Coprime factorizations and stabilizability of infinite-dimensional linear systems

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Abstract

We extend some classical results on coprime factorizations to proper transfer functions whose values are allowed to be bounded operators between Hilbert spaces of any dimensions. We also present some new equivalences.

Any proper transfer function can be realized as a well-posed linear system (aka. WPLS or Salamon–Weiss system). We give sufficient and/or necessary conditions for a proper transfer function to have 1. a quasi–right coprime, 2. doubly coprime factorization, in terms of A. quotients of stable transfer functions, B. (state-feedback) stabilizability and/or detectability of realizations, C. dynamic (or "internal") stabilizability of realizations, D. the ranges of the Hankel and Toeplitz operators of the transfer function.

For example, we show that a proper transfer function has a quasi-right-coprime factorization iff it has an output-stabilizable realization, and that it has a doubly coprime factorization if it has a stabilizing controller. Part of the results have already been known for the case of matrix-valued transfer functions and most of the rest for rational transfer functions.

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In Section 1, we present part of "A." & "C." & "D.", our pure frequency domain results. In Section 2, we recall the definitions of WPLSs and state feedback. In Sections 3 and 4, we present "B." and extend the others. The proofs and details can be found in [M04b].

1 Frequency-domain results

By $\mathbb{R}$ (resp. $\mathbb{C}$) we denote the set of real (resp. complex) numbers; $i\mathbb{R}$ denotes the imaginary axis, $\mathbb{R}_+ := \{t \in \mathbb{R} \,\big|\, t \geq 0\}$, $\mathbb{R}_- := \{t \in \mathbb{R} \,\big|\, t \leq 0\}$. By U, H, Y we denote Hilbert spaces of arbitrary dimension and by $\mathcal{B}(U,Y)$ bounded linear operators $U \to Y$; $\mathcal{B}(U) := \mathcal{B}(U,U)$, $\mathbb{C}^+_\omega := \{s \in \mathbb{C} \,\big|\, \operatorname{Re} s > \omega\}$. Given $\omega \in \mathbb{R}$, by $\mathrm{H}^\infty(\mathbb{C}^+_\omega; \mathcal{B}(U,Y))$ (or $\mathrm{H}^\infty_\omega(U,Y)$) we denote the set of bounded holomorphic functions $\mathbb{C}^+_\omega \to \mathcal{B}(U,Y)$. The elements of $\mathrm{H}^\infty_\infty := \cup_{\omega\in\mathbb{R}} \mathrm{H}^\infty_\omega$ are called (well-posed or) *proper transfer functions*, and those of H^∞_0 *stable*. (We do not study improper plants, though improper controllers will be presented in Section 4.) We identify any holomorphic function on $\mathbb{C}^+_\omega$ with its restrictions to $\mathbb{C}^+_\alpha$ $(\alpha > \omega)$.

We first extend a classical result; the terminology will be explained below:

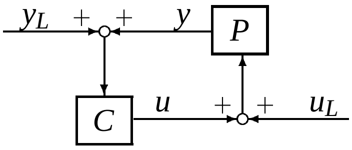


Figure 1: Controller C for the plant P

Theorem 1.1 (D.c.f.) *Any dynamically stabilizable proper transfer function has a d.c.f. If the input and output dimensions are finite, then also the converse holds (and the controller can be taken stable).*

We shall extend this to a long chain of related implications in Theorem 4.1.

A proper transfer function P ("the plant") is called *dynamically stabilizable* (aka. "internally stabilizable") if there is a proper transfer function C ("the controller") such that $\left[\begin{smallmatrix} I & -P \\ -C & I \end{smallmatrix}\right]^{-1}$ exists (at some point) and is stable (it follows that the inverse exists on any $\mathbb{C}^+_\omega$ on which P and C exist (for $\omega \geq 0$)).

This is the case iff the maps $\left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in Figure 1 are (proper and) stable (i.e., L^2 is mapped into L^2), since, obviously, $\left[\begin{smallmatrix} \widehat{u} \\ \widehat{y} \end{smallmatrix}\right] = \left(\left[\begin{smallmatrix} I & -P \\ -C & I \end{smallmatrix}\right]^{-1} - \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]\right) \left[\begin{smallmatrix} \widehat{u_L} \\ \widehat{y_L} \end{smallmatrix}\right]$, where $\widehat{\ }$ denotes the Laplace transform (see (3)).

We say that P has a *d.c.f.* (doubly coprime factorization) if there are $f, g, \widetilde{f}, \widetilde{g}, F, G, \widetilde{F}, \widetilde{G} \in \mathrm{H}^\infty_0$ s.t. $g^{-1} \in \mathrm{H}^\infty_\infty$, $P = fg^{-1}$ and $\left[\begin{smallmatrix} g & F \\ f & G \end{smallmatrix}\right] \left[\begin{smallmatrix} \widetilde{G} & -\widetilde{F} \\ -\widetilde{f} & \widetilde{g} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I_U & 0 \\ 0 & I_Y \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{G} & -\widetilde{F} \\ -\widetilde{f} & \widetilde{g} \end{smallmatrix}\right] \left[\begin{smallmatrix} g & F \\ f & G \end{smallmatrix}\right]$ on $\mathbb{C}^+_0$. (It follows that $\widetilde{g}^{-1} \in \mathrm{H}^\infty_\infty$ and $P = \widetilde{g}^{-1}\widetilde{f}$. Note that $f \in \mathrm{H}^\infty(\mathbb{C}^+_0; \mathcal{B}(U,Y))$ and $g \in \mathrm{H}^\infty(\mathbb{C}^+_0; \mathcal{B}(U))$.)

If P and C are as above, then we can choose $f, g, \widetilde{f}, \widetilde{g}, F, G, \widetilde{F}, \widetilde{G}$ so that they, in addition, satisfy $G^{-1}, \widetilde{G}^{-1} \in \mathrm{H}^\infty_\infty$, $C = FG^{-1} = \widetilde{G}^{-1}\widetilde{F}$.

For matrix-valued transfer functions, (i.e., for $\dim U, \dim Y < \infty$), the first claim of Theorem 1.1 is due to [S89] and the converse due to [Q04] (they cover also some improper transfer functions). The first author extended the result for infinite-dimensional U and Y in [M04b] using the integral Riccati equation theory of [M04a] (the extensibility of the converse is open).

By $\mathrm{H}^2(\mathbb{C}^+_\omega; U)$ (or $\mathrm{H}^2_\omega(U)$) we denote Hilbert space of holomorphic functions $h : \mathbb{C}^+_\omega \to U$ for which

$$\|h\|^2_{\mathrm{H}^2} := \sup_{r>\omega} \int_{-\infty}^{\infty} \|h(r+it)\|^2_U \, dt < \infty.$$

Functions $f \in \mathrm{H}^\infty(\mathbb{C}^+_0; \mathcal{B}(U,Y))$ and $g \in \mathrm{H}^\infty(\mathbb{C}^+_0; \mathcal{B}(U))$ are called *q.r.c.* (quasi–right coprime) if $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] h \in \mathrm{H}^2_0 \;\Rightarrow\; h \in \mathrm{H}^2_0$ for all $h \in \mathrm{H}^2_\infty(U) := \cup_\omega \mathrm{H}^2_\omega(U)$.

(A sufficient but not necessary condition for this is that f, g are *r.c. (right-coprime)*, i.e., that $\left[\begin{smallmatrix} \widetilde{F} & -\widetilde{G} \end{smallmatrix}\right] \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \equiv I$ (on $\mathbb{C}^+_0$) for some $\left[\begin{smallmatrix} \widetilde{F} & -\widetilde{G} \end{smallmatrix}\right] \in \mathrm{H}^\infty_0$. Another one is the Corona condition (Theorem 4.1(C)).)

Any transfer function having a "(stable) right factorization" $P = fg^{-1}$ has a "q.r.c. factorization" (i.e., f, g can be taken q.r.c.):

Theorem 1.2 (Quasi–right-coprime factorization) *Given any $f \in \mathrm{H}^\infty(\mathbb{C}^+_0; \mathcal{B}(U,Y))$, $g \in \mathrm{H}^\infty(\mathbb{C}^+_0; \mathcal{B}(U))$ such that g^{-1} exists and is bounded on some right half-plane, there are f_2, g_2 that satisfy the same conditions, $fg^{-1} = f_2 g_2^{-1}$, and, in addition, f_2 and g_2 are q.r.c. and $\left[\begin{smallmatrix} f_2 \\ g_2 \end{smallmatrix}\right]$ is inner.*

Thus, we can "cancel any common zeros of f and g on $\mathbb{C}^+_0$". Indeed, when f, g are q.r.c., then $f(s)u = 0 \;\Rightarrow\; g(s)u \neq 0$ for any $u \in U$. We call $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ *inner* if (the boundary function) $\|\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] u\|_{U \times Y} = \|u\|_U$ a.e. on $i\mathbb{R}$ for any $u \in U$, equivalently, $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ is an isometry on H^2_0.

We note that a right factorization fg^{-1} is q.r.c. iff any properly-invertible common right factor of f and g is a unit (in H^∞). However, if $\dim U = \infty$, then any two maps have common right factors that do not have even proper inverses, hence the property "weakly coprime" of [S89] is not interesting in this generality.

Next we reformulate Theorem 1.2 and add a third condition:

Theorem 1.3 *Let $\omega \in \mathbb{R}$, $P \in \mathrm{H}^\infty(\mathbb{C}^+_\omega; \mathcal{B}(U,Y))$. Then the following are equivalent:*

(i) *P has a right factorization;*

(ii) *P has a q.r.c. factorization;*

(iii) $\mathrm{Ran}(H_{P,\alpha}) \subset \mathrm{Ran}(T_{P,0}) + \mathrm{H}^2_0$ *for some* $\alpha \geq \omega$.

By Corollary 3.3, a fourth equivalent condition is that P has an output-stabilizable realization.

Here $\mathrm{Ran}(T_{P,0}) := \{Ph \mid h \in \mathrm{H}^2(\mathbb{C}_0^+;U)\}$ is the range of the *Toeplitz operator* of P (restricted to H^2_0) and $\mathrm{Ran}(H_{P,\alpha}) := \{\Pi Ph \mid h \in \mathrm{H}^2(\mathbb{C}_\alpha^-;U)\}$ that of the *Hankel operator* of P (projected to $\mathrm{H}^2(\mathbb{C}_\alpha^+;U)$; here $\mathbb{C}_\alpha^- := \{s \in \mathbb{C} \mid \mathrm{Re}\, s < \alpha\}$ and $\mathrm{H}^2(\mathbb{C}_\alpha^-;U)$ is the set of holomorphic functions $h : \mathbb{C}_\alpha^- \to U$ for which $\|h\|^2_{\mathrm{H}^2(\mathbb{C}_\alpha^-;U)} := \sup_{r<\alpha} \int_{-\infty}^{\infty} \|h(r+it)\|_U^2\, dt < \infty$). Sometimes the Hankel operator is defined by adding a reflection: "$H_{P,\alpha} := \Pi P \breve{\ }$" (on $\mathrm{H}^2(\mathbb{C}^+_{-\alpha};U)$), where $\breve{h}(s) := h(-s)$, but this does not affect the range. The alternative definition $(I-\Pi)P\Pi$ is not compatible with Theorem 1.3.

For $h \in \mathrm{H}^2(\mathbb{C}_\alpha^-;U)$, Ph is an element of $\mathrm{L}^2(\alpha + i\mathbb{R};Y)$, and hence $\Pi Ph \in \mathrm{H}^2(\mathbb{C}_\alpha^+;U)$, where Π is the orthogonal projection $\mathrm{L}^2(\alpha + i\mathbb{R};Y) \to \mathrm{H}^2(\mathbb{C}_\alpha^+;U)$. An equivalent way to interpret (iii) is that for any $h \in \mathrm{H}^2(\mathbb{C}_\alpha^-;U)$, there is $\widetilde{h} \in \mathrm{H}^2(\mathbb{C}_0^+;U)$ s.t. $Ph - P\widetilde{h}$ is the restriction of a H^2_0 function to the domain of $Ph - P\widetilde{h}$ (i.e., to $\{s \in \mathbb{C} \mid \max\{0,\omega\} < \mathrm{Re}\, s < \alpha\}$).

2 Well-Posed Linear Systems

It is well-known that any rational transfer function has a d.c.f. and a jointly exponentially stabilizable realization. The proper transfer function $P(s) = (s-1)^{-1/2}$ has neither, and in general, a proper transfer function has a d.c.f. iff it has a jointly stabilizable and detectable realization [S98a]. To be able to formulate this and further necessary and sufficient conditions on factorizability in terms of realizations, we must define the realizations of transfer functions.

For this purpose, we use *WPLSs* (well-posed linear systems), since their transfer functions are proper transfer functions, and conversely, any proper transfer function is the transfer function of a WPLS. The WPLSs are equivalent to Lax–Phillips scattering systems and to the operator-based models of Béla Sz.-Nagy and Ciprian Foiaş [S04].

A LTI system is typically governed by

$$\dot{x}(t) = Ax(t) + Bu(t), \;\; y(t) = Cx(t) + Du(t) \quad (1)$$

(for $t \geq 0$), $x(0) = x_0$, where the *generators* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U, H \times Y)$ are matrices, or more generally, linear operators on Hilbert spaces (U,H,Y) of arbitrary dimensions.

Given x_0 and u, the *state* x and *output* y equal

$$x(t) = \mathscr{A}^t x_0 + \mathscr{B}^t u, \quad y = \mathscr{C} x_0 + \mathscr{D} u, \quad (2)$$

$$\mathscr{A}^t = e^{At}, \quad \mathscr{B}^t u = \int_0^t \mathscr{A}^{t-s} Bu(s)\, ds,$$

$$(\mathscr{C} x_0)(t) = C\mathscr{A}^t x_0, \; (\mathscr{D} u)(t) = C\mathscr{B}^t u + Du(t).$$

We study *Well-Posed Linear Systems (WPLSs)* ("Salamon–Weiss class"), i.e., time-invariant systems of form (2), with $\mathscr{A}^t, \mathscr{B}^t, \mathscr{C}, \mathscr{D}$ linear, bounded, compatible with each other and continuous on $H \times \mathrm{L}^2_{\mathrm{loc}}$. It follows that $\mathscr{A}$ is a strongly continuous semigroup and A,B,C exist to satisfy $\dot{x} = Ax + Bu$ (and $y = Cx$ when $u = 0$), but A,B,C may be unbounded. [M04a]

By $\mathrm{L}^2_\omega(\mathbb{R};U) = e^{-\omega\cdot}\mathrm{L}^2(\mathbb{R};U)$ we denote the Hilbert space of (equivalence classes of Bochner-)measurable functions $u := \mathbb{R} \to U$ for which $\|u\|^2_{\mathrm{L}^2_\omega} := \int_\mathbb{R} e^{-\omega t} \|u(t)\|^2_U\, dt < \infty$. We set $\mathrm{L}^2 := \mathrm{L}^2_0$, $(\tau^t u)(s) := u(t+s)$ and $\pi_\pm u := \chi_{\mathbb{R}_\pm} u$, where $\chi_E(t) := \begin{cases} 1, & t \in E; \\ 0, & t \notin E \end{cases}$. We also consider π_+ as the projection $\mathrm{L}^2(\mathbb{R};U) \to \mathrm{L}^2(\mathbb{R}_+;U)$ or as its adjoint.

We now recall the exact definition of WPLSs:

Definition 2.1 (WPLS and stability) *Let* $\omega \in \mathbb{R}$. *An* ω-stable well-posed linear system on (U,H,Y) *is a quadruple* $\Sigma = \left[\begin{smallmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{smallmatrix}\right]$*, where* $\mathscr{A}^t$*,* $\mathscr{B}$*,* $\mathscr{C}$*, and* $\mathscr{D}$ *are bounded linear operators of the following type:*

1. $\mathscr{A}^\cdot : H \to H$ *is a strongly continuous semigroup of bounded linear operators on H satisfying* $\sup_{t\geq 0} \|e^{-\omega t}\mathscr{A}^t\|_H < \infty$;

2. $\mathscr{B}$: $\mathrm{L}^2_\omega(\mathbb{R};U) \to H$ *satisfies* $\mathscr{A}^t \mathscr{B} u = \mathscr{B}\tau^t \pi_- u$ *for all* $u \in \mathrm{L}^2_\omega(\mathbb{R};U)$ *and* $t \in \mathbb{R}_+$;

3. $\mathscr{C}$: $H \to \mathrm{L}^2_\omega(\mathbb{R};Y)$ *satisfies* $\mathscr{C}\mathscr{A}^t x = \pi_+ \tau^t \mathscr{C} x$ *for all* $x \in H$ *and* $t \in \mathbb{R}_+$;

4. $\mathscr{D} \colon \mathrm{L}^2_\omega(\mathbb{R};U) \to \mathrm{L}^2_\omega(\mathbb{R};Y)$ *satisfies* $\tau^t \mathscr{D} u = \mathscr{D}\tau^t u$, $\pi_- \mathscr{D} \pi_+ u = 0$, *and* $\pi_+ \mathscr{D} \pi_- u = \mathscr{C}\mathscr{B}u$ *for all* $u \in \mathrm{L}^2_\omega(\mathbb{R};U)$ *and* $t \in \mathbb{R}$.

We say that $\mathscr{A}$ *(resp.* $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$*) is* α-stable *if 1. (resp. 2., 3., 4.) holds for* $\omega = \alpha$. Stable *means* 0*-stable;* exponentially stable *means* ω*-stable for some* $\omega < 0$. *The system is* output stable *(resp.* SOS-stable*) if* $\mathscr{C}$ *(resp.* $\mathscr{C}$ *and* $\mathscr{D}$*) is stable.*

Given any x_0, u, *we define* x, y *by (2), where* $\mathscr{B}^t := \mathscr{B}\tau^t\pi_+$.

By $\widehat{u}$ we denote the *Laplace transform* of u:

$$\widehat{u}(s) := \int_{\mathbb{R}} \mathrm{e}^{-st} u(t)\, dt \qquad (s \in \mathbb{C}^+_\omega). \tag{3}$$

The Laplace transform is an isometric (modulo $(2\pi)^{1/2}$) isomorphism of L^2_ω onto H^2_ω. This corresponds to an isometric isomorphism of H^∞_ω [on]to the [time-invariant and causal] maps on L^2_ω; similarly, every proper transfer function has a realization:

Theorem 2.2 *Let* $\omega \in \mathbb{R}$. *For any* ω*-stable WPLS* $\left[\begin{smallmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{smallmatrix}\right]$, *there is a unique* transfer function $\widehat{\mathscr{D}} \in \mathrm{H}^\infty_\omega$ *s.t.* $\widehat{\mathscr{D}u} = \widehat{\mathscr{D}}\widehat{u}$ *on* $\mathbb{C}^+_\omega$ *for any* $u \in \mathrm{L}^2_\omega(\mathbb{R}_+;U)$.

Conversely, any $\widehat{\mathscr{D}} \in \mathrm{H}^\infty(\mathbb{C}^+_\omega; \mathcal{B}(U,Y))$ *is the transfer function of some* ω*-stable WPLSs (which is called the* realization *of* $\widehat{\mathscr{D}}$*).*

For systems having bounded generators, we have $\widehat{\mathscr{D}}(s) = D + C(s-A)^{-1}B$ (actually this is valid for rather unbounded generators, once we use the Yosida extension of C). For such systems, state feedback means using $u(t) = Kx(t)$ as the input (for some $K \in \mathcal{B}(H,U)$). Substitution of this into (1) leads to the *closed-loop system* $\dot{x}(t) = (A+BK)x(t)$, $y(t) = (C+DK)x(t)$, or, if we allow for the *external input* $u_\circlearrowleft$, i.e., $u(t) = Kx(t) + u_\circlearrowleft(t)$, to $\dot{x}(t) = (A+BK)x(t) + Bu_\circlearrowleft(t)$, $y(t) = (C+DK)x(t) + Du_\circlearrowleft(t)$.

To generalize this to arbitrary WPLSs, one adds an additional output $v := \mathscr{K}x_0 + \mathscr{F}u$ and uses $u = v + u_\circlearrowleft$ as the input (i.e, $(I-\mathscr{F})u = \mathscr{K}x_0 + u_\circlearrowleft$), as in Figure 2. Such *state feedback* is called *admissible* if the map $(I-\mathscr{F}) : u \to u_\circlearrowleft$ has a well-posed inverse (bounded on L^2_ω

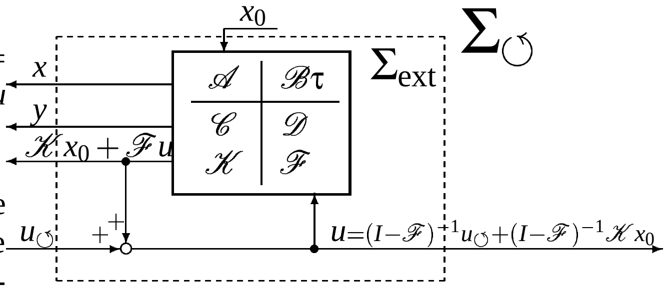


Figure 2: State-feedback connection $u(t) = Kx(t) + Fu(t)$

for some $\omega \in \mathbb{R}$; this is always the case if the feedback is generated by some $K \in \mathcal{B}(H,U)$), equivalently, if $(I - \widehat{\mathscr{F}})^{-1} \in \mathrm{H}^\infty_\infty$:

Definition 2.3 ($\Sigma_\circlearrowleft, K, [\,\mathscr{K} \mid \mathscr{F}\,]$**)** *A pair* $[\,\mathscr{K} \mid \mathscr{F}\,]$ *is called an* admissible state-feedback pair *for* $\left[\begin{smallmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{smallmatrix}\right]$ *if the extended system*

$$\Sigma_{\text{ext}} := \left[\begin{array}{c|c} \mathscr{A} & \mathscr{B}\tau \\ \hline \mathscr{C} & \mathscr{D} \\ \mathscr{K} & \mathscr{F} \end{array}\right] \tag{4}$$

is a WPLS *(on* $(U, H, Y \times U)$*) and* $(I - \widehat{\mathscr{F}})^{-1} \in \mathrm{H}^\infty_\infty(U)$.

We set $\mathscr{M} := (I - \mathscr{F})^{-1}$, $\mathscr{N} := \mathscr{D}\mathscr{M}$ *and denote the corresponding closed-loop system (see Figure 2)*

$$\Sigma_\circlearrowleft = \left[\begin{array}{c|c} \mathscr{A}_\circlearrowleft & \mathscr{B}_\circlearrowleft\tau \\ \hline \mathscr{C}_\circlearrowleft & \mathscr{D}_\circlearrowleft \\ \mathscr{K}_\circlearrowleft & \mathscr{F}_\circlearrowleft \end{array}\right] = \left[\begin{array}{c|c} \mathscr{A} + \mathscr{B}\tau\mathscr{M}\mathscr{K} & \mathscr{B}\mathscr{M}\tau \\ \hline \mathscr{C} + \mathscr{D}\mathscr{M}\mathscr{K} & \mathscr{D}\mathscr{M} \\ \mathscr{M}\mathscr{K} & \mathscr{M} - I \end{array}\right] \tag{5}$$

$$= \Sigma_{\text{ext}} \begin{bmatrix} I & 0 \\ -\mathscr{K} & I - \mathscr{F} \end{bmatrix}^{-1} \tag{6}$$

$$= \Sigma_{\text{ext}} \begin{bmatrix} I & 0 \\ \mathscr{M}\mathscr{K} & \mathscr{M} \end{bmatrix} : \begin{bmatrix} x_0 \\ u_\circlearrowleft \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ u - u_\circlearrowleft \end{bmatrix}.$$

We call $[\,\mathscr{K} \mid \mathscr{F}\,]$ stabilizing *if* $\Sigma_\circlearrowleft$ *is stable. If there exists a stabilizing state-feedback pair for* Σ, *then* Σ *is called* stabilizable *(similarly for exponentially, SOS- or output-stabilizing etc.).*

(To allow for formulas similar to (6), we often denote the WPLS $\left[\begin{smallmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} \mathscr{A} & \mathscr{B}\tau \\ \mathscr{C} & \mathscr{D} \end{smallmatrix}\right]$.)

3 Quasi-coprime factorizations

We start by noting that "optimizability" or the *state-FCC* ("Finite Cost Condition") is equivalent to exponential stabilizability:

Theorem 3.1 *A WPLS $[\frac{\mathscr{A}}{\mathscr{C}}\big|\frac{\mathscr{B}}{\mathscr{D}}]$ is exponentially stabilizable iff for each initial state $x_0 \in H$ there is a stable input ($u \in \mathrm{L}^2(\mathbb{R}_+;U)$) such that the state $x := \mathscr{A}x_0 + \mathscr{B}\tau u$ is stable ($x \in \mathrm{L}^2$).*

If B is *bounded* ($B \in \mathcal{B}(U,H)$), then a third equivalent condition is that there exists (a *state feedback operator*) $K \in \mathcal{B}(H,U)$ such that the semigroup generated by $A+BK$ is exponentially stable, and a fourth one that some $\mathcal{P} \geq 0$ solves the Riccati equation (*RE*) $\mathcal{P}BB^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + I$ (in which case we can have $K = -B^*\mathcal{P}$); this case has been well known.

The theorem was proved first in M04c], by using resolvent REs (built on reciprocal RE theory), and later, more directly, in [M04a], by using integral REs in place of the above algebraic RE. In both proofs, it was shown that the control minimizing the *cost* $\|x\|_2^2 + \|u\|_2^2$ corresponds to an admissible state-feedback pair (note that the state-FCC says that the above cost can be made finite).

The corresponding equivalence also holds for (and similar comments apply to) output-stabilizability:

Theorem 3.2 *Let $[\frac{\mathscr{A}}{\mathscr{C}}\big|\frac{\mathscr{B}}{\mathscr{D}}]$ be a WPLS. Then the following are equivalent:*

(i) *For each $x_0 \in H$ there is $u \in \mathrm{L}^2(\mathbb{R}_+;U)$ such that $y \in \mathrm{L}^2$.*

(ii) *There is an output-stabilizing state-feedback pair $[\ \mathscr{K} \mid \mathscr{F}\ ]$ for Σ.*

(iii) *There is a SOS-stabilizing state-feedback pair $[\ \mathscr{K} \mid \mathscr{F}\ ]$ for Σ such that $\mathscr{N},\mathscr{M}$ are q.r.c. and $\mathscr{N}^*\mathscr{N} + \mathscr{M}^*\mathscr{M} = I$.*

(iv) *The corresponding integral RE has a nonnegative solution.*

Note from Definition 2.3 that $\mathscr{D} = \mathscr{N}\mathscr{M}^{-1}$ (equivalently, $\widehat{\mathscr{D}} = \widehat{\mathscr{N}}\widehat{\mathscr{M}^{-1}}$) and that $\left[\begin{smallmatrix}\mathscr{N}\\ \mathscr{M}\end{smallmatrix}\right]$ maps $u_\circlearrowleft \mapsto \left[\begin{smallmatrix}y\\ u\end{smallmatrix}\right]$ (when $x_0 = 0$). By (iii), actually, $\mathscr{C}_\circlearrowleft, \mathscr{D}_\circlearrowleft, \mathscr{K}_\circlearrowleft, \mathscr{F}_\circlearrowleft$ are all stable.

The pair in (iii) is the one that (strictly) minimizes the cost $\int_0^\infty (\|y(t)\|_Y^2 + \|u(t)\|_U^2)\,dt$ for each x_0 (in Figure 2, for $u_\circlearrowleft = 0$; note that the *output-FCC* (i) above says that this cost can be made finite). This minimizing pair is unique modulo an invertible operator $E \in \mathcal{B}(U)$; and all q.r.c. factorizations of $\mathscr{D}$ are given by $(\mathscr{N}E)(\mathscr{M}E)^{-1}$; the operator E must be unitary to satisfy (iii).

From Theorem 3.2 one can derive an extension of Theorem 1.3 (set $\widehat{\mathscr{D}} := P$, $f := \widehat{\mathscr{N}}$, $g := \widehat{\mathscr{M}}$):

Corollary 3.3 (q.r.c.f.) *Let $P \in \mathrm{H}^\infty_\omega(U,Y)$, $\omega \in \mathbb{R}$, and let $\mathscr{D}$ denote the (I/O) map for which $\widehat{\mathscr{D}} = P$. Then following are equivalent:*

(i) *$P = fg^{-1}$, where $f,g \in \mathrm{H}^\infty_0$, $g^{-1} \in \mathrm{H}^\infty_\infty(U)$.*

(ii) *$P = fg^{-1}$, where $f,g \in \mathrm{H}^\infty_0$ are q.r.c., $g^{-1} \in \mathrm{H}^\infty_\infty(U)$, and $\left[\begin{smallmatrix}f\\ g\end{smallmatrix}\right]$ is inner.*

(iii) *For some $\alpha \geq \omega$ and any $v \in \mathrm{L}^2_\alpha(\mathbb{R}_-;U)$ there exists $u \in \mathrm{L}^2(\mathbb{R}_+;U)$ s.t. $\pi_+\mathscr{D}(v+u) \in \mathrm{L}^2$ (equivalently, $\mathscr{D}(v+u) \in \mathrm{L}^2$).*

(iv) *There is an output-stabilizable realization of P.*

(v) *There is a stabilizable realization of P.*

Condition (iii) says that $\mathrm{Ran}_\alpha(\pi_+\mathscr{D}\pi_-) \subset \mathrm{L}^2 + \mathrm{Ran}(\pi_+\mathscr{D}\pi_+)$, which is the state-space formulation of Theorem 1.3(iii).

4 Coprime factorizations

In Theorem 4.1 we list several conditions for a proper transfer function and known and new implications for them. We suspect that actually all these implications may be equivalent (except (C)); at least this is the case for rational transfer functions. The terminology will be explained below the theorem (see [M02] for further details).

The implications (SC)⇒(dcf), and (IL)⇒($\left[\begin{smallmatrix}\mathscr{D} & 0\\ 0 & I\end{smallmatrix}\right]$)⇒(FCC&FCC$^\mathrm{d}$)⇒(ext.s&d) are new, from [M04b]. As explained above, under $\dim U, \dim Y < \infty$ (SC)⇒(dcf) is due to [S89], the converse due to [Q04]; (cf)⇔(dcf) is due to the Tolokonnikov's Lemma and (cf)⇔(C) is the Corona Theorem. The implications (dcf)⇔(j.s&d)⇔(j.ext.s&d)⇒(ext.s&d) are essentially due to [S98a], and (cf)⇔(CC)⇒(IL) due to [CWW01]. Implications (dcf)⇒(cf)⇒(C) and (FCC&FCC$^\mathrm{d}$)⇐(ext.s&d) are obvious and the Youla parameterization is well known.

Theorem 4.1 (D.c.f.⇔...) *For any $P \in \mathrm{H}^\infty_\infty(U,Y)$, the implications below hold for the following properties:*

(SC) *P has a stabilizing (dynamic) controller.*

(dcf) *P has a d.c.f.*

(cf) *P has a r.c.f. or a l.c.f.*

(CC) *P has a stabilizing canonical controller.*

(IL) *P has a stabilizing controller with internal loop.*

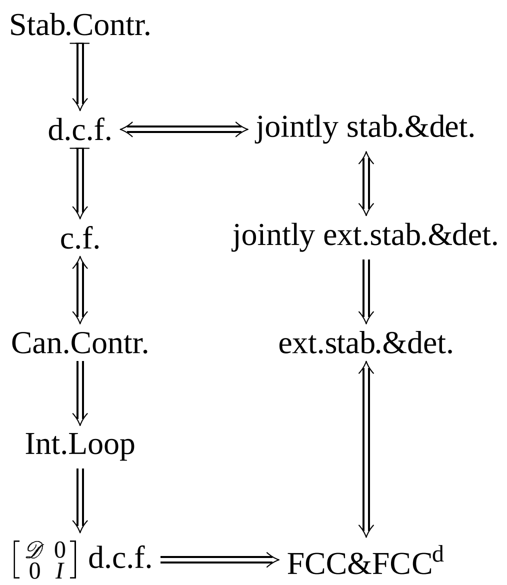
($\left[\begin{smallmatrix}\mathscr{D} & 0\\ 0 & I\end{smallmatrix}\right]$) $\left[\begin{smallmatrix}P & 0\\ 0 & I\end{smallmatrix}\right]$ *has a d.c.f.*

(FCC&FCC$^\mathrm{d}$) *P has a realization Σ s.t. the output-FCC holds for Σ and for its dual.*

(ext.s&d) *P has an externally stabilizable and externally detectable realization.*

(j.ext.s&d) *P has a jointly externally stabilizable and externally detectable realization.*

(j.s&d) *P has a jointly stabilizable and detectable realization.*



Condition (IL) is equivalent to the following:

(ILreal) *P has a realization that is stabilizable by controller (system) with internal loop.*

Analogous claims hold for (SC) and (CC) too.

If $\dim U, \dim Y < \infty$*, then also the converse implications hold for the arrows starting with "$\models$", and (cf) is equivalent also to the following* Corona condition*:*

(C) $P = FG^{-1}$ *with* $F,G \in \mathrm{H}^\infty$, $F^*F + G^*G \geq \varepsilon I$ *on* $\mathbb{C}^+$ *for some* $\varepsilon > 0$, *and* $\det G \not\equiv 0$.

Given a d.c.f. $\left[\begin{smallmatrix}g & F\\ f & G\end{smallmatrix}\right] = \left[\begin{smallmatrix}\widetilde{G} & -\widetilde{F}\\ -\widetilde{f} & \widetilde{g}\end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty(U \times Y)$ *of P, all stabilizing controllers with internal loop for P are obtained from the standard Youla parameterization* $(F+gQ)(G+fQ)^{-1}$*, where* $Q \in \mathrm{H}^\infty(Y,U)$ *is arbitrary (the controller is proper iff* $(G+fQ)^{-1} \in \mathrm{H}^\infty_\infty(U)$*).*

We call $P = \widetilde{g}^{-1}\widetilde{f}$ a *l.c.f.* (left-coprime factorization) of P iff $\widetilde{g}^{-1} \in \mathrm{H}^\infty_\infty(Y)$ and $\widetilde{f}, \widetilde{g} \in \mathrm{H}^\infty$ are *l.c.* (i.e., $[\,\widetilde{f}\ \widetilde{g}\,] \in \mathrm{H}^\infty$ is right-invertible on H^∞; equivalently, $[\,\widetilde{f}\ \widetilde{g}\,]\left[\begin{smallmatrix}F\\ -G\end{smallmatrix}\right] = I$ for some $F, G \in \mathrm{H}^\infty$).

We say that $\mathscr{O}$ is a *stabilizing controller with internal loop* for $\mathscr{D}$ (or $\widehat{\mathscr{D}}$) if $\mathscr{O} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ for some Hilbert space Ξ and $(I - \mathscr{D}^o)^{-1} \in \mathrm{TIC}$, where $\mathscr{D}^o = \left[\begin{smallmatrix}0 & \mathscr{O}_{11} & \mathscr{O}_{12}\\ \mathscr{D} & 0 & 0\\ 0 & \mathscr{O}_{21} & \mathscr{O}_{22}\end{smallmatrix}\right]$. Note from Figure 3 that $\mathscr{D}^o_I := (I - \mathscr{D}^o)^{-1} - I$ maps $\left[\begin{smallmatrix}u_L\\ y_L\\ \xi_L\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}u\\ y\\ \xi\end{smallmatrix}\right]$. Thus, $\mathscr{O}$ is stabilizing iff the maps $\left[\begin{smallmatrix}u_L\\ y_L\\ \xi_L\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}u\\ y\\ \xi\end{smallmatrix}\right]$ are well-posed and stable (equivalently, iff the corresponding transfer functions are in H^∞). An equivalent condition is that $\left[\begin{smallmatrix}I & -C_0\\ -P_0 & I\end{smallmatrix}\right]^{-1} \in \mathrm{H}^\infty$, where $C_0 := \widehat{\mathscr{O}}$ and $P_0 := \left[\begin{smallmatrix}P & 0\\ 0 & I\end{smallmatrix}\right]$.

The equivalent condition (ILreal) is formally stronger: it means (the existence of Σ and $\widetilde{\Sigma}$ for this fixed $\mathscr{D}$ such) that all 25 maps from initial states and external inputs to states and outputs in Figure 3 are stable (i.e., that the combined closed-loop system Σ^o_I is stable, where Σ^o is given by (7.21) of [M02]). See Section 7.2 of [M02] for further details.

If $\mathscr{Y} \in \mathrm{TIC}(Y,U)$ and $\mathscr{X} \in \mathrm{TIC}(U)$ are r.c., then $\mathscr{O} := \left[\begin{smallmatrix}0 & \mathscr{Y}\\ I & I-\mathscr{X}\end{smallmatrix}\right]$ is called a (right) *canonical controller* (see [CWW01] or [M02]); in [M02], the term *controller with a coprime internal loop*

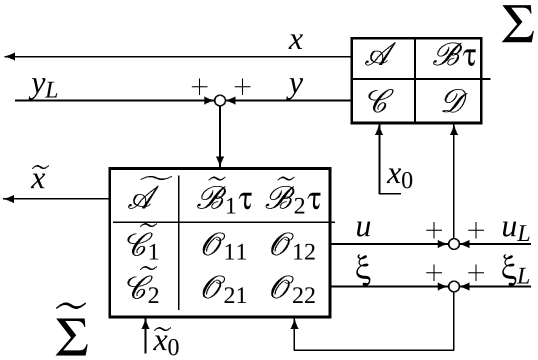


Figure 3: DF-controller $\widetilde{\Sigma}$ with internal loop for $\Sigma \in \mathrm{WPLS}(U,H,Y)$

was used. Sometimes we denote it by $\mathscr{Y}\mathscr{X}^{-1}$, as in the Youla parameterization above. Similarly, $\left[\begin{smallmatrix} 0 & I \\ \widetilde{\mathscr{Y}} & I-\widetilde{\mathscr{X}} \end{smallmatrix}\right]$ is called a (left) canonical controller if $\widetilde{\mathscr{Y}}$ and $\widetilde{\mathscr{X}}$ are l.c.

A (dynamic feedback) controller $\mathscr{O}$ (resp. $\widetilde{\Sigma}$) with internal loop is *proper* or well-posed (i.e., "with internal loop" can be dropped) iff $I - \mathscr{O}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$. In that case we can redefine $\mathscr{O}$ (resp. $\widetilde{\Sigma}$) so as to have $\mathscr{O}_{12}, \mathscr{O}_{21}, \mathscr{O}_{22} = 0$ (resp. $\mathscr{O}_{12}, \mathscr{O}_{21}, \mathscr{O}_{22}, \widetilde{\mathscr{B}}_2, \widetilde{\mathscr{C}}_2 = 0$), as in the classical definition of a controller.

The Youula parameterization covers all stabilizing controllers with internal loop in the sense that any other controller with internal loop defines the same closed-loop map $\left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ as exactly one of these (modulo $\left[\begin{smallmatrix} \mathscr{Y}' \\ \mathscr{X}' \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathscr{Y} \\ \mathscr{X} \end{smallmatrix}\right] \mathscr{E}$ for some $\mathscr{E} \in \mathcal{G}\mathrm{TIC}$), although the maps from ξ_L and to ξ (the internal loops) may differ. In particular, this parameterization contains all well-posed stabilizing controllers.

The *dual* of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right)$; and this relation can be extended to general WPLSs. As noted in Theorem 3.2, condition (FCC&FCC$^{\mathrm{d}}$) implies that we can add a new output to Σ and feed it back to the original input so that the closed-loop system becomes SOS-stable. The dual obviously implies that by feeding the original output back to a suitable new input, the dual of SOS-stability is achieved. It actually follows that not merely the output and I/O maps but also the input maps become stable, i.e., that the closed-loop systems become *externally stable* (only the semigroup may be unstable), so this leads to the condition (ext.s&d). The stronger condition (j.ext.s&d) says that, in addition, the new output row and the new input column fit *simultaneously* to the same system with the original one (namely $\Sigma_{\mathrm{Total}} := \left(\begin{smallmatrix} A & B & T \\ C & D & 0 \\ K & 0 & 0 \end{smallmatrix}\right)$, if, e.g., B and C are not too unbounded so that the optimal state-feedback operators K and T^* exist for Σ and its dual, respectively), and that both feedbacks (i.e., operators $A+BK$ and $A+TC$ in place of A) mentioned above make the whole Σ_{Total} externally stable (including the bottom-right element, whose closed-loop transfer function under K becomes $K(s-A-BK)^{-1}T$)).

Since external stability does not imply stability, the implication from (j.ext.s&d)⇒(j.s&d) may require one to change the realization.

Finally, we note that if (ext.s&d) implies (SC), then actually all conditions in Theorem 4.1 are equivalent (except (C), which is strictly weaker if $\dim U = \infty = \dim Y$). The weaker implication from (ext.s&d) to (j.ext.s&d) would have the same result except for (SC). At present the validity of neither implication is known without some additional assumptions. on the realization.

Theorem 4.1 leads to its "exponential variant", where stabilization is replaced by exponential stabilization etc. (cf. Definition 2.1). In this case, the exponential version of (ext.s&d)⇒(j.ext.s&d) only requires one to show that the bottom-right transfer function mentioned above is proper (this is always the case if B and C are not too unbounded). Some sufficient conditions to this and the above two implications are given in [S98b] and [M02], and further ones are to be expected from Ruth Curtain and Mark Opmeer.

In Theorem 4.1 (resp. Corollary 3.3), we listed several sufficient and/or necessary conditions for a proper transfer function to have a r.c.f. (resp. q.r.c.f.). If P has a r.c.f., then any q.r.c.f. of P is a r.c.f., but not all maps having a q.r.c.f. have a r.c.f. (let $f, g \in \mathrm{H}^\infty(\mathbb{C}^+;\mathbb{C})$ be Blaschke products with zeros att n^{-2} and $(n^2+1)^{-1}$ for $n = 2,3,\ldots$). Moreover, not all proper transfer functions have a q.r.c.f. (e.g., $P = (s-1)^{-1/2}$ is not meromorphic on $\mathbb{C}^+$, hence not of form f/g).

Sometimes one wants to stabilize a system dynamically through *partial feedback* (measurement-feedback), where the controller can measure only to a part of the input and affect only a part of the output. If the implication from (ext.s&d) to (j.ext.s&d) in Theorem 4.1 holds, then the numerous classical sufficient and necessary factorization conditions for partial feedback stabilizability of a rational transfer function are in fact necessary for any proper transfer function (they are known to be sufficient; see, e.g., [M02], Section 7.3, where also the necessity is given in the case that $\dim U, \dim Y < \infty$).

Similarly, if a system is exponentially stabilizable with internal loop through partial feedback, then so is the subsystem that is at the reach of the feedback controller. Also the converse holds if (ext.s&d) implies (j.ext.s&d).

All these results are well-known for finite-dimensional systems (or rational transfer functions), and an extension for smoothing $\mathscr{A}$ (i.e., $\mathscr{A}B_1u_0, \mathscr{A}^*C_2^*y_0$ locally integrable for all $u_0 \in U$, $y_0 \in Y$) with constructive formulas this was shown in Theorem 7.3.12(c) of [M02]. Since the result reduces the problem to the dynamic feedback stabilization of Σ_{21}, one more often only treats dynamic stabilization, and for that problem similar though stronger sufficient conditions have been given in, e.g., [WC97], which also provides further historical remarks (see also [M02]).

5 Generalizations and notes

Naturally, all above results have analogies for exponential stabilization (and exponential coprime factorizations: maps belong to H^∞_ω for some $\omega < 0$). The factorizations (and controllers) can be found by solving corresponding algebraic or integral Riccati equations. All results and proofs are given in [M04a], but [M02] contains some further details and historical notes.

Section 2 is well known (see, e.g., [M04a] or [M02] for historical remarks). Theorem 3.1 and "(i)⇔(ii)⇔(iv)" of Theorem 3.2 have been known for bounded B (an extension was given in Section 9.2 of [M02], and the parabolic case can be found in, e.g., [LT00]). Also Corollary 3.3 is new.

Quasi-coprimeness was presented in [M02], motivated by the fact that q.r.c. output-stabilization is the weakest form of stabilization that allows the reduction of optimal control problems to the output-stable case. By Theorem 3.2, this reduction is available for any solvable problems. However, the theory in [M04a] also allows for direct solutions.

The sources of the implications in Theorem 4.1 were listed above the theorem. The non-r.c.f. and non-q.r.c.f. examples are due to Sergei Treil and Olof Staffans. See [M04a] and Sections 7.1–7.2 of [M02] for further details and results on coprime factorizations and dynamic stabilization.

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