system corresponding to a $d_0 \neq 0$ can be interpreted as being obtained from the system with $d_0 = 0$ by closing a boundary torque feedback loop and adding a boundary torque input for the beam, because the last term in the above equation is the torque acting at $\xi = 0$.

A similar analysis can be done for the other end of the beam.

We mention that if we assume that $\tilde{\kappa} = \delta_x$, where $x \in (0, l)$ (a Dirac mass at x), then we obtain a free beam with a local force feedback at x . If we assume that $\tilde{\kappa} = \delta'_x$, then we obtain a free beam with a local torque feedback at x .

Example 2.6 (Maxwell’s equations). In this example we consider Maxwell’s equations on a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary Γ . We denote the electric and magnetic fields by \mathbf{E} and \mathbf{H} , respectively. We consider the system described by the Maxwell equations on Ω , assuming that the materials in Ω are linear, homogeneous, and isotropic, they have no conductivity (hence, there are no currents), and there are no charges and no external sources of electric field:

$$(2.7) \quad \mu \frac{\partial \mathbf{H}}{\partial t} = -\text{rot } \mathbf{E}, \quad \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \text{rot } \mathbf{H},$$

$$(2.8) \quad \text{div}(\mu \mathbf{H}) = 0, \quad \text{div}(\varepsilon \mathbf{E}) = 0.$$

Here ε (the electric permittivity) and μ (the magnetic permeability) are positive numbers. We denote by ν the unit normal outward vector field on Γ (this is defined almost everywhere on Γ). We denote by $\gamma_0 \mathbf{E}$ and $\gamma_0 \mathbf{H}$ the traces of \mathbf{E} and \mathbf{H} on Γ and denote the tangential component of $\gamma_0 \mathbf{E}$ by $\pi_\tau \mathbf{E}$ so that

$$\pi_\tau \mathbf{E} = (\nu \times \gamma_0 \mathbf{E}) \times \nu.$$

Note that $\nu \times \gamma_0 \mathbf{E}$ is the same as $\pi_\tau \mathbf{E}$ rotated 90° around the normal direction to Γ . We define the input function u and the output function y by (cf. [28])

$$(2.9) \quad \begin{aligned} u &= \frac{1}{\sqrt{2}} \left(\nu \times \gamma_0 \mathbf{H} + \pi_\tau \mathbf{E} \right), \\ y &= \frac{1}{\sqrt{2}} \left(\nu \times \gamma_0 \mathbf{H} - \pi_\tau \mathbf{E} \right). \end{aligned}$$

In this brief example we consider only the case when $\mu = 1$ and $\varepsilon = 1$, because this case fits directly into the framework of Theorem 1.1. (The generalization to other values of μ and ε is easy, and for nonconstant μ and ε it becomes more tricky; a much more general and detailed treatment of Maxwell’s equations is in [44].)

All the spaces that we use to analyze the Maxwell equations are real Hilbert spaces, consisting of real-valued functions. The input and output space consists of tangential vector fields on Γ :

$$(2.10) \quad U = \{u \in L^2(\Gamma, \mathbb{R}^3) \mid u \cdot \nu = 0\}.$$

The state space is $X = E \oplus E$, where $E = L^2(\Omega; \mathbb{R}^3)$. Thus, for a state $x = [\begin{smallmatrix} \mathbf{H} \\ \mathbf{E} \end{smallmatrix}]$, the expression $\|x\|^2 = \|\mathbf{H}\|^2 + \|\mathbf{E}\|^2$ is twice the physical energy. Systems described by Maxwell’s equations were considered by many authors (see, for instance, [11, 20, 26, 28]), but we are not aware of works that define the input and output variables in a similar way to (2.9).

For any $\mathbf{E} \in E$, $\text{rot } \mathbf{E}$ is defined in the sense of distributions on Ω . We know from Theorem 2 on p. 204 of Dautray and Lions [8] that if $\mathbf{E} \in E$ is such that $\text{rot } \mathbf{E} \in E$, then its tangential trace \mathbf{E}_τ is well defined as an element of $\mathcal{H}^{-\frac{1}{2}}(\Gamma; \mathbb{R}^3)$. Hence, it makes sense to define a dense subspace of E as

$$E_0 = \{\mathbf{E} \in E \mid \text{rot } \mathbf{E} \in E, \pi_\tau \mathbf{E} \in U\},$$

which is a Hilbert space with the norm

$$\|\mathbf{E}\|_{E_0}^2 = \|\mathbf{E}\|_E^2 + \|\text{rot } \mathbf{E}\|_E^2 + \|\pi_\tau \mathbf{E}\|_U^2.$$

We define $L \in \mathcal{L}(E_0, E)$ and $K \in \mathcal{L}(E_0, U)$ by

$$L\mathbf{E} = \text{rot } \mathbf{E}, \quad K\mathbf{E} = \sqrt{2} \pi_\tau \mathbf{E}.$$

From the way the space E_0 and the operators L and K are defined, it is easy to see that $[\begin{smallmatrix} L \\ K \end{smallmatrix}]$ is a closed operator, as required in (1.7). It is also not difficult to see that L itself is not closed. We take $G = 0$. According to Theorems 1.1 and 1.3 (with $H = E$), these operators determine via (1.13) a scattering conservative system node S_{sca} . It can be shown (see [44]) that the equations of (1.17) that are satisfied by sufficiently smooth trajectories of S_{sca} are equivalent to (2.7) and (2.9). (The proof is not straightforward.) The equations (2.8) do not follow from (1.17), but it does follow that $\text{div } \mathbf{H}$ and $\text{div } \mathbf{E}$ are constant in time. In particular, if they are zero at the initial time $t = 0$, then they remain zero for all $t \geq 0$.

For much more detail on this example we refer the reader to our article [44], which can be regarded as a continuation of the present article. That article treats a more general case, where a part of the boundary is superconductive and the remaining part plays the same role as the full boundary Γ does above. In addition, in [44] the material in Ω is allowed to have a nonzero conductivity and nonconstant coefficients ε and μ , and the formulas (2.9) contain an additional coefficient function r .

3. System nodes, well-posed systems, scattering passive systems, and conservative systems. First we recall some simple facts about strongly continuous semigroups. Let $\mathbb{T} = (\mathbb{T}_t)_{t>0}$ be a strongly continuous semigroup of bounded linear operators on the Hilbert space X with generator A . We define on X a new norm by

$$(3.1) \quad \|x\|_{-1} = \|(\beta I - A)^{-1}x\|,$$

where $\beta \in \rho(A)$ is fixed. The choice of β is not important, because different choices lead to equivalent norms. The generator A determines two additional Hilbert spaces as follows: X_1 is $\mathcal{D}(A)$ with the norm $\|x\|_1 = \|(\beta I - A)x\|$ (this norm is equivalent to the graph norm), while X_{-1} is the completion of X with respect to the norm $\|\cdot\|_{-1}$. It is possible to extend A to an operator $A_{-1} \in \mathcal{L}(X, X_{-1})$ which generates a strongly continuous semigroup \mathbb{T}_{-1} on X_{-1} . For every $t \geq 0$, $\mathbb{T}_{-1,t}$ is an extension of \mathbb{T}_t . The extended semigroup is isomorphic to \mathbb{T} via the unitary operator $\beta I - A_{-1} \in \mathcal{L}(X, X_{-1})$. Often we denote these extensions of A and of \mathbb{T}_t by the same symbols A and \mathbb{T}_t . (We refer to our books [34, 39] for more details.)

The first aim of this section is to recall some facts about system nodes following [34, 18, 17]. The idea of system nodes goes back to Smuljan in 1986, using a different terminology. The concept was formalized in Staffans [30, 34].

DEFINITION 3.1. *Let U, X , and Y be Hilbert spaces. An operator*

$$S : \mathcal{D}(S) \rightarrow X \oplus Y \quad \text{with } \mathcal{D}(S) \subset X \oplus U$$

is called a system node on (U, X, Y) if it has the following properties:

1. *S is closed (as an operator from $X \oplus U$ to $X \oplus Y$).*
2. *We partition $S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$. The operator $A : \mathcal{D}(A) \rightarrow X$ defined by*

$$(3.2) \quad Ax = A\&B \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad \mathcal{D}(A) = \{x \in X \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{D}(S)\}$$

is the generator of a strongly continuous semigroup on X .

3. *The operator $A\&B$ (with $\mathcal{D}(A\&B) = \mathcal{D}(S)$) can be extended to an operator $\begin{bmatrix} A_{-1} & B \end{bmatrix} \in \mathcal{L}(X \oplus U, X_{-1})$, where X_{-1} is defined as above.*
4. *$\mathcal{D}(S) = \{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \oplus U \mid A_{-1}x + Bu \in X \}$.*

It is easy to see that if S is a system node on (U, X, Y) , then $\mathcal{D}(S)$ is dense in $X \oplus U$ and $A\&B$ is closed (with domain $\mathcal{D}(S)$). Hence, the graph norm on $\mathcal{D}(S)$ is equivalent to the graph norm of the operator $A\&B$ on the same domain, defined by

$$(3.3) \quad \left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{\mathcal{D}(S)}^2 = \|x\|^2 + \|u\|^2 + \|A_{-1}x + Bu\|^2.$$

The operator A is called the *semigroup generator* of S and B is called the *control operator* of S . The operator $C \in \mathcal{L}(X_1, Y)$ defined by

$$(3.4) \quad Cx = C\&D \begin{bmatrix} x \\ 0 \end{bmatrix} \quad \forall x \in \mathcal{D}(A)$$

is called the *observation operator* of S . In this paper we usually write A instead of A_{-1} . The *transfer function* of S is the $\mathcal{L}(U, Y)$ -valued analytic function defined by

$$(3.5) \quad \mathbf{G}(s) = C\&D \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \quad \forall s \in \rho(A).$$

It is easy to see that for all $s, \beta \in \rho(A)$ we have

$$(3.6) \quad \mathbf{G}(s) - \mathbf{G}(\beta) = C [(sI - A)^{-1} - (\beta I - A)^{-1}] B.$$

Combining (3.4) with (3.5) we easily get the useful formula

$$(3.7) \quad C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C [x - (sI - A)^{-1}Bu] + \mathbf{G}(s)u,$$

valid for all $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{D}(S)$ and all $s \in \rho(A)$. This, together with $A\&B \begin{bmatrix} x \\ u \end{bmatrix} = Ax + Bu$, shows that S is completely determined by A, B, C , and $\mathbf{G}(s)$ (for a single s).

Define the space

$$(3.8) \quad Z = D(A) + (\beta I - A)^{-1}BU,$$

which is a Hilbert space with the norm

$$(3.9) \quad \|z\|_Z^2 = \inf \{ \|x\|_1^2 + \|v\|^2 \mid x \in X_1, v \in U, z = x + (\beta I - A)^{-1}Bv \}.$$

Note that if $\begin{bmatrix} x \\ v \end{bmatrix} \in \mathcal{D}(S)$, then $x \in Z$ and $\|x\|_Z \leq m \|\begin{bmatrix} x \\ v \end{bmatrix}\|_{\mathcal{D}(S)}$ for some $m > 0$ independent of x and v . The system node is called *compatible* if C has a continuous extension to an operator $\overline{C} \in \mathcal{L}(Z, Y)$. In this case, we may define the operator $D \in \mathcal{L}(U, Y)$ by $D = \mathbf{G}(\beta) - \overline{C}(\beta I - A)^{-1}B$ and it follows from (3.6) that D is independent of $\beta \in \rho(A)$. Then $C\&D$ and S can be split to take their form, which is familiar from finite-dimensional systems theory,

$$C\&D \begin{bmatrix} x \\ v \end{bmatrix} = \overline{C}x + Dv, \quad S = \begin{bmatrix} A & B \\ \overline{C} & D \end{bmatrix},$$

and we have

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B + D \quad \forall s \in \rho(A).$$

A system node S is usually associated with the equation

$$(3.10) \quad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad \forall t \geq 0,$$

or equivalently, using the notation A, B , and $C\&D$ from Definition 3.1,

$$(3.11) \quad \dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = C\&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad \forall t \geq 0.$$

DEFINITION 3.2. *Let S be a closed linear operator from $X \oplus U$ to $X \oplus Y$ with domain $\mathcal{D}(S)$ (but S need not be a system node).*

A triple (x, u, y) is called a classical solution of (3.10) on $[0, \infty)$ if

- (a) $x \in C^1([0, \infty); X)$,
- (b) $u \in C([0, \infty); U)$, $y \in C([0, \infty); Y)$,
- (c) $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S)$ for all $t \geq 0$,
- (d) equation (3.10) holds.

A triple (x, u, y) is called a generalized solution of (3.10) on $[0, \infty)$ if

- (e) $x \in C([0, \infty); X)$,
- (f) $u \in L^2_{loc}([0, \infty); U)$, $y \in L^2_{loc}([0, \infty); Y)$,
- (g) there exists a sequence (x_k, u_k, y_k) of classical solutions of (3.10) such that $x_n \rightarrow x$ in $C([0, \infty); X)$, $u_k \rightarrow u$ in $L^2_{loc}([0, \infty); U)$, $y_k \rightarrow y$ in $L^2_{loc}([0, \infty); Y)$.

We remark that it follows easily from conditions (a)–(d) above that every classical solution of (3.10) on $[0, \infty)$ also satisfies

$$(h) \begin{bmatrix} x \\ u \end{bmatrix} \in C([0, \infty); \mathcal{D}(S)),$$

where the continuity is with respect to the graph norm of S on $\mathcal{D}(S)$.

The following proposition guarantees that for a system node, we have plenty of classical solutions of the system equation (3.10).

PROPOSITION 3.3. *Let S be a system node on (U, X, Y) . If $u \in C^2([0, \infty); U)$ and $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathcal{D}(S)$, then (3.10) has a unique classical solution (x, u, y) satisfying $x(0) = x_0$. Moreover, this classical solution satisfies*

$$x \in C^2([0, \infty); X_{-1}).$$

For the proof we refer to Lemma 4.7.8 in [34] or Proposition 4.2.11 in [39]. (Various versions of (parts of) this proposition can be found in the literature.)

For any $\tau \geq 0$ and $u \in L^2([0, \infty); U)$, let us denote by $\mathbf{P}_\tau u$ the restriction of u to $[0, \tau]$. Let us denote by \mathcal{D} the space of all the pairs $(x_0, u) \in X \oplus L^2([0, \infty); U)$ which satisfy the assumptions of Proposition 3.3. Notice that \mathcal{D} is dense in $X \oplus L^2([0, \infty); U)$. Hence, the corresponding space \mathcal{D}_τ of pairs $(x_0, \mathbf{P}_\tau u)$ is dense in $X \oplus L^2([0, \tau]; U)$. The last proposition allows us to define the operators Σ_τ from \mathcal{D}_τ to $X \oplus L^2([0, \tau], Y)$ such that for any solution of (3.10) and for any $\tau \geq 0$,

$$(3.12) \quad \begin{bmatrix} x(\tau) \\ \mathbf{P}_\tau y \end{bmatrix} = \Sigma_\tau \begin{bmatrix} x_0 \\ \mathbf{P}_\tau u \end{bmatrix}.$$

DEFINITION 3.4. *The system node S is called well-posed if for some (hence, for every) $\tau > 0$, the operator Σ_τ from (3.12) has a continuous extension*

$$\Sigma_\tau \in \mathcal{L}(X \oplus L^2([0, \tau], U), X \oplus L^2([0, \tau], Y)).$$

In this case, the family $(\Sigma_\tau)_{\tau \geq 0}$ is called a well-posed linear system.

For such systems we refer to [34] and the references therein. Every well-posed system node is compatible; see Theorem 3.4 in [35]. For well-posed system nodes Proposition 3.3 can be modified to obtain a stronger statement, as follows.

PROPOSITION 3.5. *Let S be a well-posed system node on (U, X, Y) . Assume that $u \in \mathcal{H}_{\text{loc}}^1((0, \infty); U)$ and $\begin{bmatrix} x_0 \\ u(0) \end{bmatrix} \in \mathcal{D}(S)$. Then (3.10) has a unique classical solution (x, u, y) satisfying $x(0) = x_0$. Moreover, we have*

$$y \in \mathcal{H}_{\text{loc}}^1((0, \infty); Y).$$

For the proof see Theorem 4.6.11 in [34] or Theorem 3.1 in [35].

DEFINITION 3.6. *The system node S is called scattering passive if all the classical solutions of (3.10) satisfy*

$$\frac{d}{dt} \|x(t)\|^2 \leq \|u(t)\|^2 - \|y(t)\|^2 \quad \forall t \geq 0.$$

An equivalent condition is that all the generalized solutions of (3.10) satisfy

$$(3.13) \quad \|x(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 dt \leq \|x(0)\|^2 + \int_0^\tau \|u(t)\|^2 dt \quad \forall t \geq 0.$$

A third equivalent condition is that the operators Σ_τ from (3.12) are *contractions*. In this case, the well-posed linear system $(\Sigma_\tau)_{\tau \geq 0}$ is called a *scattering passive linear system*. Such systems have been studied in [4, 24, 31, 32, 34, 35] and other references. (In [24] and [35] such systems were called dissipative.)

The system node S is called *scattering energy preserving* if the power balance equation

$$\frac{d}{dt} \|x(t)\|^2 = \|u(t)\|^2 - \|y(t)\|^2 \quad \forall t \geq 0$$

holds for all classical solutions of (3.10). Clearly, such systems nodes are scattering passive. The corresponding scattering passive system $(\Sigma_\tau)_{\tau \geq 0}$ is then *scattering energy preserving*, which means that all its trajectories satisfy (3.13) with equality. In other words, the operators Σ_τ are isometric.

The *dual* of a system node S on (U, X, Y) is simply its adjoint S^* . It can be verified that S^* is a system node on (Y, X, U) . The semigroup generator A^d , the control operators B^d , the observation operator C^d , and transfer functions \mathbf{G}^d of the dual system node S^* are related to the corresponding operators for S as follows:

$$(3.14) \quad A^d = A^*, \quad B^d = C^*, \quad C^d = B^*, \quad \mathbf{G}^d(s) = \mathbf{G}(\bar{s})^*.$$

The system node S^* is well-posed (or scattering passive) if and only if S is well-posed (or scattering passive). This follows from Theorems 3.4 and 3.5 in [36].

The system node S is called *scattering conservative* if both S and S^* are scattering energy preserving. The corresponding scattering-passive system $(\Sigma_\tau)_{\tau \geq 0}$ is then also called *scattering conservative*. For such a system, the operators Σ_τ are unitary. For scattering conservative systems we refer to [4, 21, 24, 28, 31, 36, 38, 45, 46] and the references therein. In particular, relatively simple necessary and sufficient conditions for a system node to be scattering conservative have been established in [24].

4. The internal Cayley transformation. We denote by \mathbb{C}_+ the open right half-plane in \mathbb{C} .

DEFINITION 4.1. *The internal Cayley transform of a system node S with parameter $\alpha \in \rho(A) \cap \mathbb{C}_+$ is the following operator from $X \oplus U$ to $X \oplus Y$:*

$$(4.1) \quad \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} (\bar{\alpha}I + A)(\alpha I - A)^{-1} & \sqrt{2\operatorname{Re} \alpha}(\alpha I - A)^{-1}B \\ \sqrt{2\operatorname{Re} \alpha}C(\alpha I - A)^{-1} & \mathbf{G}(\alpha) \end{bmatrix},$$

where A, B, C are as in Definition 3.1 and \mathbf{G} is the transfer function of S .

The internal Cayley transformation has been used in [4, 24, 25, 31, 32, 33, 34, 35] and other references. It should not be confused with the usual *Cayley transform* of the operator S with parameter $\alpha \in \rho(S) \cap \mathbb{C}_+$, which is $S^C = (\bar{\alpha}I + S)(\alpha I - S)^{-1}$.

The operator defined in (4.1) can be interpreted as a discrete-time system node on (U, X, Y) , which determines a discrete-time system with input space U , state space X , and output space Y via the equations

$$(4.2) \quad x_{k+1} = \mathbf{A}x_k + \mathbf{B}u_k, \quad y_k = \mathbf{C}x_k + \mathbf{D}u_k.$$

It is easy to check that if $\alpha > 0$, then we have

$$\begin{bmatrix} \frac{x_{k+1} - x_k}{h} \\ \frac{y_k}{\sqrt{h}} \end{bmatrix} = S \begin{bmatrix} \frac{x_{k+1} + x_k}{2} \\ \frac{u_k}{\sqrt{h}} \end{bmatrix}, \quad \text{where } h = \frac{2}{\alpha}.$$

The transformation that leads from the system S to the discrete-time system described in (4.2) (with $\alpha > 0$) is called *Tustin discretization* with time step h in the engineering literature. The transfer function of this discrete-time system is

$$\mathbf{C}(zI - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \mathbf{G}\left(\frac{\alpha z - \bar{\alpha}}{z + 1}\right) \quad \text{for } |z| > 1.$$

PROPOSITION 4.2. *Let $S = \begin{bmatrix} A \& B \\ C \& D \end{bmatrix}$ be a system node with semigroup generator A and let $\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ be its internal Cayley transform with parameter $\alpha \in \rho(A) \cap \mathbb{C}_+$.*

Then the operator \mathbf{A} does not have -1 as an eigenvalue, and \mathbf{S} can alternatively be computed from S in the following way: The operator

$$E = \begin{bmatrix} \frac{I}{\sqrt{2\operatorname{Re} \alpha}} & 0 \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} \alpha I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A \& B \\ 0 & 0 \end{bmatrix} \right)$$

maps $\mathcal{D}(S)$ one-to-one onto $X \oplus U$ and

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} -I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sqrt{2\operatorname{Re}\alpha}I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} E^{-1}.$$

The inverse of E , denoted by $\mathbf{E} = E^{-1}$, is given by

$$(4.3) \quad \mathbf{E} = \begin{bmatrix} \frac{I}{\sqrt{2\operatorname{Re}\alpha}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I + \mathbf{A} & \mathbf{B} \\ 0 & I \end{bmatrix}$$

and S can be recovered from \mathbf{S} via the formula

$$(4.4) \quad \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} \alpha I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \sqrt{2\operatorname{Re}\alpha}I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -I & 0 \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \mathbf{E}^{-1}.$$

Conversely, suppose that $\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \in \mathcal{L}(X \oplus U; X \oplus Y)$, that \mathbf{A} does not have -1 as an eigenvalue, and that for some $\alpha \in \mathbb{C}_+$ the operator

$$(4.5) \quad A = (\alpha\mathbf{A} - \bar{\alpha}I)(\mathbf{A} + I)^{-1}, \quad \mathcal{D}(A) = \operatorname{Ran}(\mathbf{A} + I)$$

generates a strongly continuous semigroup on X . Then $\alpha \in \rho(A)$ and \mathbf{E} from (4.3) is injective and has dense range. Denote the range of \mathbf{E} by $\mathcal{D}(S)$, and on this domain define the operator $S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ by (4.4).

Then S is a system node whose semigroup generator is A from (4.5). The internal Cayley transform of S with parameter α is \mathbf{S} .

For the proof we refer to Propositions 5.1 and 5.2 in [4] or Lemma 7.1 in [32]. Note that according to the above proposition, the formulas (4.3)–(4.4) define the inverse internal Cayley transformation.

PROPOSITION 4.3. *Let S be a scattering passive system node. Then for every $\alpha \in \mathbb{C}_+$, the internal Cayley transform of S is a contraction.*

Conversely, suppose that $\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ is a contraction and that -1 is not an eigenvalue of \mathbf{A} . Then for every $\alpha \in \mathbb{C}_+$, the operator S defined by (4.3)–(4.4), with $\mathcal{D}(S)$ being the range of \mathbf{E} , is a scattering passive system node.

For the proof we refer to Theorem 7.1(ii) in [32]. (Various parts of this can also be found in [4], [24], and [34].)

PROPOSITION 4.4. *If S is a scattering passive system node, then S is scattering conservative if and only if its internal Cayley transform \mathbf{S} is unitary.*

For the proof we refer again to [4], [24], [32], or [34].

5. The external Cayley transformation. A concept that is closely related to the concept of a scattering passive (or conservative) system node is the concept of an *impedance passive* (or *conservative*) system node. Still following the terminology of [31, 32], these are system nodes which have equal input and output spaces and for which the trajectories (the solutions of (3.10)) satisfy

$$\|x(\tau)\|^2 - \|x(0)\|^2 \leq 2 \int_0^\tau \operatorname{Re} \langle e(t), f(t) \rangle dt \quad \forall \tau \in [0, T)$$

(or the corresponding equality). Here, we have denoted the input signal by e (sometimes called the effort) and the output signal by f (sometimes called the flow), and of course (3.10) should be written with these signals in place of u and y .

It is always possible to transform an impedance passive or conservative system node S_{imp} into a scattering passive or conservative system node S_{sca} by the *external*

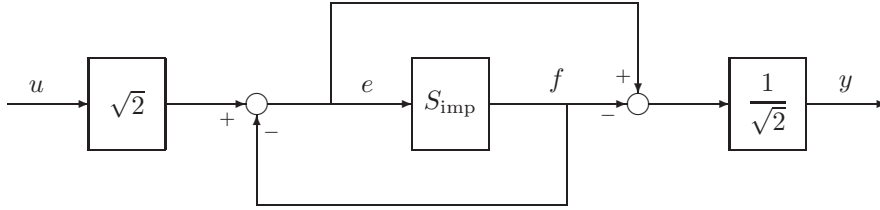


FIG. 1. The system node S_{sca} with input u and output y , as obtained from the system node S_{imp} (with input e and output f) via (1.11).

Cayley transformation (sometimes called the diagonal transformation) which redefines the input and the output as in (1.11). The inverse transformation is given by the same formulas, only with the places of u, y and e, f reversed, as is easy to see. The external Cayley transformation has been employed in many works; see, for example, Macchelli et al. [22], Staffans [31, 32, 33], and Weiss [42]. In the literature, often an extra parameter $\beta \in \mathbb{C}_+$ is included in the definition, but here we have taken $\beta = 1$. The external Cayley transformation can be understood also as an output feedback transformation (combined with a feed-forward term and a rescaling), as Figure 1 (approximately reproduced from [42]) shows. It is easy to see from this figure that the relationship between the transfer functions of S_{imp} and S_{sca} is

$$\mathbf{G}_{\text{sca}} = (I - \mathbf{G}_{\text{imp}})(I + \mathbf{G}_{\text{imp}})^{-1}.$$

In the papers [31, 32] the precise relationship between the system nodes S_{imp} and S_{sca} has been determined, and this is conveyed in the following proposition.

PROPOSITION 5.1. *Suppose that $S_{\text{imp}} = \begin{bmatrix} [A\&B]_{\text{imp}} \\ [C\&D]_{\text{imp}} \end{bmatrix}$ is an impedance passive system node. Then the operator*

$$(5.1) \quad E_{\text{imp}} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\text{imp}} \end{bmatrix} \right)$$

with domain $\mathcal{D}(S_{\text{imp}})$ is injective. Denote the range of this operator by $\mathcal{D}(S_{\text{sca}})$, and define the operator S_{sca} (with domain $\mathcal{D}(S_{\text{sca}})$) by

$$(5.2) \quad S_{\text{sca}} = \begin{bmatrix} [A\&B]_{\text{sca}} \\ [C\&D]_{\text{sca}} \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \left(\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}I \end{bmatrix} + \begin{bmatrix} [A\&B]_{\text{imp}} \\ 0 & 0 \end{bmatrix} \right) E_{\text{imp}}^{-1}.$$

Then S_{sca} is a scattering passive system node. The inverse of E_{imp} is $E_{\text{imp}}^{-1} = E_{\text{sca}}$, where

$$(5.3) \quad E_{\text{sca}} := \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\text{sca}} \end{bmatrix} \right),$$

and S_{imp} can be recovered from S_{sca} via

$$(5.4) \quad S_{\text{imp}} = \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix} + \left(\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2}I \end{bmatrix} + \begin{bmatrix} [A\&B]_{\text{sca}} \\ 0 & 0 \end{bmatrix} \right) E_{\text{sca}}^{-1}.$$

Notice that S_{sca} is obtained from S_{imp} by the same formulas by which S_{imp} is obtained from S_{sca} . However, there is a hidden asymmetry here: the external Cayley

transformation will not yield every possible scattering passive system node. The range of the external Cayley transformation (when applied to impedance conservative system nodes) will be characterized in Remark 5.4. This range is contained in those scattering passive system nodes for which the operator $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ from (4.1) does not have -1 as an eigenvalue. When the backward transformation (5.3)–(5.4) is applied to a scattering passive system node that does not satisfy the above eigenvalue condition, then it results in a “multivalued operator.” (It follows from (5.7) below that E_{sca} is injective if and only if $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ does not have -1 as an eigenvalue.) Both the single-valued case and the multivalued case can be analyzed with the methods described in Arov, Kurula, and Staffans [1, 2], Behrndt, Hassi, and de Snoo [6], Derkach et al. [9], Hassi, Malamud, and Mogilevskii [14], Kurula [17], Kurula and Staffans [18], Kurula et al. [19], etc. Here we shall restrict ourselves to the single-valued case.

To prove Theorem 1.1, we need the following generalization of Proposition 5.1, which can be applied to an operator S_{imp} that is not necessarily a system node. The proof of this theorem is based on the fact that if one applies to a system node S both the internal Cayley transform with parameter $\alpha = 1$ and the external Cayley transform (in any order), then the resulting operator is $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} (I+S)(I-S)^{-1}$. (Here $(I+S)(I-S)^{-1}$ is the usual Cayley transform of S with parameter $\alpha = 1$.)

THEOREM 5.2. *Let $S_{\text{imp}} = \begin{bmatrix} [A\&B]_{\text{imp}} \\ [C\&D]_{\text{imp}} \end{bmatrix}$ be an operator in $X \oplus U$ with domain $\mathcal{D}(S_{\text{imp}})$ such that $T := \begin{bmatrix} [A\&B]_{\text{imp}} \\ -[C\&D]_{\text{imp}} \end{bmatrix}$ (with the same domain) is m -dissipative. Then the operator E_{imp} from (5.1) is injective on $\mathcal{D}(S_{\text{imp}})$. We denote its range by $\mathcal{D}(S_{\text{sca}})$ and we define S_{sca} (with domain $\mathcal{D}(S_{\text{sca}})$) by (5.2). Then S_{sca} is a scattering passive system node and $E_{\text{imp}}^{-1} = E_{\text{sca}}$ from (5.3).*

We denote by A_{sca} , B_{sca} , and C_{sca} the semigroup generator, the control operator, and the observation operator of S_{sca} , and we denote by \mathbf{G}_{sca} its transfer function. Then, for all $s \in \mathbb{C}_+$,

$$(5.5) \quad \begin{bmatrix} (sI - A_{\text{sca}})^{-1} & \frac{1}{\sqrt{2}}(sI - A_{\text{sca}})^{-1}B_{\text{sca}} \\ \frac{1}{\sqrt{2}}C_{\text{sca}}(sI - A_{\text{sca}})^{-1} & \frac{1}{2}(I + \mathbf{G}_{\text{sca}}(s)) \end{bmatrix} = \left(\begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{imp}} \\ -[C\&D]_{\text{imp}} \end{bmatrix} \right)^{-1}.$$

The operator S_{imp} can be recovered from S_{sca} via the formulas (5.3)–(5.4).

The system node S_{sca} is scattering conservative if and only if T is skew-adjoint.

Proof. Define \mathbf{S} to be the Cayley transform of T with parameter 1, meaning that $\mathbf{S} = (I + T)(I - T)^{-1}$, so that

$$(5.6) \quad \frac{1}{2}(\mathbf{S} + I) = \frac{1}{2} \left(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) = \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{imp}} \\ -[C\&D]_{\text{imp}} \end{bmatrix} \right)^{-1}.$$

Then according to classical results about the Cayley transformation (see Theorem 3.4.9 in [34] or p. 167 in [37]), \mathbf{S} is a contraction, and \mathbf{S} is unitary if and only if T is skew-adjoint. Moreover, -1 is not an eigenvalue of \mathbf{S} and hence also not of \mathbf{A} . Indeed, if there would exist a nonzero $x \in X$ such that $\mathbf{A}x = -x$, then from $\|\mathbf{S}\| \leq 1$ we see that $\mathbf{C}x = 0$ and hence $\mathbf{S} \begin{bmatrix} x \\ 0 \end{bmatrix} = - \begin{bmatrix} x \\ 0 \end{bmatrix}$, a contradiction.

Since \mathbf{S} is contractive and \mathbf{A} does not have -1 as an eigenvalue, it follows from the converse part of Proposition 4.3 that \mathbf{S} has an inverse internal Cayley transform with parameter $\alpha = 1$, denoted S_{sca} , and this is a scattering passive system node. According to the last part of Proposition 4.2, the semigroup generator of S_{sca} is

$$A_{\text{sca}} = (\mathbf{A} - I)(\mathbf{A} + I)^{-1}, \quad \mathcal{D}(A_{\text{sca}}) = \text{Ran}(\mathbf{A} + I)$$

and the internal Cayley transform of S_{sca} is \mathbf{S} .

Denote the transfer function of S_{sca} by \mathbf{G}_{sca} . If in formula (4.1) we take $\alpha = 1$, add the identity on $X \oplus U$ to both sides, and divide by 2, then we get

$$\frac{1}{2} \left(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) = \begin{bmatrix} (I - A_{\text{sca}})^{-1} & \frac{1}{\sqrt{2}}(I - A_{\text{sca}})^{-1}B_{\text{sca}} \\ \frac{1}{\sqrt{2}}C_{\text{sca}}(I - A_{\text{sca}})^{-1} & \frac{1}{2}(I + \mathbf{G}_{\text{sca}}(1)) \end{bmatrix}.$$

This combined with (5.6) gives (5.5) with $s = 1$.

Now we show that S_{sca} (which in the proof we have defined differently from in the theorem) is indeed given by (5.2). We factor the right-hand side above, obtaining

$$(5.7) \quad \frac{1}{2} \left(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) = E_{\text{sca}} \begin{bmatrix} (I - A_{\text{sca}})^{-1} & (I - A_{\text{sca}})^{-1}B_{\text{sca}} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix},$$

where E_{sca} is defined as in (5.3). This factorization is checked by direct computation, using (3.4) and (3.5). The last factor on the right-hand side maps $X \oplus U$ one-to-one onto itself, and the second factor maps $X \oplus U$ one-to-one onto $\mathcal{D}(S_{\text{sca}})$, whereas, according to (5.6), the left-hand side maps $X \oplus U$ one-to-one onto $\mathcal{D}(S_{\text{imp}})$. Consequently, E_{sca} maps $\mathcal{D}(S_{\text{sca}})$ one-to-one onto $\mathcal{D}(S_{\text{imp}})$. Inverting both sides of (5.5) with $s = 1$ we get the following identity, valid on $\mathcal{D}(S_{\text{imp}})$:

$$(5.8) \quad \begin{bmatrix} I & 0 \\ 0 & \sqrt{2}I \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{sca}} \\ 0 & 0 \end{bmatrix} \right) E_{\text{sca}}^{-1} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{imp}} \\ -[C\&D]_{\text{imp}} \end{bmatrix}.$$

The bottom row of this identity implies that the bottom row of E_{sca}^{-1} is equal to $\frac{1}{\sqrt{2}}([0 \ I] + [C\&D]_{\text{imp}})$. Trivially, $[I \ 0] E_{\text{sca}} = [I \ 0]$, and hence the top row of E_{sca}^{-1} is $[I \ 0]$. Thus, $E_{\text{sca}}^{-1} = E_{\text{imp}}$, as defined in (5.1). In particular, E_{imp} is injective and maps $\mathcal{D}(S_{\text{imp}})$ one-to-one onto $\mathcal{D}(S_{\text{sca}})$. If we multiply (5.8) by E_{imp}^{-1} to the right and discard the bottom row, then we get the top row of (5.2) (since $[I \ 0] E_{\text{imp}}^{-1} = [I \ 0] E_{\text{sca}} = [I \ 0]$). That the bottom row of (5.2) also holds follows from the fact that $E_{\text{imp}}^{-1} = E_{\text{sca}}$. Thus, S_{sca} is indeed given by (5.2).

Now we show that S_{imp} can be recovered from S_{sca} via (5.3)–(5.4). It is easy to see that the top row of (5.4) follows from the top row of (5.2) (since $E_{\text{sca}} = E_{\text{imp}}^{-1}$). The bottom row of (5.4) follows from $E_{\text{sca}}^{-1} = E_{\text{imp}}$ and (5.1).

Recall (from this proof) that T is skew-adjoint if and only if \mathbf{S} is unitary. According to Proposition 4.4, \mathbf{S} is unitary if and only if S_{sca} is scattering conservative.

It remains to prove (5.5) for general $s \in \mathbb{C}_+$. Let us denote by $\mathcal{R}(s)$ the left-hand side of (5.5). Using (3.4), (3.5), and (5.3) we factor

$$\begin{aligned} \mathcal{R}(s) &= E_{\text{sca}} \begin{bmatrix} (sI - A_{\text{sca}})^{-1} & (sI - A_{\text{sca}})^{-1}B_{\text{sca}} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \\ &= E_{\text{sca}} \left(\begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{sca}} \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix}. \end{aligned}$$

Using (5.4) we obtain

$$\begin{aligned} \mathcal{R}(s) &= \left(\begin{bmatrix} I & 0 \\ 0 & \sqrt{2}I \end{bmatrix} \left(\begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{sca}} \\ 0 & 0 \end{bmatrix} \right) E_{\text{sca}}^{-1} \right)^{-1} \\ &= \left(\begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{imp}} \\ -[C\&D]_{\text{imp}} \end{bmatrix} \right)^{-1}. \quad \square \end{aligned}$$

Remark 5.3. By examining the proof of Theorem 5.2 we find that the converse of this theorem is also true in the following form: If S_{sca} is a scattering passive system node, and if its internal Cayley transform \mathbf{S} with parameter $\alpha = 1$ does not have -1 as an eigenvalue, then its external Cayley transform S_{imp} given by (5.3)–(5.4) is well defined, and it satisfies the assumption of Theorem 5.2.

Remark 5.4. It follows from Theorem 5.2 and Proposition 4.11 in Ball and Staffans [5] that a scattering conservative system node S_{sca} with input and output space U and state space X is the image of an impedance conservative system node under the external Cayley transformation if and only if the operator E_{sca} from (5.3) is injective and the projection of the range of E_{sca} onto U is all of U .

PROPOSITION 5.5. *We use the notation and the assumption of Theorem 5.2. The triple (x, u, y) is a classical (or generalized) solution of $\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S_{\text{sca}} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ on $[0, \infty)$ if and only if the triple (x, e, f) , where e, f satisfy (1.11), is a classical (or generalized) solution of $\begin{bmatrix} \dot{x}(t) \\ f(t) \end{bmatrix} = S_{\text{imp}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$ on $[0, \infty)$.*

Proof. First we prove the claim about the classical solutions. Clearly the transformation (1.11) preserves continuity, so that conditions (a) and (b) in Definition 3.2 are equivalent for the two systems. Thus, it suffices to prove that for any fixed $t \geq 0$, $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S_{\text{sca}})$ if and only if $\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \in \mathcal{D}(S_{\text{imp}})$ and that $\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S_{\text{sca}} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ if and only if $\begin{bmatrix} \dot{x}(t) \\ f(t) \end{bmatrix} = S_{\text{imp}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$.

Suppose that the vectors $x(t), \dot{x}(t) \in X$ and $u(t), y(t) \in U$ are such that $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{D}(S_{\text{sca}})$ and $\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S_{\text{sca}} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$. Then by (1.11),

$$\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) + y(t) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C \& D]_{\text{sca}} \end{bmatrix} \right) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

We have obtained that $\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = E_{\text{sca}} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$, and hence $\begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \in \mathcal{D}(S_{\text{imp}})$. Combining $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = E_{\text{sca}}^{-1} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$ with (5.4) we obtain

$$S_{\text{imp}} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} 0 \\ -e(t) \end{bmatrix} + \begin{bmatrix} \dot{x}(t) \\ \sqrt{2}u(t) \end{bmatrix} = \begin{bmatrix} \dot{x}(t) \\ f(t) \end{bmatrix},$$

as claimed in the proposition. The converse direction is proved similarly.

The claim about the generalized solutions of the two equations follows from the corresponding relationship between the classical solutions of these equations. \square

The above proposition shows that the transformation described by (5.1) and (5.2) is indeed the external Cayley transformation (defined earlier via (1.11)).

PROPOSITION 5.6. *We use the notation and the assumption of Theorem 5.2. The adjoint operator S_{imp}^* has an external Cayley transform, denoted S_{sca}^d , and this is a scattering passive system node. Moreover,*

$$(5.9) \quad S_{\text{sca}}^d = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} S_{\text{sca}}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

Thus, S_{sca}^d is scattering conservative if and only if T is skew-adjoint.

Proof. The operator

$$(5.10) \quad T^d = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} S_{\text{imp}}^* = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} T^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

is m-dissipative since T is m-dissipative. Thus, by Theorem 5.2, the external Cayley transform of S_{imp}^* , denoted by S_{sca}^d , is a scattering passive system node. To establish the connection (5.9) between S_{sca}^d and S_{sca}^* we use (5.5). We denote by A_{sca}^d , B_{sca}^d , and C_{sca}^d the semigroup generator, the control operator and the observation operator of S_{sca}^d , and we denote by $\mathbf{G}_{\text{sca}}^d$ its transfer function. Then according to (5.5) we have

$$\begin{bmatrix} (I - A_{\text{sca}}^d)^{-1} & \frac{1}{\sqrt{2}}(I - A_{\text{sca}}^d)^{-1}B_{\text{sca}}^d \\ \frac{1}{\sqrt{2}}C_{\text{sca}}^d(I - A_{\text{sca}}^d)^{-1} & \frac{1}{2}(I + \mathbf{G}_{\text{sca}}^d(1)) \end{bmatrix} = (I - T^d)^{-1}.$$

Using here (5.10) and again (5.5), we obtain

$$\begin{aligned} & \begin{bmatrix} (I - A_{\text{sca}}^d)^{-1} \frac{1}{\sqrt{2}}(I - A_{\text{sca}}^d)^{-1}B_{\text{sca}}^d & \\ \frac{1}{\sqrt{2}}C_{\text{sca}}^d(I - A_{\text{sca}}^d)^{-1} & \frac{1}{2}(I + \mathbf{G}_{\text{sca}}^d(1)) \end{bmatrix} \\ &= \begin{bmatrix} (I - A_{\text{sca}}^*)^{-1} & -\frac{1}{\sqrt{2}}(I - A_{\text{sca}}^*)^{-1}C_{\text{sca}}^* \\ -\frac{1}{\sqrt{2}}B_{\text{sca}}^*(I - A_{\text{sca}}^*)^{-1} & \frac{1}{2}(I + \mathbf{G}_{\text{sca}}^*(1)) \end{bmatrix}. \end{aligned}$$

This shows that

$$(5.11) \quad A_{\text{sca}}^d = A_{\text{sca}}^*, \quad B_{\text{sca}}^d = -C_{\text{sca}}^*, \quad C_{\text{sca}}^d = -B_{\text{sca}}^*, \quad \mathbf{G}_{\text{sca}}^d(1) = \mathbf{G}_{\text{sca}}^*(1).$$

We introduce the system node \tilde{S} by $\mathcal{D}(\tilde{S}) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \mathcal{D}(S_{\text{sca}})$,

$$\tilde{S} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} S_{\text{sca}} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

It is easy to verify that the semigroup generator of \tilde{S} is $\tilde{A} = A_{\text{sca}}$, its control operator is $\tilde{B} = -B_{\text{sca}}$, its observation operator is $\tilde{C} = -C_{\text{sca}}$, and its transfer function is $\tilde{\mathbf{G}}(s) = \mathbf{G}_{\text{sca}}(s)$. As mentioned after (3.7), S_{sca}^d is completely determined by A_{sca}^d , B_{sca}^d , C_{sca}^d , and $\mathbf{G}_{\text{sca}}^d(1)$. Comparing (5.11) with (3.14), we see that S_{sca}^d must be the dual of \tilde{S} , i.e., $S_{\text{sca}}^d = \tilde{S}^*$. From here we get that (5.9) holds.

Finally, we know from Theorem 5.2 that S_{sca}^d is scattering conservative if and only if T^d is skew-adjoint. We see from (5.10) that T^d is skew-adjoint if and only if T is. \square

6. The proof of Theorem 1.1. We start with a lemma about computing the adjoint of a certain type of matrix of operators, which then leads to a result about matrices of operators that are m-dissipative. Much useful material about matrices of operators can be found in the unpublished book of Engel [10] (but the specific results that we need here do not seem to follow from those in [10]). For the concept of the dual space with respect to a pivot space and related concepts we refer, for example, to section 2.9 in [39]. As in [39], the pairing between a Hilbert space V and its dual V' with respect to the pivot space H is considered to be linear in the first component and antilinear in the second, so that if both elements in the pairing belong to H , then the pairing coincides with their inner product in H .

LEMMA 6.1. *Let H_1 and H_2 be Hilbert spaces that are identified with their dual spaces. Let $\mathcal{D}(\Lambda)$ be a dense subspace of H_2 and let $\Lambda : \mathcal{D}(\Lambda) \rightarrow H_1$ be closed. We regard $\mathcal{D}(\Lambda)$ as a Hilbert space with the graph norm of Λ , and let $\mathcal{D}(\Lambda)'$ be the dual of $\mathcal{D}(\Lambda)$ with respect to the pivot space H_2 . Then $\Lambda \in \mathcal{L}(\mathcal{D}(\Lambda), H_1)$ and its adjoint (in the sense of bounded operators) is $\Lambda^* \in \mathcal{L}(H_1, \mathcal{D}(\Lambda)')$. Let $G \in \mathcal{L}(\mathcal{D}(\Lambda), \mathcal{D}(\Lambda)')$. We define*

$$T = \begin{bmatrix} 0 & -\Lambda \\ \Lambda^* & G \end{bmatrix}$$

with domain

$$\mathcal{D}(T) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_1 \times \mathcal{D}(\Lambda) \mid \Lambda^*z + Gw \in H_2 \right\}.$$

Then $\mathcal{D}(T)$ is dense in $H_1 \oplus H_2$ and the adjoint of T (as an unbounded operator on $H_1 \oplus H_2$) is

$$(6.1) \quad T^* = \begin{bmatrix} 0 & \Lambda \\ -\Lambda^* & G^* \end{bmatrix}$$

with domain

$$(6.2) \quad \mathcal{D}(T^*) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in H_1 \times \mathcal{D}(\Lambda) \mid -\Lambda^*z_0 + G^*w_0 \in H_2 \right\}.$$

Proof. It is clear that $\Lambda \in \mathcal{L}(\mathcal{D}(\Lambda), H_1)$ and its adjoint Λ^* (in the sense of bounded operators) belongs to $\mathcal{L}(H_1, \mathcal{D}(\Lambda)')$. When we regard Λ as an unbounded operator from H_2 to H_1 , then it has another adjoint (in the sense of unbounded operators) which maps $\mathcal{D}(\Lambda^*) \subset H_1$ to H_2 . The adjoint in the sense of bounded operators is an extension of the adjoint in the sense of unbounded operators, as is easy to check. For this reason, we use the same notation Λ^* for both of them. Since Λ is closed, we have that $\mathcal{D}(\Lambda^*)$ is dense in H_1 .

Now we show that $\mathcal{D}(T)$ is dense in $H_1 \oplus H_2$. Let $\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in \mathcal{D}(T)^\perp$. Since $\mathcal{D}(\Lambda^*) \times \{0\} \subset \mathcal{D}(T)$, we must have $\langle z, z_0 \rangle = 0$ for all $z \in \mathcal{D}(\Lambda^*)$, and since $\mathcal{D}(\Lambda^*)$ is dense in H_1 we get that $z_0 = 0$. By the construction of the space $\mathcal{D}(\Lambda)'$, the operator $I + \Lambda^*\Lambda$ is a continuous bijection of $\mathcal{D}(\Lambda)$ onto $\mathcal{D}(\Lambda)'$, and consequently

$$\begin{bmatrix} I & \Lambda^* \end{bmatrix} \begin{bmatrix} I \\ \Lambda \end{bmatrix} (I + \Lambda^*\Lambda)^{-1} = I.$$

Thus, $\begin{bmatrix} I & \Lambda^* \end{bmatrix}$ maps $\mathcal{D}(\Lambda) \times H_1$ onto $\mathcal{D}(\Lambda)'$, and so for every $w \in \mathcal{D}(\Lambda)$ there exists some $w_2 \in \mathcal{D}(\Lambda)$ and $z \in H_1$ such that $Gw = w_2 - \Lambda^*z$. Hence, for every $w \in \mathcal{D}(\Lambda)$ there is some $z \in H_1$ such that $\Lambda^*z + Gw \in H_2$, i.e., $\begin{bmatrix} z \\ w \end{bmatrix} \in \mathcal{D}(T)$. Thus, the condition that $\begin{bmatrix} 0 \\ w_0 \end{bmatrix}$ is orthogonal to every $\begin{bmatrix} z \\ w \end{bmatrix} \in \mathcal{D}(T)$ implies that w_0 is orthogonal to $\mathcal{D}(\Lambda)$, and hence $w_0 = 0$. This proves that $\mathcal{D}(T)$ is dense in $H_1 \oplus H_2$.

By the definition of the adjoint of an unbounded operator, an element $x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}$ from $H_1 \oplus H_2$ belongs to $\mathcal{D}(T^*)$ if and only if the functional $F(x) = \langle Tx, x_0 \rangle$, which is defined for $x = \begin{bmatrix} z \\ w \end{bmatrix} \in \mathcal{D}(T)$, has a continuous extension to all $H_1 \oplus H_2$. We have

$$F \left(\begin{bmatrix} z \\ w \end{bmatrix} \right) = -\langle \Lambda w, z_0 \rangle_{H_1} + \langle \Lambda^*z + Gw, w_0 \rangle_{H_2}.$$

Suppose that $\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in \mathcal{D}(T^*)$. Since for $z \in \mathcal{D}(\Lambda^*)$ and $w = 0$ we have

$$F \left(\begin{bmatrix} z \\ 0 \end{bmatrix} \right) = \langle \Lambda^*z, w_0 \rangle$$

and this is a continuous function of z , it follows that $w_0 \in \mathcal{D}(\Lambda^{**}) = \mathcal{D}(\Lambda)$. It follows that for any $\begin{bmatrix} z \\ w \end{bmatrix} \in \mathcal{D}(T)$,

$$(6.3) \quad \begin{aligned} F \left(\begin{bmatrix} z \\ w \end{bmatrix} \right) &= -\langle \Lambda w, z_0 \rangle_{H_1} + \langle \Lambda^*z + Gw, w_0 \rangle_{\mathcal{D}(\Lambda)', \mathcal{D}(\Lambda)} \\ &= \langle w, -\Lambda^*z_0 + G^*w_0 \rangle_{\mathcal{D}(\Lambda), \mathcal{D}(\Lambda)'} + \langle z, \Lambda w_0 \rangle_{H_1}. \end{aligned}$$

Since F has a continuous extension to $H_1 \oplus H_2$, it follows that $-\Lambda^*z_0 + G^*w_0 \in H_2$. Thus we have shown that

$$\mathcal{D}(T^*) \subset \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in H_1 \times \mathcal{D}(\Lambda) \mid -\Lambda^*z_0 + G^*w_0 \in H_2 \right\}.$$

Conversely, suppose that $x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in H_1 \times \mathcal{D}(\Lambda)$ is such that $-\Lambda^*z_0 + G^*w_0 \in H_2$. Then the expression in (6.3) depends continuously on $\begin{bmatrix} z \\ w \end{bmatrix} \in H_1 \oplus H_2$. Doing the computations leading to (6.3) backward, we obtain that $F(x)$ (defined for $x \in \mathcal{D}(T)$) has a continuous extension to $H_1 \oplus H_2$, so that $x_0 \in \mathcal{D}(T^*)$. Thus, we have proved (6.2). Notice that the expression in (6.3) is in fact $\langle x, T^*x_0 \rangle$ with T^* as in the lemma. This confirms the formula for T^* given in the lemma. \square

PROPOSITION 6.2. *With the assumptions of Lemma 6.1, assume additionally that*

$$(6.4) \quad \operatorname{Re} \langle Gw_0, w_0 \rangle \leq 0 \quad \forall w_0 \in \mathcal{D}(\Lambda).$$

Then T is m -dissipative (as an unbounded operator on $H_1 \oplus H_2$).

Moreover, T is skew-adjoint if and only if we always have equality in (6.4).

Proof. According to Lemma 6.1, T^* is an operator with a similar structure as T , so that we can apply the lemma to T^* , obtaining that $\mathcal{D}(T^*)$ is dense and $T^{**} = T$. This implies that T is closed. (So far we have not used (6.4).) It is easy to check that both T and T^* are dissipative (as unbounded operators on $H_1 \oplus H_2$). By, e.g., Proposition 3.1.11 in [39], T is m -dissipative.

To prove the last part of the proposition, first we note that if T is skew-adjoint, then so is G , so that $\operatorname{Re} \langle Gw_0, w_0 \rangle = 0$ for all $w_0 \in \mathcal{D}(\Lambda)$. Conversely, if G is such that we always have equality in (6.4), then we have $\langle Gw_1, w_2 \rangle = -\langle w_1, Gw_2 \rangle$ for all $w_1, w_2 \in \mathcal{D}(\Lambda)$. (See the proof of Proposition 3.2.2 in [39] for the easy argument.) Since G is a bounded operator from $\mathcal{D}(\Lambda)$ to $\mathcal{D}(\Lambda)'$, it follows that $G^* = -G$. Now we see from (6.1) that $T^* = -T$. \square

Proof of Theorem 1.1. We use the notation and the assumptions in the theorem. Introduce

$$K_0 = \frac{1}{\sqrt{2}}K, \quad \Lambda = \begin{bmatrix} L \\ K_0 \end{bmatrix};$$

then Λ is a closed operator from E to $H \oplus U$ with domain $\mathcal{D}(\Lambda) = E_0$. Notice that Λ and G satisfy the assumptions of Proposition 6.2 with $H \oplus U$ and E in place of H_1 and H_2 . According to Proposition 6.2, the operator

$$\tilde{T} = \begin{bmatrix} 0 & -\Lambda \\ \Lambda^* & G \end{bmatrix} = \begin{bmatrix} 0 & 0 & -L \\ 0 & 0 & -K_0 \\ L^* & K_0^* & G \end{bmatrix}$$

with domain

$$\mathcal{D}(\tilde{T}) = \left\{ \begin{bmatrix} z_0 \\ u_0 \\ w_0 \end{bmatrix} \in H \times U \times E_0 \mid L^*z_0 + K_0^*u_0 + Gw_0 \in E \right\}$$

is m -dissipative on $H \oplus U \oplus E$. Moreover, \tilde{T} is skew-adjoint if and only if G is such that we always have equality in (1.2). We interchange the places of U and E in the space $H \oplus U \oplus E$; then the entries of \tilde{T} get permuted accordingly (we interchange

the last two rows and then the last two columns). This gives that the operator

$$T = \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \\ 0 & -K_0 & 0 \end{bmatrix}$$

is m-dissipative on $H \oplus E \oplus U$ with domain

$$\mathcal{D}(T) = \left\{ \begin{bmatrix} z_0 \\ w_0 \\ u_0 \end{bmatrix} \in H \times E_0 \times U \mid L^*z_0 + Gw_0 + K_0^*u_0 \in E \right\}.$$

T is skew-adjoint if and only if G is such that we always have equality in (1.2).

We want to use the above T as the m-dissipative operator appearing in Theorem 5.2. For this, we partition T horizontally into the operators

$$(6.5) \quad [A\&B]_{\text{imp}} = \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \end{bmatrix}, \quad -[C\&D]_{\text{imp}} = \begin{bmatrix} 0 & -K_0 & 0 \end{bmatrix},$$

both with domain $\mathcal{D}(T)$, and we define

$$(6.6) \quad S_{\text{imp}} = \begin{bmatrix} [A\&B]_{\text{imp}} \\ [C\&D]_{\text{imp}} \end{bmatrix}, \quad \mathcal{D}(S_{\text{imp}}) = \mathcal{D}(T).$$

Then according to Theorem 5.2, the operator S_{sca} defined in (5.2) is a scattering passive system node. We partition S_{sca} as in (5.2),

$$S_{\text{sca}} = \begin{bmatrix} [A\&B]_{\text{sca}} \\ [C\&D]_{\text{sca}} \end{bmatrix}$$

and we want to express these in terms of L, K , and G . For this, according to (5.2), we have to compute the inverse (on its range) of E_{imp} from (5.1):

$$\begin{aligned} E_{\text{imp}} &= \begin{bmatrix} I & 0 \\ 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ [C\&D]_{\text{imp}} \end{bmatrix} \right) \\ &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{I}{\sqrt{2}} \end{bmatrix} \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & K_0 & 0 \end{bmatrix} \right) = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \frac{K_0}{\sqrt{2}} & \frac{I}{\sqrt{2}} \end{bmatrix}, \end{aligned}$$

which is defined on $\mathcal{D}(T)$. It is easy to see that E_{imp} is indeed injective (as stated in Theorem 5.2) and its range $\mathcal{D}(S_{\text{sca}}) = E_{\text{imp}}\mathcal{D}(T)$ is given by (1.14). It is also easy to check that

$$E_{\text{imp}}^{-1} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -K_0 & \sqrt{2}I \end{bmatrix}.$$

Substituting this and (6.5) into (5.2), we obtain that on $\mathcal{D}(S_{\text{sca}})$,

$$\begin{aligned} \begin{bmatrix} [A\&B]_{\text{sca}} \\ [C\&D]_{\text{sca}} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{bmatrix} + \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \\ 0 & 0 & \sqrt{2}I \end{bmatrix} \cdot \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -K_0 & \sqrt{2}I \end{bmatrix} \\ &= \begin{bmatrix} 0 & -L & 0 \\ L^* & G - K_0^*K_0 & \sqrt{2}K_0^* \\ 0 & -\sqrt{2}K_0 & 2I - I \end{bmatrix} = \begin{bmatrix} 0 & -L & 0 \\ L^* & G - \frac{1}{2}K^*K & K^* \\ 0 & -K & I \end{bmatrix}. \end{aligned}$$

According to Theorem 5.2, S_{sca} is a scattering passive system node with state space

$H \oplus E$ and with input and output space U . We also know from the same theorem that S_{sca} is scattering conservative if and only if G is skew-adjoint. \square

Note that in the above proof, we have shown a bit more than what was required. Indeed, we have also shown (without any extra effort) that S_{sca} is scattering conservative if and only if G satisfies (1.2). This also follows from Theorem 1.3.

Remark 6.3. Our assumption (1.7) that the operator $\begin{bmatrix} L \\ K \end{bmatrix}$ is closed is not far from being a necessary condition for the conclusion of Theorem 1.1 to be valid, in the following sense. Being a system node, the operator S_{sca} is closed. As can easily be seen, this implies that also its external Cayley transform S_{imp} is closed, which implies in particular that the second column of S_{imp} is closed. This second column is $\begin{bmatrix} -L \\ G \\ K_0 \end{bmatrix}$; see (6.5) and (6.6). If, in addition, we assume that $G \in \mathcal{L}(E)$, then the domain of this column is E_0 and it follows that $\begin{bmatrix} L \\ K \end{bmatrix}$ has to be closed (as a densely defined operator from E to $H \oplus U$), which is (1.7).

Proof of Proposition 1.2. All the claims in this proposition follow from standard properties of a well-posed system node. See, in particular, (3.3), claim (h) after Definition 3.2, and Proposition 3.5. \square

Proof of Theorem 1.3. We shall use the operator $S_{\text{imp}} = \begin{bmatrix} [A\&B]_{\text{imp}} \\ [C\&D]_{\text{imp}} \end{bmatrix}$, whose components $[A\&B]_{\text{imp}}$ and $[C\&D]_{\text{imp}}$ are as introduced in the proof of Theorem 1.1 (see (6.5)). According to the same proof, S_{sca} is obtained from S_{imp} via the external Cayley transformation (5.1)–(5.2). If $(\begin{bmatrix} z \\ w \end{bmatrix}, u, y)$ is a classical solution of $\begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \\ y(t) \end{bmatrix} = S_{\text{sca}} \begin{bmatrix} z(t) \\ w(t) \\ u(t) \end{bmatrix}$ and if we define the signals e, f by (1.11), then by Proposition 5.5, $(\begin{bmatrix} z \\ w \end{bmatrix}, e, f)$ is a classical solution of $\begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \\ f(t) \end{bmatrix} = S_{\text{imp}} \begin{bmatrix} z(t) \\ w(t) \\ e(t) \end{bmatrix}$. Thus,

$$\begin{bmatrix} \dot{z}(t) \\ \dot{w}(t) \\ -f(t) \end{bmatrix} = \begin{bmatrix} 0 & -L & 0 \\ L^* & G & K_0^* \\ 0 & -K_0 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ w(t) \\ e(t) \end{bmatrix} \quad \forall t \geq 0.$$

By taking the real part of the inner product of both sides in this equation with the vector $[z(t) \ w(t) \ f(t)]^T$ and recalling that $w(t) \in E_0$, we get that for all $t \geq 0$,

$$\frac{d}{dt} \|z(t)\|^2 + \frac{d}{dt} \|w(t)\|^2 - 2\text{Re} \langle e(t), f(t) \rangle = 2\text{Re} \langle Gw(t), w(t) \rangle.$$

Here $2\text{Re} \langle e(t), f(t) \rangle = \|u(t)\|^2 - \|y(t)\|^2$, and (1.18) follows.

The formula for the adjoint system follows from Proposition 5.6. The operator S_{imp}^* is of the same form as S_{imp} , apart from the fact that L and G are replaced by $-L$ and G^* . When S_{sca}^d is multiplied by $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ to the left and right this has the effect of replacing K with $-K$. \square

7. The resolvent and the transfer function. In this section we continue to investigate the special class of systems described in section 1, and we prove Theorem 1.4. We begin with two preliminary propositions.

PROPOSITION 7.1. *Consider the spaces $H, E, E_0, E'_0, X_0, X'_0$ as introduced at the beginning of section 1. If the operators L, K , and G are as in (1.1), (1.2), and (1.7), then for every $s \in \mathbb{C}_+$ the operator $P(s) \in \mathcal{L}(E_0, E'_0)$ defined by*

$$(7.1) \quad P(s) = sI + \frac{1}{2}K^*K - G + \frac{1}{s}L^*L$$

has an inverse $V(s) \in \mathcal{L}(E'_0, E_0)$.

Proof. Because of condition (1.7), the space E_0 is complete with the graph norm of $\begin{bmatrix} L \\ K \end{bmatrix}$, which is

$$\|w\|_0^2 = \|w\|_E^2 + \|Lw\|_H^2 + \|Kw\|_U^2 \quad \forall w \in E_0.$$

Using the closed graph theorem, it follows that the original norm $\|\cdot\|_{E_0}$ on E_0 is equivalent to the above graph norm. In this proof, we may therefore replace $\|\cdot\|_{E_0}$ with $\|\cdot\|_0$, but we still keep the old notation $\|\cdot\|_{E_0}$ for the norm on E_0 and the notation $\|\cdot\|_{E'_0}$ for the corresponding dual norm on E'_0 .

For every $w \in E_0$ and $s \in \mathbb{C}_+$ we have

$$\begin{aligned} \|P(s)w\|_{E'_0} \cdot \|w\|_{E_0} &\geq \operatorname{Re} \langle P(s)w, w \rangle_{E'_0, E_0} \\ &= (\operatorname{Re} s) \|w\|_E^2 + \frac{1}{2} \|Kw\|_U^2 + \left(\operatorname{Re} \frac{1}{s} \right) \|Lw\|_H^2 - \langle Gw, w \rangle_{E'_0, E_0} \\ &\geq (\operatorname{Re} s) \|w\|_E^2 + \frac{1}{2} \|Kw\|_U^2 + \left(\operatorname{Re} \frac{1}{s} \right) \|Lw\|_H^2 \geq c \|w\|_{E_0}^2, \end{aligned}$$

where $c = \min\{\operatorname{Re} s, \frac{1}{2}, \operatorname{Re} \frac{1}{s}\}$. This implies that $P(s)$ is injective and has closed range. The operator

$$P(s)^* = \bar{s}I + \frac{1}{2}K^*K - G^* + \frac{1}{\bar{s}}L^*L$$

has the same structure as $P(s)$, so that by the same argument as in the first step, it is also injective and has closed range. Since $[\operatorname{Ran} P(s)]^\perp = \operatorname{Ker} P(s)^*$ (the orthogonal complement is with respect to the duality pairing; see, for instance, Theorem 4.12 in Rudin [27]), this implies that $\operatorname{Ran} P(s)$ is dense in E'_0 . Hence, $\operatorname{Ran} P(s) = E'_0$, and thus $P(s)$ has a bounded inverse, which we denote by $V(s)$. \square

PROPOSITION 7.2. *We use the notation and the assumptions of Proposition 7.1 and S_{sca} is the system node from Theorem 1.1. Let A be the semigroup generator of S_{sca} , let B and C be the control and observation operators of S_{sca} , and let \mathbf{G} be its transfer function. Then for every $s \in \mathbb{C}_+$, we have*

$$\begin{aligned} (7.2) \quad (sI - A)^{-1} &= \begin{bmatrix} \frac{1}{s}I - \frac{1}{s^2}LV(s)L^* & -\frac{1}{s}LV(s) \\ \frac{1}{s}V(s)L^* & V(s) \end{bmatrix} \\ (sI - A)^{-1}B &= \begin{bmatrix} -\frac{1}{s}LV(s) \\ V(s) \end{bmatrix} K^*, \\ C(sI - A)^{-1} &= -K \begin{bmatrix} -\frac{1}{s}V(s)L^* & V(s) \end{bmatrix} \\ \mathbf{G}(s) &= I - KV(s)K^*, \end{aligned}$$

where $V(s) = P(s)^{-1}$ is the operator defined in Proposition 7.1.

Proof. Define S_{imp} as in (6.5) and (6.6), so that S_{imp} satisfies the assumption in Theorem 5.2. It is easy to see that on $\mathcal{D}(S_{\text{imp}})$ we have

$$\begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{imp}} \\ -[C\&D]_{\text{imp}} \end{bmatrix} = \left[\begin{array}{cc|c} sI & L & 0 \\ -L^* & sI - G & -K_0^* \\ \hline 0 & K_0 & I \end{array} \right],$$

where $K_0 = \frac{1}{\sqrt{2}}K$. The operator on the right-hand side above has a natural extension to a bounded linear operator from $X_0 \oplus U$ to $X'_0 \oplus U$. This extended operator can be

factored as

$$\begin{aligned} & \left[\begin{array}{cc|c} sI & L & 0 \\ -L^* & sI - G & -K_0^* \\ \hline 0 & K_0 & I \end{array} \right] \\ &= \left[\begin{array}{cc|c} I & 0 & 0 \\ -\frac{1}{s}L^* & I & -K_0^* \\ \hline 0 & 0 & I \end{array} \right] \left[\begin{array}{cc|c} sI & 0 & 0 \\ 0 & P(s) & 0 \\ \hline 0 & 0 & I \end{array} \right] \left[\begin{array}{cc|c} I & \frac{1}{s}L & 0 \\ 0 & I & 0 \\ \hline 0 & K_0 & I \end{array} \right], \end{aligned}$$

where the first factor is a boundedly invertible operator on $X'_0 \oplus U$, the second factor is a boundedly invertible operator from $X_0 \oplus U$ to $X'_0 \oplus U$, and the last factor is a boundedly invertible operator on $X_0 \oplus U$ (we have used Proposition 7.1). By inverting the sides of the last formula, we get that

$$\begin{aligned} & \left(\begin{bmatrix} sI & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} [A\&B]_{\text{imp}} \\ -[C\&D]_{\text{imp}} \end{bmatrix} \right)^{-1} \\ &= \left[\begin{array}{cc|c} I & -\frac{1}{s}L & 0 \\ 0 & I & 0 \\ \hline 0 & -K_0 & I \end{array} \right] \left[\begin{array}{cc|c} \frac{1}{s}I & 0 & 0 \\ 0 & V(s) & 0 \\ \hline 0 & 0 & I \end{array} \right] \left[\begin{array}{cc|c} I & 0 & 0 \\ \frac{1}{s}L^* & I & K_0^* \\ \hline 0 & 0 & I \end{array} \right]. \end{aligned}$$

We have seen in the proof of Theorem 1.1 that S_{sca} is obtained from S_{imp} via the external Cayley transformation (5.1) and (5.2). Using (5.5) (where now $A_{\text{sca}} = A$, $B_{\text{sca}} = B$, $C_{\text{sca}} = C$, and $\mathbf{G}_{\text{sca}} = \mathbf{G}$) we get (7.2). \square

Remark 7.3. It is well known that if A is the generator of a semigroup of contractions, then $\|(sI - A)^{-1}\| \leq \frac{1}{\text{Re } s}$ for all $s \in \mathbb{C}_+$. Combining this fact with (7.2), we obtain that

$$\|V(s)\|_{\mathcal{L}(E)} \leq \frac{1}{\text{Re } s} \quad \forall s \in \mathbb{C}_+.$$

Remark 7.4. With the assumptions and the notation of Proposition 7.2, the space Z from (3.8), (3.9) is a subspace of X_0 with continuous embedding. Moreover, the operator \tilde{C} from (1.3), when restricted to Z , is in $\mathcal{L}(Z, U)$. All this is easy to prove using Proposition 7.1 and the expression of $(I - A)^{-1}B$ from (7.2).

Proof of Theorem 1.4. The given formulas for the operators A and C follow directly from (1.13), (3.2), and (3.4).

Now we prove (1.20). The continuity and density of the embeddings $X_0 \subset X \subset X'_0$ follow directly from the continuity and density of the embeddings $E_0 \subset X \subset E'_0$.

By the definition of $\mathcal{D}(A)$ we have that $X_1 \subset X_0$. We claim that X_1 is dense in X_0 . To prove this assume that $\begin{bmatrix} z' \\ w' \end{bmatrix} \in X'_0$ and $\langle \begin{bmatrix} z' \\ w' \end{bmatrix}, (I - A)^{-1} \begin{bmatrix} z \\ w \end{bmatrix} \rangle_{X'_0, X_0} = 0$ for all $\begin{bmatrix} z \\ w \end{bmatrix} \in X$. We have to show that this implies $\begin{bmatrix} z' \\ w' \end{bmatrix} = 0$. By (7.2),

$$\begin{aligned} & \left\langle \begin{bmatrix} z' \\ w' \end{bmatrix}, (I - A)^{-1} \begin{bmatrix} z \\ w \end{bmatrix} \right\rangle_{X'_0, X_0} \\ &= \left\langle \begin{bmatrix} z' \\ w' \end{bmatrix}, \begin{bmatrix} I - LV(1)L^* & -LV(1) \\ V(1)L^* & V(1) \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \right\rangle_{X'_0, X_0} \\ &= \langle z', z - LV(1)L^*z - LV(1)w \rangle_H + \langle w', V(1)L^*z + V(1)w \rangle_{E'_0, E_0} \\ &= \langle z' - LV(1)^*L^*z' + LV(1)^*w', z \rangle_H + \langle -V(1)^*L^*z' + V(1)^*w', w \rangle_E. \end{aligned}$$

Thus $z' - LV(1)^*L^*z' + LV(1)^*w' = 0$ and $-V(1)^*L^*z' + V(1)^*w' = 0$. The second of these equations gives $L^*z' = w'$, which substituted in the first equation gives $z' = 0$. Thus, $\begin{bmatrix} z' \\ w' \end{bmatrix} = 0$ and this proves that X_1 is dense in X_0 .

Since both X_1 and X_0 are continuously embedded in X , it follows that the embedding from X_1 to X_0 is a closed operator from X_1 to X_0 . By the closed graph theorem, the embedding $X_1 \hookrightarrow X_0$ is continuous.

Recall that A^* has the same structure as A with L , K , and G replaced with $-L$, $-K$, and G^* . Therefore, by the same argument as above, if we replace A by A^* and denote the domain of A^* by X_1^d , then X_1^d is continuously and densely embedded in X_0 . Recall from Proposition 2.10.2 in [39] that X_{-1} is the dual of X_1^d with respect to the pivot space X . Therefore, by duality X'_0 is densely and continuously embedded in X_{-1} . This completes the proof of (1.20).

Let $A_{-1} \in \mathcal{L}(X, X_{-1})$ be the usual extension of the generator A , as explained at the beginning of section 3. Since X_0 is continuously embedded in X , the restriction of A_{-1} to X_0 is in $\mathcal{L}(X_0, X_{-1})$. Since X'_0 is continuously embedded in X_{-1} , the operator \overline{A} may also be regarded as an operator in $\mathcal{L}(X_0, X_{-1})$. Since the restrictions of these two operators to X_1 are equal and since X_1 is dense in X_0 , it follows that these two operators are in fact equal, as stated in the theorem.

The two remaining claims about B and $\mathbf{G}(s)$ follow from Proposition 7.2. \square

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