Well-Posed State/Signal Systems in Continuous Time

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Abstract. We introduce a new class of linear systems, the L^p-well-posed state/signal systems in continuous time, we establish the foundations of their theory and we develop some tools for their study. The principal feature of a state/signal system is that the external signals of the system are not a priori divided into inputs and outputs. We relate state/signal systems to the better-known class of well-posed input/state/output systems, showing that state/signal systems are more flexible than input/state/output systems but still have enough structure to provide a meaningful theory. We also give some examples which point to possibilities for further study.

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1. Introduction

In this work we introduce a new class of linear systems, the well-posed state/signal systems (shortly written s/s systems) in continuous time. Our approach differs from classical control theory in the sense that the systems under consideration have *no fixed inputs or outputs*, but instead a *combined external signal*, which can be decomposed into inputs and outputs *in different ways.*

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In order to make this idea more concrete, let us consider a continuous-time input/state/output system (i/s/o system) in differential form with state x, input u and output y:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \quad x(0) = x_0 \text{ given}, \ t \geq 0. \tag{1.1}$$

Here $\dot{x}$ denotes the derivative of x with respect to t, $x(t) \in \mathcal{X}$, $u(t) \in \mathcal{U}$ and $y(t) \in \mathcal{Y}$. We call $\mathcal{X}$ the *state space*, $\mathcal{U}$ the *input space* and $\mathcal{Y}$ the *output space* and, at the moment, we assume that all these spaces are finite dimensional for simplicity.

Example 1.1. *In the system* (1.1)*, we might instead want to consider the signal y as input and the signal u as output, thus* inverting the flow *of the system. If D is invertible, then this is indeed possible and we obtain the new system*

$$\begin{cases} \dot{x}(t) = (A - BD^{-1}C)x(t) + BD^{-1}y(t) \\ u(t) = -D^{-1}Cx(t) + D^{-1}y(t) \end{cases}, \quad x(0) = x_0 \text{ given}, \ t \geq 0, \tag{1.2}$$

which is of the same type as the system in (1.1)*.*

The idea to ignore the distinction between inputs and outputs can be formalised as follows. Consider the product space $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$, which we call the *combined external signal space*. We can identify the subspaces $\left[\begin{smallmatrix} \mathcal{Y} \\ \{0\} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \{0\} \\ \mathcal{U} \end{smallmatrix}\right]$ of $\mathcal{W}$ with $\mathcal{Y}$ and $\mathcal{U}$, respectively. In this way we can view $\mathcal{U}$ and $\mathcal{Y}$ as subspaces of $\mathcal{W}$ and add elements of $\mathcal{U}$ and $\mathcal{Y}$ in $\mathcal{W}$: $u + y = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$. In this way $\mathcal{W}$ can be identified with the direct sum $\mathcal{U} \dotplus \mathcal{Y}$.

Defining the *combined external signal* of (1.1) by $w(t) := u(t) + y(t)$, we may now write (1.1) equivalently as

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \ x(0) = x_0, \ t \geq 0, \quad \text{where} \quad V = \begin{bmatrix} A & B \\ 1_{\mathcal{X}} & 0 \\ C & D + 1_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}. \tag{1.3}$$

The triple $(V; \mathcal{X}, \mathcal{W})$ is called the *state/signal node* (s/s node) of the system.

Returning to Example 1.1, we note that although the equations (1.1) and (1.2) are different, they describe the same physical system, because the relations between the different signals are preserved. This is reflected in the fact that the s/s node is invariant under flow inversion:

$$\begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 1_{\mathcal{X}} & 0 \\ -D^{-1}C & D^{-1} + 1 \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 1_{\mathcal{X}} & 0 \\ -D^{-1}C & D^{-1} + 1_{\mathcal{U}} \end{bmatrix}$$
$$\times \begin{bmatrix} 1_{\mathcal{X}} & 0 \\ C & D \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} = \begin{bmatrix} A & B \\ 1_{\mathcal{X}} & 0 \\ C & D + 1_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix},$$

since $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ C & D \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ when D is invertible.

By choosing different decompositions of the external signal into inputs and outputs we get different input/output behaviours. Indeed, the *i/s/o representation* in (1.1) corresponds to the particular *input/output space pair (i/o pair)* $\left(\begin{bmatrix}\{0\}\\ \mathcal{U}\end{bmatrix},\begin{bmatrix}\mathcal{Y}\\ \{0\}\end{bmatrix}\right)$ while (1.2) corresponds to the i/o pair $\left(\begin{bmatrix}\mathcal{Y}\\ \{0\}\end{bmatrix},\begin{bmatrix}\{0\}\\ \mathcal{U}\end{bmatrix}\right)$.

Example 1.2. *Assume for the moment that the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ in* (1.1) *coincide. The operation of choosing the signal $u^\times := (u+y)/\sqrt{2}$ as input and the signal $y^\times := (u-y)/\sqrt{2}$ as output is called the "diagonal transformation" in e.g.* [Sta02b]. *It turns out that the system in* (1.1) *is diagonally transformable if and only if $1 + D$ is invertible.*

Making the diagonal transformation corresponds to decomposing $\mathcal{W}$ into another direct sum $\mathcal{W} = \mathcal{U}^\times \dot{+} \mathcal{Y}^\times$, where $\mathcal{U}^\times = \begin{bmatrix}1_{\mathcal{U}}\\ 1_{\mathcal{U}}\end{bmatrix}\mathcal{U}$ and $\mathcal{Y}^\times = \begin{bmatrix}-1_{\mathcal{Y}}\\ 1_{\mathcal{Y}}\end{bmatrix}\mathcal{U}$. The s/s node $(V;\mathcal{X},\mathcal{W})$ is invariant under the diagonal transformation as well, in a sense which we make precise in Example 6.8.

The state/signal setting is advantageous when one considers interconnection, where the interconnection determines which signals of the interconnected subsystems may act as inputs and which signals are outputs. See e.g. V. Belevitch's classic work [Bel68] on circuit theory. A particularly unrealistic assumption in the i/s/o formulation is that the load on the output has no influence on the modelled system. For an electrical circuit this means that the output impedance of the system is zero or that the load impedance is infinite, which in practice never is the case. The s/s approach is related to the behavioural framework developed for finite-dimensional systems by J. W. Polderman and J. C. Willems in [PW98].

After this general motivation for our approach, let us now describe in more detail what we mean by a state/signal system (s/s system). Let the state space $\mathcal{X}$ and the external signal space $\mathcal{W}$ be finite-dimensional vector spaces. (Later we allow these spaces to be Banach spaces.) Let V be a closed subspace of $\begin{bmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{bmatrix}$ which we call the *generating subspace.* A *classical s/s trajectory* generated by V on the time interval $I \subset \mathbb{R}$ is a pair $\begin{bmatrix}x\\ w\end{bmatrix}$ of functions in $\begin{bmatrix}C^1(I;\mathcal{X})\\ C(I;\mathcal{W})\end{bmatrix}$, which satisfies

$$\begin{bmatrix}\dot{x}(t)\\ x(t)\\ w(t)\end{bmatrix} \in V, \quad t \in I, \tag{1.4}$$

with one-sided derivatives at any end points of I. We denote the space of classical trajectories on I generated by V by $\mathfrak{V}(I)$.

In order for V to generate a reasonable linear system through (1.4), we need to assume that V has some additional technical properties. In the finite-dimensional case $\mathcal{W}$ should have a decomposition $\mathcal{W} = \mathcal{U} \dot{+} \mathcal{Y}$ into an i/o pair $(\mathcal{U},\mathcal{Y})$, such that V generates a unique classical trajectory on $\mathbb{R}^+$ for all given initial states $x(0)$ in $\mathcal{X}$ and all given input signals u in $C(\mathbb{R}^+;\mathcal{U})$. That is, denoting the pointwise projection of $\mathcal{W}$ onto $\mathcal{U}$ along $\mathcal{Y}$ by $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}$, the condition

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t \geq 0,\ x(0) = x_0,\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = u \tag{1.5}$$

should be satisfied by a unique classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\mathfrak{V}(\mathbb{R}^+)$.

We denote the closure of $\mathfrak{V}(\mathbb{R}^+)$ in $\begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ by $\mathfrak{W}^p$ and call its elements the L^p *trajectories generated by* V. By the L^p-well-posed *state/signal system (s/s system)* generated by V we mean the triple $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ obtained in the manner described above.

Thus (1.5) should be thought of as an abstract differential equation and the trajectories as its solutions. In this sense a s/s *node* $(V;\mathcal{X},\mathcal{W})$ is a *static object*, which generates a system by specifying its evolution at any given time t. The *system* $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ *is defined as the set of all trajectories*, which are functions of time, and thus dynamic objects. This idea applies to i/s/o nodes and systems, which we need later in this article, as well.

In this paper we take (1.5) as the starting point instead of (1.1), and we do not at the outset care about whether V can be written in the form (1.3) or not. Our approach is motivated by the input/output invariance of the s/s node $(V;\mathcal{X},\mathcal{W})$, which we demonstrated above. We use well-established notation whenever possible and we refer the reader to the appendix for some definitions and notation.

A theory for infinite-dimensional s/s systems in discrete time is already well under way in a series [AS05], [AS07a], [AS07b], [AS07c] and [AS08] of articles written by D. Z. Arov and the second author. In our current paper we study infinite-dimensional systems in continuous time, letting $\mathcal{X}$ and $\mathcal{W}$ be Banach spaces. The construction above generalises to infinite dimensions, but the formulations become more technical than in the discrete-time and the finite-dimensional cases. Often these difficulties are related to the fact that typical applications in continuous time (partial differential equations) demand that some important operators are unbounded. For example, both in discrete and continuous time we can write V as the graph of some operator S, in a way similar to (1.3). In the discrete-time setting this operator S is bounded, but in the continuous-time setting it may be unbounded.

The class of L^p-well-posed i/s/o systems plays a very central role in this paper. This class has been studied in e.g. [Sal87], [Sal89], [Wei89a], [Wei89b], [Wei89c], [CW89], [Wei94], [WST01], [SW02], [SW04] and many other articles. The book [Sta05] collects most of the background we need on L^p-well-posed i/s/o systems and for simplicity we often cite results from [Sta05]. The reader may consult this source for further references to the original versions of the various results.

Passive systems, i.e., systems that do not have any internal energy sources, are one of the main motivations for our study of s/s systems. Our framework applies particularly well to this important class of systems and we will develop their theory in a future paper. Passive i/s/o systems in continuous time have previously been studied in e.g. [Aro95], [AN96], [Aro99], [WST01], [Sta02a], [Sta02b], [TW03], [MS06], [MS07] and [MSW06].

This paper is structured as follows. In Section 2 we define the notion of a continuous-time well-posed s/s node. The most fundamental properties of s/s systems are studied in Section 3, chiefly using L^p trajectories. In Section 4 we study the admissibility of given i/o pairs for a s/s system and give the corresponding well-posed i/s/o representations. Section 5 is devoted to a short study of i/s/o-system nodes and their relation to the associated i/s/o systems. In Section 6 we prove the existence and uniqueness of a maximal generating subspace of any given s/s system. We end the paper by giving two examples of how the s/s theory can be applied in order to model some systems which are ill-posed in the i/s/o setting.

2. Construction of well-posed state/signal nodes

In this section we introduce well-posed state/signal nodes by taking the abstract differential-equation approach, which we outlined in the introduction. Trajectories and the subspaces V that generate them are thus the main objects to be studied in this section.

Definition 2.1. *Let I be a subinterval of $\mathbb{R}^+$ with positive length, let $\mathcal{X}$ and $\mathcal{W}$ be Banach spaces and let V be a subspace of $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ with the norm*

$$\left\| \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right\|_V = \|z\|_{\mathcal{X}} + \|x\|_{\mathcal{X}} + \|w\|_{\mathcal{W}}. \tag{2.1}$$

By a classical trajectory *generated by V on I we mean a pair $\begin{bmatrix} x \\ w \end{bmatrix}$ in $\begin{bmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{W}) \end{bmatrix}$ that satisfies:*

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad for\ t \in I, \tag{2.2}$$

with one-sided derivatives at any end points of I.

We denote the set of classical trajectories on I by $\mathfrak{V}(I)$. For brevity we write $\mathfrak{V}[a,b] := \mathfrak{V}([a,b])$ and $\mathfrak{V} := \mathfrak{V}[0,\infty)$.

By τ^c we denote the bilateral shift operator, which shifts its argument function to the left by a distance c. The operator which restricts the domain of its argument function to the interval I is denoted by ρ_I. The function $f \bowtie_c g$ coincides with f on the interval $(-\infty, c)$ and with g on $[c,\infty)$. See the appendix for precise definitions of these operators.

Lemma 2.2. *Let I be a subinterval of $\mathbb{R}$. Then the following claims are valid:*

(i) *A pair $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ lies in $\mathfrak{V}(I)$ if and only if $\left[\begin{smallmatrix} \dot x \\ x \\ w \end{smallmatrix}\right] \in C(I;V)$.*

(ii) *For all $-\infty < a < b < \infty$ and $c \in \mathbb{R}$ we have*

$$\mathfrak{V}[a,b] = \tau^c \mathfrak{V}[a+c, b+c] \quad \text{and} \quad \mathfrak{V}[a,\infty) = \tau^c \mathfrak{V}[a+c,\infty).$$

(iii) *For all subintervals I' of I we have*

$$\rho_{I'}\mathfrak{V}(I) \subset \mathfrak{V}(I').$$

(iv) *Let $c \in (a,b)$, $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \in \mathfrak{V}[a,c]$ and $\begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{V}[c,b]$. Then*

$$\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{V}[a,b]$$

if and only if

$$\dot x^1(c) = \dot x^2(c), \quad x^1(c) = x^2(c) \quad \text{and} \quad w^1(c) = w^2(c). \tag{2.3}$$

Proof. (i) Obviously $\begin{bmatrix} \dot x(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for $t \in I$ with $\dot x$, x and w continuous on I if and only if $\begin{bmatrix} \dot x \\ x \\ w \end{bmatrix} \in C(I;V)$, because $\begin{bmatrix} \dot x(t) \\ x(t) \\ w(t) \end{bmatrix} \to \begin{bmatrix} \dot x(t_0) \\ x(t_0) \\ w(t_0) \end{bmatrix}$ in V if and only if $\dot x(t) \to \dot x(t_0)$, $x(t) \to x(t_0)$ in $\mathcal{X}$ and $w(t) \to w(t_0)$ in $\mathcal{W}$, cf. (2.1).

(ii) Trivially e.g. $\tau^c C([a+c,b+c];V) = C([a,b];V)$.

(iii) The restriction to I' of a function in $C(I;V)$ lies in $C(I';V)$.

(iv) If (2.3) holds, then

$$\lim_{t\to c^-} \left(\begin{bmatrix} \dot x^1 \\ x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} \dot x^2 \\ x^2 \\ w^2 \end{bmatrix} \right)(t) = \begin{bmatrix} \dot x^1(c) \\ x^1(c) \\ w^1(c) \end{bmatrix} = \begin{bmatrix} \dot x^2(c) \\ x^2(c) \\ w^2(c) \end{bmatrix}, \tag{2.4}$$

because of continuity of $\begin{bmatrix} \dot x^1 \\ x^1 \\ w^1 \end{bmatrix}$ on $[a,c]$. As $\begin{bmatrix} \dot x^2 \\ x^2 \\ w^2 \end{bmatrix}$ is continuous on $[c,b]$ it is clear that $\begin{bmatrix} \dot x^1 \\ x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} \dot x^2 \\ x^2 \\ w^2 \end{bmatrix}$ is continuous on $[a,b]$.

Conversely, if $\begin{bmatrix} \dot x^1 \\ x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} \dot x^2 \\ x^2 \\ w^2 \end{bmatrix}$ is continuous on $[a,b]$, then (2.4), and therefore (2.3), holds. □

In the following definition we introduce the notion of a s/s node $(V;\mathcal{X},\mathcal{W})$ by adding a number of conditions on the subspace V in Definition 2.1. As we will show in Lemma 2.4 below, the main feature of a s/s node is that its trajectories always can be extended in the forward-time direction.

Definition 2.3. *Let* $\mathcal{X}$ *and* $\mathcal{W}$ *be Banach spaces and let* $V \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$. *We say that* $(V; \mathcal{X}, \mathcal{W})$ *is a* state/signal node (s/s node) *if* V *has the following properties:*

(i) *The space* V *is closed (in the norm* (2.1)*).*

(ii) *The space* V *has the property* $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Longrightarrow z = 0$.

(iii) *There exists some* $T > 0$ *such that*

$$\forall \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V \; \exists \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T] : \quad \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}. \tag{2.5}$$

We remark that property (ii) of Definition 2.3 implies that two classical trajectories $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix}$ and $\begin{bmatrix} x^2 \\ w^2 \end{bmatrix}$ generated by a s/s node can be concatenated at c if and only if $x^1(c) = x^2(c)$ and $w^1(c) = w^2(c)$. Indeed, in this case

$$\begin{bmatrix} \dot{x}^1(c) - \dot{x}^2(c) \\ x^1(c) - x^2(c) \\ w^1(c) - w^2(c) \end{bmatrix} = \begin{bmatrix} \dot{x}^1(c) - \dot{x}^2(c) \\ 0 \\ 0 \end{bmatrix} \in V,$$

which implies that $\dot{x}^1(c) = \dot{x}^2(c)$.

Lemma 2.4. *Condition (iii) of Definition 2.3 holds for some* $T > 0$ *if and only if it holds for all* $T > 0$. *In this case*

$$V = \left\{ \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T] \right\} \quad \text{and} \tag{2.6}$$

$$\forall \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V \; \exists \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V} : \quad \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}. \tag{2.7}$$

Proof. First we show that if (2.5) holds for some $T > 0$ then it also holds for T replaced by any $T' \in (0,T)$. Assume therefore that $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T]$ satisfies $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$. Then $\begin{bmatrix} x' \\ w' \end{bmatrix} := \rho_{[0,T']} \begin{bmatrix} x \\ w \end{bmatrix}$ lies in $\mathfrak{V}[0,T']$, by Lemma 2.2(iii), and moreover $\begin{bmatrix} \dot{x}'(0) \\ x'(0) \\ w'(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$.

We proceed by showing that if Definition 2.3(iii) holds for some $T > 0$ then the same condition also holds for T replaced by $2T$. By assumption, for any $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$ there is a trajectory $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \in \mathfrak{V}[0,T]$ with $\begin{bmatrix} \dot{x}^1(0) \\ x^1(0) \\ w^1(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$. According to Definition 2.1, $\begin{bmatrix} \dot{x}(T) \\ x(T) \\ w(T) \end{bmatrix} \in V$ and by letting $\begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{V}[0,T]$ be such that $\begin{bmatrix} \dot x^2(0) \\ x^2(0) \\ w^2(0) \end{bmatrix} = \begin{bmatrix} \dot x^1(T) \\ x^1(T) \\ w^1(T) \end{bmatrix}$, we obtain from Lemma 2.2 that the function $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_T \left(\tau^{-T} \begin{bmatrix} x^2 \\ w^2 \end{bmatrix}\right)$ is a classical trajectory on $[0, 2T]$, which by construction starts from $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$. By induction we have that Definition 2.3(iii) holds with T replaced by $2^n T$, for any $n \in \mathbb{Z}^+$. Letting $n \to \infty$, we get a function $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$ which satisfies (2.7), cf. Definition A.2(iii).

Now we prove the last claim. By Definition 2.1, any $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ in particular satisfies $\begin{bmatrix} \dot x(0) \\ x(0) \\ w(0) \end{bmatrix} \in V$. Conversely, by (2.6), for any $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$, there exists a classical trajectory $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T]$ with $\begin{bmatrix} \dot x(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$. □

The preceding lemma and its proof shows that for s/s nodes claim (iii) of Lemma 2.2 can be sharpened to

$$\forall b' \in (a,b]: \ \rho_{[a,b']}\mathfrak{V}[a,b] = \mathfrak{V}[a,b'] \quad \text{and} \quad \forall b' > a: \ \rho_{[a,b']}\mathfrak{V}[a,\infty) = \mathfrak{V}[a,b']. \tag{2.8}$$

This is because every trajectory in $\mathfrak{V}[a,b']$ can be extended to a trajectory on $[a,\infty)$, i.e., in addition to Lemma 2.2(iii) we also have $\rho_{[a,b']}\mathfrak{V}[a,\infty) \supset \mathfrak{V}[a,b']$.

Definition 2.5. *The pair* $(\mathcal{U},\mathcal{Y})$ *is a* (direct-sum) decomposition *of the Banach space* $\mathcal{W}$ *if* $\mathcal{U}$ *and* $\mathcal{Y}$ *are closed subspaces of* $\mathcal{W}$ *and* $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, *i.e., every vector in* $\mathcal{W}$ *can be written as the sum of unique elements* $u \in \mathcal{U}$ *and* $y \in \mathcal{Y}$.

The corresponding (bounded) projection onto $\mathcal{U}$ *along* $\mathcal{Y}$ *is denoted* $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}$ *and the complementary projection is* $\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}$. *By this we mean that if* $w = u + y$, *where* $u \in \mathcal{U}$ *and* $y \in \mathcal{Y}$, *then* $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = u$ *and* $\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w = (1 - \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}})w = y$.

We apply $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}$ *to a function* $f \in \mathcal{W}^I$ *pointwise, i.e.* $(\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} f)(t) = \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} f(t)$, $t \in I$.

If $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, then we identify $w = u + y$, $u \in \mathcal{U}$ and $y \in \mathcal{Y}$, with $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ through $\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}(u+y)$ and $u + y = \begin{bmatrix} \mathcal{I}_{\mathcal{Y}} & \mathcal{I}_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$, where $\mathcal{I}_{\mathcal{Y}}$ and $\mathcal{I}_{\mathcal{U}}$ are the injection operators from $\mathcal{Y}$ and $\mathcal{U}$ to $\mathcal{W}$, respectively. In particular, if we have two decompositions $\mathcal{W} = \mathcal{U}_1 \dotplus \mathcal{Y}_1 = \mathcal{U}_2 \dotplus \mathcal{Y}_2$ then we identify

$$\begin{bmatrix} \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1} w \\ \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w \end{bmatrix} = w = \begin{bmatrix} \mathcal{P}_{\mathcal{Y}_2}^{\mathcal{U}_2} w \\ \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} w \end{bmatrix}. \tag{2.9}$$

We have the following standard result.

Lemma 2.6. *The Cartesian product* p*-norm* $\|\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\|_{\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]} = (\|y\|_{\mathcal{Y}}^p + \|u\|_{\mathcal{U}}^p)^{1/p}$ *is equivalent to the norm on* $\mathcal{W}$ *for any* $1 \le p < \infty$ *and any decomposition* $\mathcal{W} = \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$*, i.e. there exists a constant* $k \ge 1$*, which depends on* p*,* $\mathcal{U}$ *and* $\mathcal{Y}$*, such that*

$$\forall w \in \mathcal{W} : \quad \frac{1}{k}(\|\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w\|^p + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|^p)^{1/p} \le \|w\|_{\mathcal{W}} \le k(\|\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w\|^p + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|^p)^{1/p}. \tag{2.10}$$

We now add significant structure to s/s nodes by introducing the concept of well-posedness.

Definition 2.7. *Let* $1 \le p < \infty$*. The s/s node* $(V; \mathcal{X}, \mathcal{W})$ *is* L^p well posed *if there exists a* $T > 0$ *and a direct sum decomposition* $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$*, such that* $\mathfrak{V}[0,T]$ *satisfies the following conditions:*

(i) *The space* $\left\{x(0) \;\middle|\; \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]\right\}$ *is dense in* $\mathcal{X}$*.*

(ii) *The operator* $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ *maps the space*

$$\mathfrak{V}_0[0,T] := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T] \;\middle|\; \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} = 0 \right\} \tag{2.11}$$

densely into $L^p([0,T]; \mathcal{U})$*.*

(iii) *There exists a* $K_T > 0$*, such that all* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ *satisfy*

$$\|x(t)\|_{\mathcal{X}} + \|w\|_{L^p([0,t];\mathcal{W})} \le K_T \left(\|x(0)\|_{\mathcal{X}} + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|_{L^p([0,t];\mathcal{U})}\right), \tag{2.12}$$

for all $t \in [0,T]$*.*

In this case we call $(\mathcal{U}, \mathcal{Y})$ *an* L^p-admissible input/output space pair (admissible i/o pair) *of the* s/s node $(V; \mathcal{X}, \mathcal{W})$*.*

In this work we only consider L^p-admissible i/o pairs, because this is the natural notion of admissibility for L^p-well-posed s/s systems. For other classes of s/s systems, however, admissibility of an i/o pair might mean something else. In the sequel we shortly write "admissible i/o pair". Similarly, we also usually talk about "well-posed systems", meaning "L^p-well-posed systems", because this is the only relevant notion of well-posedness here and the value of p is usually clear from the context.

Remark 2.8. *The defining properties of a discrete-time s/s node in* [AS05, Def. 2.1] *have the following counterparts in the continuous-time setting:*

(i) *The space* V *is closed.*

(ii) *The set*

$$\left\{ x \in \mathcal{X} \mid \exists z \in \mathcal{X}, w \in \mathcal{W} : \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}$$

is a dense subspace of $\mathcal{X}$*.*

(iii) *If* $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$ *then* $z = 0$*.*

Out of these necessary, but not sufficient, conditions, (i) and (iii) are identical to the corresponding discrete-time conditions. In the discrete case the set defined in (ii) is all of $\mathcal{X}$.

Property (iii) implies that the space V can be written as the graph

$$V = \begin{bmatrix} F \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \operatorname{Dom}(F)$$

of some linear operator F. Property (i) says that F is closed. However, its domain

$$\operatorname{Dom}(F) = \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \;\middle|\; \exists z : \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \right\}$$

needs not be closed as in the discrete case and, therefore, F need not be bounded in the continuous case.

The main significance of (2.12) is that the classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ depends continuously on the initial state $x(0)$ and the "input" $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$. This property is the essence of well-posedness in continuous time and it will be heavily exploited in the coming sections. The following technical lemma explains the other two conditions that we impose on well-posed s/s nodes.

Lemma 2.9. *Let $(V;\mathcal{X},\mathcal{W})$ be a s/s node, let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ and let $T > 0$ be such that condition (ii) of Definition 2.7 is satisfied. Then the following claims are true:*

(i) *For all $\varepsilon > 0$, $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$ and $u \in L^p([0,T];\mathcal{U})$, there exists a trajectory $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T]$ with $\begin{bmatrix} \dot x(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$ and $\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w - u\|_{L^p([0,T];\mathcal{U})} < \varepsilon$.*

(ii) *If in addition to condition (ii), condition (i) of Definition 2.7 is also met, then the space*

$$\mathcal{D}^T := \left\{ \begin{bmatrix} x(0) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T] \right\} \tag{2.13}$$

is dense in $\begin{bmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}$.

Proof. (i) By Definition 2.3(iii) and Lemma 2.4 we may let $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \in \mathfrak{V}[0,T]$ be such that $\begin{bmatrix} \dot x^1(0) \\ x^1(0) \\ w^1(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$. Thereafter, by Definition 2.7(ii), we can find an $\begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{V}[0,T]$ such that $x^2(0) = 0$, $w^2(0) = 0$ and

$$\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w^2 - (u - \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w^1)\| < \varepsilon.$$

By Definitions 2.1 and 2.3(ii) we then also have $\dot{x}^2(0) = 0$. Thus the function $\begin{bmatrix} x \\ w \end{bmatrix} := \begin{bmatrix} x^1 + x^2 \\ w^1 + w^2 \end{bmatrix}$ lies in $\mathfrak{V}[0,T]$ and satisfies $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$ and $\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w - u\| < \varepsilon$.

(ii) Fix $\varepsilon > 0$, $x_0 \in \mathcal{X}$ and $u \in L^p([0,T];\mathcal{U})$. If condition (i) of Definition 2.7 is met, then we can find a classical trajectory $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$, which satisfies $\|\widetilde{x}(0) - x_0\| < \varepsilon/2$. Moreover, $\left[\begin{smallmatrix} \dot{\widetilde{x}}(0) \\ \widetilde{x}(0) \\ \widetilde{w}(0) \end{smallmatrix}\right] \in V$, and by the first part of this lemma there then exists a classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ with $x(0) = \widetilde{x}(0)$ and $\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w - u\| < \varepsilon/2$. This trajectory satisfies $\left\| \begin{bmatrix} x(0) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} - \begin{bmatrix} x_0 \\ u \end{bmatrix} \right\| < \varepsilon.$ □

We now prove the important fact that the conditions in Definition 2.7, and therefore also the claims in Lemma 2.9, are independent of $T > 0$.

Lemma 2.10. *Assume that $(V;\mathcal{X},\mathcal{W})$ is a s/s node. Any of the claims (i) – (iii) in Definition 2.7 is valid for some $T > 0$ if and only if the respective claim is valid for all $T > 0$.*

Proof. Again, if one of the conditions (ii) or (iii) holds for some $T > 0$ then it is easy to see that it holds also for T replaced by any $T' \in (0,T)$. We show that if claim (ii) or (iii) is valid for some $T > 0$ then it is valid for T replaced by $2T$, cf. the proof of Lemma 2.4.

(i) Lemma 2.4 yields that we independently of $T > 0$ have

$$\left\{ x(0) \mid \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T] \right\} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} V.$$

(ii) Let $\varepsilon > 0$ and $u_0 \in L^p([0,2T];\mathcal{U})$ be arbitrary. By assumption we can find a trajectory $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \in \mathfrak{V}[0,T]$, such that $x^1(0) = 0$, $w^1(0) = 0$ and

$$\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w^1 - \rho_{[0,T]} u_0\|_{L^p([0,T];\mathcal{U})} < \varepsilon/2.$$

In particular $\begin{bmatrix} \dot{x}^1(T) \\ x^1(T) \\ w^1(T) \end{bmatrix} \in V$ and by Lemma 2.9(i) there exists a trajectory $\begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{V}[0,T]$, such that $\dot{x}^2(0) = \dot{x}^1(T)$, $x^2(0) = x^1(T)$, $w^2(0) = w^1(T)$ and

$$\left\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w^2 - \rho_{[0,T]} \tau^T u_0\right\|_{L^p([0,T];\mathcal{U})} < \varepsilon/2.$$

In this way we obtain that

$$\begin{bmatrix} x \\ w \end{bmatrix} := \begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_T \tau^{-T} \begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{V}[0,2T],$$

by Lemma 2.2, and $x(0) = x^1(0) = 0$, $w(0) = 0$ and

$$\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w - u_0\|_{L^p([0,2T];\mathcal{U})} < \varepsilon.$$

(iii) We assume that (2.12) is true for $t \in [0,T]$. Thus we may without loss of generality take $t \in [T,2T]$ and $K_T \geq 1$. Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,2T]$ be arbitrary and note that $\rho_{[0,T]}\tau^T \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ by Lemma 2.2. Writing $u = \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ and making heavy use of $1 \leq K_T \leq K_T^2$ we obtain:

$$\begin{aligned}
\|x(t)\|_{\mathcal{X}} + \|w\|_{L^p([0,t];\mathcal{W})} &\leq \|w\|_{L^p([0,T];\mathcal{W})} + \|x(t)\|_{\mathcal{X}} + \|w\|_{L^p([T,t];\mathcal{W})} \\
&\leq \|w\|_{L^p([0,T];\mathcal{W})} + K_T \left(\|x(T)\|_{\mathcal{X}} + \|u\|_{L^p([T,t];\mathcal{U})}\right) \\
&\leq K_T \left(\|x(0)\|_{\mathcal{X}} + \|u\|_{L^p([0,T];\mathcal{U})}\right) \\
&\qquad + K_T^2 \left(\|x(0)\|_{\mathcal{X}} + \|u\|_{L^p([0,T];\mathcal{U})}\right) + K_T \|u\|_{L^p([T,t];\mathcal{U})} \\
&\leq 2K_T^2 \left(\|x(0)\|_{\mathcal{X}} + \|u\|_{L^p([0,T];\mathcal{U})}\right) + 2K_T^2 \|u\|_{L^p([T,t];\mathcal{U})} \\
&\leq 2K_T^2 \left(\|x(0)\|_{\mathcal{X}} + \|u\|_{L^p([0,t];\mathcal{U})}\right).
\end{aligned}$$

□

The following proposition requires the operator δ_a, which evaluates its argument function at a, and the space $L^p_{loc}(\mathbb{R}^+;\mathcal{U})$ of functions that locally lie in L^p. See Definitions A.1 and A.3 in the appendix for more details.

Lemma 2.11. *If $(V;\mathcal{X},\mathcal{W})$ is an L^p-well-posed s/s node then the space*

$$\mathcal{D}^+ := \left\{ \begin{bmatrix} x(0) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V} \right\} \tag{2.14}$$

is dense in $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}(\mathbb{R}^+;\mathcal{U}) \end{bmatrix}$.

Proof. Let $x_0 \in \mathcal{X}$ and $u \in L^p_{loc}(\mathbb{R}^+;\mathcal{U})$ be arbitrary. We construct a sequence $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}$, such that $\begin{bmatrix} x_n(0) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \end{bmatrix} \to \begin{bmatrix} x_0 \\ u \end{bmatrix}$. By Lemma 2.9(ii) and Lemma 2.10 there for all $n \geq 1$ exists a pair $\begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix} \in \mathfrak{V}[0,n]$, such that

$$\forall n \geq 1 : \quad \left\| \begin{bmatrix} \widetilde{x}_n(0) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w}_n \end{bmatrix} - \begin{bmatrix} x_0 \\ \rho_{[0,n]} u \end{bmatrix} \right\| < 1/n.$$

Moreover, according to (2.8) there for all $n \geq 1$ exist $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}$, such that $\begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix} = \rho_{[0,n]} \begin{bmatrix} x_n \\ w_n \end{bmatrix}$. Now

$$\|x_n(0) - x_0\| = \|\widetilde{x}_n(0) - x_0\| < 1/n \to 0 \quad \text{and}$$

$$\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n - u\|_n = \|\rho_{[0,n]}(\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n - u)\|_{L^p([0,n];\mathcal{U})} = \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w}_n - \rho_{[0,n]} u\|_{L^p([0,n];\mathcal{U})} \to 0$$

for all seminorms $\|\cdot\|_n$ on $L^p_{loc}(\mathbb{R}^+;\mathcal{U})$, cf. Definition A.3(ii). This implies that the sequence $\begin{bmatrix} x_n(0) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \end{bmatrix}$ in $\mathcal{D}^+$ tends to $\begin{bmatrix} x_0 \\ u \end{bmatrix}$ in $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}(\mathbb{R}^+;\mathcal{U}) \end{bmatrix}$. □

It is now time to proceed to the next section, where we are finally able to define the notion of a well-posed state/signal system.

3. Well-posed state/signal systems

In the study of well-posed input/state/output systems the state trajectory is only required to be continuous and the external signals are allowed to belong to $L^p_{loc}([a,\infty);\mathcal{W})$, see e.g. [Sta05]. We now extend the space of trajectories of s/s systems in order to include trajectories of this type.

Recall that we for bounded $[a,b]$ have $L^p_{loc}([a,b];\mathcal{W}) = L^p([a,b];\mathcal{W})$ and that $x_n \to x$ in $C([a,\infty);\mathcal{X})$ if and only if $\rho_{[a,b]} x_n \to \rho_{[a,b]} x$ uniformly for all bounded subintervals $[a,b]$ of $[a,\infty)$. See Definitions A.2 and A.3 for more details.

Definition 3.1. *Let $\mathcal{X}$ and $\mathcal{W}$ be Banach spaces, let I be a subinterval of $\mathbb{R}$ and let V be a subspace of $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ with the norm (2.1).*

The pair $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C(I;\mathcal{X}) \\ L^p_{loc}(I;\mathcal{W}) \end{bmatrix}$ is an L^p trajectory *on I generated by V if there exists a sequence $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}(I)$ such that $x_n \to x$ in $C(I;\mathcal{X})$ and $w_n \to w$ in $L^p_{loc}(I;\mathcal{W})$. We denote the space of L^p trajectories on I by $\mathfrak{W}^p(I)$, again abbreviating $\mathfrak{W}^p[a,b] := \mathfrak{W}^p([a,b])$ and $\mathfrak{W}^p := \mathfrak{W}^p[a,\infty)$.*

Definition 3.1 says that $\mathfrak{W}^p(I)$ is the closure of $\mathfrak{V}(I)$ in $\begin{bmatrix} C(I;\mathcal{X}) \\ L^p_{loc}(I;\mathcal{W}) \end{bmatrix}$. Thus, in spite of their name, the external signal part of the L^p trajectories on $[a,\infty)$ do not lie globally in $L^p([a,\infty);\mathcal{W})$, but only locally. From now on we mainly use L^p trajectories and for brevity we assume that all trajectories are of L^p type except when we explicitly mention that a given trajectory is classical.

In the terminology of [Paz83], the classical trajectories generated by V correspond to classical solutions of the inhomogeneous Cauchy-type problem $\left[\begin{smallmatrix} \dot{x} \\ x \\ w \end{smallmatrix}\right] \in V$, whereas L^p trajectories closely resemble the corresponding mild solutions. Most of the auxiliary results cited in this section are found in [Paz83].

The following corollary to Definition 3.1 is the L^p-trajectory analogue of Lemma 2.2.

Corollary 3.2. *For all subintervals I of $\mathbb{R}$, the spaces $\mathfrak{W}^p(I)$ satisfy:*

(i) *For all $c \in \mathbb{R}$, $\mathfrak{W}^p[a,b] = \tau^c \mathfrak{W}^p[a+c, b+c]$ and $\mathfrak{W}^p[a,\infty) = \tau^c \mathfrak{W}^p[a+c,\infty)$.*

(ii) *For all subintervals I' of I:*

$$\rho_{I'} \mathfrak{W}^p(I) \subset \mathfrak{W}^p(I'). \tag{3.1}$$

(iii) *The space $\mathfrak{W}^p$ of trajectories on $\mathbb{R}^+$ is invariant under left shift on $\mathbb{R}^+$, i.e., for all $t \geq 0$ we have $\rho_+ \tau^t \mathfrak{W}^p \subset \mathfrak{W}^p$.*

Proof. (i) Let $\begin{bmatrix} x \\ w \end{bmatrix}$ be a trajectory on $I + c$ with $\begin{bmatrix} x_n \\ w_n \end{bmatrix}$ a sequence of classical trajectories approximating it. Then $\tau^c \begin{bmatrix} x_n \\ w_n \end{bmatrix}$ is a sequence of classical trajectories on I, converging to $\tau^c \begin{bmatrix} x \\ w \end{bmatrix}$ in $\begin{bmatrix} C(I;\mathcal{X}) \\ L^p_{loc}(I;\mathcal{W}) \end{bmatrix}$. By Definition 3.1, $\tau^c \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a trajectory on I.

(ii) If $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p(I)$ then, by Definition 3.1, there exist $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}(I)$ such that $x_n \to x$ uniformly on bounded intervals I and $w_n \to w$ in $L^p_{loc}(I;\mathcal{W})$. By Lemma 2.2(iii), $\rho_{I'} \begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}(I')$ and of course $\rho_{I'} x_n \to \rho_{I'} x$ uniformly on bounded intervals and $\rho_{I'} w_n \to \rho_{I'} w$ in $L^p_{loc}(I';\mathcal{W})$. This shows that $\rho_{I'} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an element of $\mathfrak{W}^p(I')$, i.e., that $\rho_{I'} \mathfrak{W}^p(I) \subset \mathfrak{W}^p(I')$.

(iii) By claim (i) we have $\tau^t \mathfrak{W}^p = \mathfrak{W}^p[-t,\infty)$ and then $\rho_+ \tau^t \mathfrak{W}^p \subset \mathfrak{W}^p$, according to claim (ii). □

We are now ready to define an L^p-well-posed s/s system.

Definition 3.3. *Let the s/s node $(V;\mathcal{X},\mathcal{W})$ be L^p-well posed with trajectories $\mathfrak{W}^p$.*

*The triple $\Sigma_{s/s} = (\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ is called the L^p-*well-posed state/signal system (well-posed s/s system) *on $(\mathcal{X},\mathcal{W})$* generated by $(V;\mathcal{X},\mathcal{W})$.

Any (not a priori well-posed) s/s node $(V';\mathcal{X},\mathcal{W})$, whose classical trajectories on some positive-length interval $[0,T]$ form a dense subspace of $\rho_{[0,T]}\mathfrak{W}^p$, is said to generate $\Sigma_{s/s} = (\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ *and V' is then called a* generating subspace *of* Σ.

An i/o pair $(\mathcal{U},\mathcal{Y})$ is admissible for the system Σ *if it is admissible for* some *of its generating s/s nodes $(V;\mathcal{X},\mathcal{W})$.*

We do not even in the well-posed case exclude the possibility that several s/s nodes generate the same s/s system. In the next few lemmas, we study the implications of the properties that we demand of a well-posed s/s node in Definition 2.7.

Lemma 3.4. *Let $1 \le p < \infty$ and $I = [a,b]$ or $I = [a,\infty)$, where $-\infty < a < b < \infty$. The following claims are true:*

(i) *The operator* $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ *maps the space* $\begin{bmatrix} C(I;\mathcal{X}) \\ L^p_{loc}(I;\mathcal{W}) \end{bmatrix}$ *continuously into the space* $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}(I;\mathcal{U}) \end{bmatrix}$.

(ii) *If the restriction of* $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ *to some closed* $W \subset \begin{bmatrix} C(I;\mathcal{X}) \\ L^p_{loc}(I;\mathcal{W}) \end{bmatrix}$ *is injective with closed range, then* $\widetilde{\mathfrak{T}} := \begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}\Big|_W^{-1}$ *is continuous.*

(iii) *If $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ is an L^p- well-posed s/s system with admissible i/o pair $(\mathcal{U},\mathcal{Y})$, then* $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ *maps $\mathfrak{W}^p(I)$ one-to-one onto* $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}(I;\mathcal{U}) \end{bmatrix}$ *and*

$$\mathfrak{T}_a^b := \begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}\Big|_{\mathfrak{W}^p[a,b]}^{-1} \quad \text{and} \quad \mathfrak{T}_a := \begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}\Big|_{\mathfrak{W}^p[a,\infty)}^{-1} \tag{3.2}$$

are both continuous.

Proof. (i) It suffices to prove that $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ is continuous at zero. Letting $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \begin{bmatrix} C(I;\mathcal{X}) \\ L^p_{loc}(I;\mathcal{W}) \end{bmatrix}$, we for all $b > a$ get:

$$\begin{aligned} \left\| \begin{bmatrix} x_n(a) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \end{bmatrix} \right\|_{\left[\begin{smallmatrix} \mathcal{X} \\ L^p([a,b];\mathcal{U}) \end{smallmatrix}\right]} &\le \sup_{t\in[a,b]} \|x_n(t)\|_{\mathcal{X}} + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}\| \|w_n\|_{L^p([a,b];\mathcal{W})} \\ &\le (1 + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}\|) \left\| \begin{bmatrix} x_n \\ w_n \end{bmatrix} \right\|_{\left[\begin{smallmatrix} C([a,b];\mathcal{X}) \\ L^p([a,b];\mathcal{W}) \end{smallmatrix}\right]}. \end{aligned}$$

Thus, if $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \to 0$, then $\begin{bmatrix} x_n(a) \\ \rho_{[a,b]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \end{bmatrix} \to 0$ for all $b > a$, which by Definition A.3(ii) implies that $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \to 0$.

(ii) The given assumptions and claim (i) yield that $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}\Big|_W$ is continuous with a closed domain, i.e., the restriction is a closed operator. Then also the inverse $\widetilde{\mathfrak{T}}$ is a closed operator, whose domain is a closed subspace of a Fréchet space. This implies that $\mathrm{Dom}\left(\widetilde{\mathfrak{T}}\right)$ is a Fréchet space and $\widetilde{\mathfrak{T}}$ is then continuous by the closed graph theorem.

(iii) Assume that $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ is well-posed with admissible i/o pair $(\mathcal{U},\mathcal{Y})$. We first show that the restriction of $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ to $\mathfrak{V}$ is injective and that the operator $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}\Big|_{\mathfrak{V}}^{-1}$ is continuous.

Let $\begin{bmatrix} \xi_n \\ u_n \end{bmatrix} \in \mathcal{D}^+$. Since $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}$ maps $\mathfrak{V}$ onto $\mathcal{D}^+$ defined in (2.14), there for every $n \geq 1$ exists an $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}$ such that $x_n(0) = \xi_n$ and $\mathcal{P}_\mathcal{U}^\mathcal{Y} w_n = u_n$. Then $\rho_{[0,T]} \begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}[0,T]$ by Lemma 2.2(iii) for all $T > 0$ and therefore, according to (2.12):

$$\|x_n\|_{C([0,T];\mathcal{X})} + \|w_n\|_{L^p([0,T];\mathcal{W})} \leq K_T(\|\xi_n\|_\mathcal{X} + \|u_n\|_{L^p([0,T];\mathcal{U})}). \tag{3.3}$$

This proves that if $\begin{bmatrix} \xi_n \\ u_n \end{bmatrix} = \begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \begin{bmatrix} x_n \\ w_n \end{bmatrix} = 0$ and $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}$, then $\begin{bmatrix} x_n \\ w_n \end{bmatrix} = 0$, i.e., that $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}\Big|_\mathfrak{V}$ is injective. If $\begin{bmatrix} \xi_n \\ u_n \end{bmatrix} \in \mathcal{D}^+$ tends to zero, then

$$\begin{bmatrix} x_n \\ w_n \end{bmatrix} := \begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}\Big|_\mathfrak{V}^{-1} \begin{bmatrix} \xi_n \\ u_n \end{bmatrix} \to 0,$$

because $\rho_{[0,T]} u_n \to 0$ for all $T > 0$. By (3.3) this implies that $\rho_{[0.T]} \begin{bmatrix} x_n \\ w_n \end{bmatrix}$ tends to zero for all $T > 0$, i.e., that $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \to 0$, cf. Definitions A.2(iii) and A.3(ii). This finishes the proof that $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}\Big|_\mathfrak{V}$ has a continuous inverse.

By Lemma 2.11, $\mathcal{D}^+$ is dense in $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}(\mathbb{R}^+;\mathcal{U}) \end{bmatrix}$ and thus the operator $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}\Big|_\mathfrak{V}^{-1}$ can be uniquely extended by continuity to an operator $\mathfrak{T}_0$, which maps the closure $\overline{\mathcal{D}^+} = \begin{bmatrix} \mathcal{X} \\ L^p_{loc}(\mathbb{R}^+;\mathcal{U}) \end{bmatrix}$ of $\mathcal{D}^+$ one-to-one onto $\overline{\mathfrak{V}}$. Definition 3.1 says that $\overline{\mathfrak{V}} = \mathfrak{W}^p$.

An analogous, but slightly simpler, argument shows that the restriction of $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}$ to $\mathfrak{V}[0, b-a]$ is injective. The inverse of this restriction can be extended to a continuous operator $\mathfrak{T}_0^{b-a}$, which maps $\overline{\mathcal{D}^{b-a}}$ one-to-one onto $\overline{\mathfrak{V}[0,b-a]} = \mathfrak{W}^p[0,b-a]$. According to Lemma 2.9(ii), $\overline{\mathcal{D}^{b-a}} = \begin{bmatrix} \mathcal{X} \\ L^p([0,b-a];\mathcal{U}) \end{bmatrix}$.

For the intervals I with left end point a we now get that

$$\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \mathfrak{W}^p(I) = \tau^{-a} \begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \tau^a \mathfrak{W}^p(I),$$

which in combination with Corollary 3.2(i) proves that $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}(I)$ one-to-one onto $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}(I;\mathcal{U}) \end{bmatrix}$. Continuity of $\mathfrak{T}_a^b$ and $\mathfrak{T}_a$ follows from claim (ii) and the fact that all the spaces $\mathfrak{W}^p[a,b]$, $\begin{bmatrix} \mathcal{X} \\ L^p([a,b];\mathcal{U}) \end{bmatrix}$, $\mathfrak{W}^p[a,\infty)$ and $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}([a,\infty);\mathcal{U}) \end{bmatrix}$ are Fréchet spaces. □

Let $-\infty < a < b < \infty$ and let $(V;\mathcal{X},\mathcal{W})$ be a s/s node. Define

$$\mathfrak{W}^p_0[a,b] := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[a,b] \;\middle|\; x(a) = 0 \right\} \tag{3.4}$$

and note that the space $\mathfrak{V}_0[0,T]$, which was defined in (2.11), is subspace of $\mathfrak{W}^p_0[0,T]$. The trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\mathfrak{W}^p_0[0,T]$ are said to be *externally generated*, because they are completely determined by the (external) input signal $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$.

Lemma 3.5. *Assume that* $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ *maps* $\mathfrak{W}^p_0[0,T]$ *one-to-one onto* $L^p([0,T];\mathcal{U})$*. Then Definition 2.7(ii) holds if and only if* $\mathfrak{V}_0[0,T]$ *is dense in* $\mathfrak{W}^p_0[0,T]$*.*

Proof. We first show that $\mathfrak{W}^p_0[0,T]$ is a closed subspace of the Banach space $\mathfrak{W}^p[0,T]$. Obviously,

$$\overline{\mathfrak{W}^p_0[0,T]} \subset \overline{\mathfrak{W}^p[0,T]} = \mathfrak{W}^p[0,T]$$

by (3.4) and Definition 3.1, respectively. Let $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{W}^p_0[0,T]$ and let $\begin{bmatrix} x_n \\ w_n \end{bmatrix}$ tend to $\begin{bmatrix} x \\ w \end{bmatrix}$ in $\begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$. Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[0,T]$, $x_n \to x$ uniformly and thus $x(0) = \lim_{n\to\infty} x_n(0) = 0$.

It is clear that $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}^p_0[0,T]$ one-to-one onto $L^p([0,T];\mathcal{U})$ if and only if $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}^p_0[0,T]$ one-to-one onto $\begin{bmatrix} \{0\} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}$. Moreover, it is easy to see that $\mathfrak{W}^p_0[0,T]$ inherits closedness from $\mathfrak{W}^p[0,T]$. Lemma 3.4 then yields that the restriction of $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ to $\mathfrak{W}^p_0[0,T]$ is continuous with a continuous inverse, which by assumption is defined on all of the Banach space $L^p([0,T];\mathcal{U})$.

Let $u \in L^p([0,T];\mathcal{U})$ and define an element of $\mathfrak{W}^p_0[0,T]$ by

$$\begin{bmatrix} x \\ w \end{bmatrix} := \left(\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix} \big|_{\mathfrak{W}^p_0[0,T]} \right)^{-1} u.$$

If $\mathfrak{V}_0$ is dense in $\mathfrak{W}^p_0$, then there exists a sequence $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}_0[0,T]$ that converges to $\begin{bmatrix} x \\ w \end{bmatrix}$ in $\begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$. Obviously $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} x_n \\ w_n \end{bmatrix} \to u$ in $L^p([0,T;\mathcal{U})$, which proves that Definition 2.7(ii) holds.

Conversely, if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p[0,T]$, then

$$\begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix} \big|_{\mathfrak{W}_0^p[0,T]} \right)^{-1} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w.$$

If Definition 2.7(ii) holds, then there exists a sequence $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}_0[0,T]$, such that $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \to \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$. Then also

$$\begin{bmatrix} x_n \\ w_n \end{bmatrix} = \left(\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix} \big|_{\mathfrak{W}_0^p[0,T]} \right)^{-1} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n$$

and by the continuity of $\left(\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix} \big|_{\mathfrak{W}_0^p[0,T]} \right)^{-1}$, we have that $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \to \begin{bmatrix} x \\ w \end{bmatrix}$ in $\mathfrak{W}_0^p[0,T]$. This proves that $\mathfrak{V}_0$ is dense in $\mathfrak{W}_0^p$. □

Let f be a function and $I \subset \mathrm{Dom}\,(f)$. In the following lemma we use the notation π_I for the operator which first restricts its argument function f to I and then extends the restriction by zero to all of $\mathbb{R}$, see Definition A.1. The lemma further illustrates the importance of bijectivity of the restriction of $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ to $\mathfrak{W}_0^p[0,T]$. We shall soon see that this bijectivity is the key to characterising the admissible i/o pairs of well-posed s/s systems.

Lemma 3.6. *Let $(V;\mathcal{X},\mathcal{W})$ be a s/s node and let $T>0$. If $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0^p[0,T]$, then $\mathfrak{W}_0^p[0,T]$ is invariant under right shift with zero padding:*

$$\forall t \ge 0: \quad \rho_{[0,T]}\tau^{-t}\pi_{[0,T]}\mathfrak{W}_0^p[0,T] \subset \mathfrak{W}_0^p[0,T]. \tag{3.5}$$

If (3.5) holds and the operator $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps the space $\mathfrak{W}_0^p[0,T]$ one-to-one onto $L^p([0,T];\mathcal{U})$, then the space $\mathfrak{W}_0^p[0,T]$ has the property:

$$\forall t \in [0,T], \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}_0^p[0,T]: \quad \rho_{[0,t]}\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0 \implies \rho_{[0,t]} \begin{bmatrix} x \\ w \end{bmatrix} = 0. \tag{3.6}$$

Proof. Let $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}_0^p[0,T]$ and let $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}_0[0,T]$ tend to $\begin{bmatrix} x \\ w \end{bmatrix}$. Then

$$\begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix} := \rho_{[0,T]}\tau^{-t}\pi_{[0,T]} \begin{bmatrix} x_n \\ w_n \end{bmatrix}$$

lies in $\mathfrak{V}_0[0,T]$, as we now show.

Lemma 2.2(ii) yields that $\tau^{-t} \begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}[t,T+t]$ with $x_n(t)=0$ and $w_n(t)=0$. This implies that $0 \bowtie_t \tau^{-t} \begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}[0,T+t]$ if 0 is the zero trajectory in $\mathfrak{V}[0,t]$, according to Lemma 2.2(iv) and Definition 2.3(ii). From Lemma 2.2(iii) we now get that

$$\begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix} = \rho_{[0,T]} \left(0 \bowtie_t \tau^{-t} \begin{bmatrix} x_n \\ w_n \end{bmatrix} \right)$$

lies in $\mathfrak{V}[0,T]$. Moreover, $\widetilde{x}_n(0)=0$ and $\widetilde{w}_n(0)=0$ by construction.

Obviously $\begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix}$ tends to

$$\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} := \rho_{[0,T]}\tau^{-t}\pi_{[0,T]}\begin{bmatrix} x \\ w \end{bmatrix} \quad \text{in} \quad \begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix},$$

which shows that $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ is an element of $\mathfrak{W}^p[0,T]$, cf. Definition 3.1. Moreover, $\widetilde{x}(0)=0$, which by (3.4) yields that $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \mathfrak{W}^p_0[0,T]$, and we have proved (3.5).

In order to prove (3.6), we suppose that $\begin{bmatrix} 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}$ maps $\mathfrak{W}^p_0[0,T]$ one-to-one onto $L^p([0,T];\mathcal{U})$ and that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p_0[0,T]$ satisfies $\rho_{[0,t]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w = 0$ for some $t \in [0,T]$. Then we have that

$$\rho_{[0,T]}\tau^{-t}\pi_{[0,T]}\tau^{t}\pi_{[0,T]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w = \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w, \tag{3.7}$$

because

$$\begin{aligned}\rho_{[0,T]}\tau^{-t}\pi_{[0,T]}\tau^{t}\pi_{[0,T]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w - \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w &= \rho_{[0,T]}\pi_{[t,T+t]}\pi_{[0,T]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w - \rho_{[0,T]}\pi_{[0,T]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w \\ &= \rho_{[0,T]}(\pi_{[t,T]} - \pi_{[0,T]})\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w = -\rho_{[0,T]}\pi_{[0,t]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w = 0.\end{aligned}$$

By the surjectivity of $\begin{bmatrix} 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}$ there exists a trajectory $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right] \in \mathfrak{W}^p_0[0,T]$, such that

$$\begin{bmatrix} 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}\begin{bmatrix} \widehat{x} \\ \widehat{w} \end{bmatrix} = \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\widehat{w} = \rho_{[0,T]}\tau^{t}\pi_{[0,T]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w. \tag{3.8}$$

From (3.5) we get that also the right-shifted trajectory

$$\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} := \rho_{[0,T]}\tau^{-t}\pi_{[0,T]}\begin{bmatrix} \widehat{x} \\ \widehat{w} \end{bmatrix}$$

belongs to $\mathfrak{W}^p_0[0,T]$. Combining (3.7) and (3.8) we get that

$$\begin{aligned}\begin{bmatrix} 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} &= \rho_{[0,T]}\tau^{-t}\pi_{[0,T]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\widehat{w} \\ &= \rho_{[0,T]}\tau^{-t}\pi_{[0,T]}\rho_{[0,T]}\tau^{t}\pi_{[0,T]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w \\ &= \begin{bmatrix} 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}\begin{bmatrix} x \\ w \end{bmatrix},\end{aligned}$$

recalling that $\pi_{[0,T]}\rho_{[0,T]} = \pi_{[0,T]}$ on $L^p_{c,loc}(\mathbb{R};\mathcal{U})$. By the injectivity of the restriction of $\begin{bmatrix} 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}$ to $\mathfrak{W}^p_0[0,T]$ we have $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ and $\rho_{[0,T]}\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] = 0$ then implies $\rho_{[0,T]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = 0$. □

The property (3.6) implies *causality*, because it says that future input does not influence past values of the trajectories. We now return to well-posed s/s systems and collect our most important findings so far in the following proposition.

Proposition 3.7. *Let $-\infty < a < b < \infty$ and let $\Sigma_{s/s} = (\mathfrak{W}^p; \mathcal{X}, \mathcal{W})$ be a well-posed s/s system with admissible i/o pair $(\mathcal{U}, \mathcal{Y})$.*

(i) *For all $x_a \in \mathcal{X}$ and $u \in L^p([a,b];\mathcal{U})$ there exists a unique $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$, such that $x(a) = x_a$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = u$ (almost everywhere).*

(ii) *For all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$ and $t \in [a,b]$ we have*

$$\|x(t)\|_{\mathcal{X}} + \|w\|_{L^p([a,t];\mathcal{W})} \le K_{b-a} \left(\|x(a)\|_{\mathcal{X}} + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|_{L^p([a,t];\mathcal{U})} \right), \tag{3.9}$$

where K_{b-a} is the constant K_T in (2.12) with $T = b-a$.

(iii) *For all $x_a \in \mathcal{X}$ and $u \in L^p_{loc}([a,\infty);\mathcal{U})$ there is a unique $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,\infty)$, such that $x(a) = x_a$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = u$ (almost everywhere).*

(iv) *Let $\mathfrak{V}_0[a,b]$ be given by (2.11) for any well-posed s/s node, which generates Σ. Then $\mathfrak{V}_0[a,b]$ is dense in the space $\mathfrak{W}_0^p[a,b]$ defined in (3.4).*

(v) *The operator $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}_0^p[a,b]$ one-to-one onto $L^p([a,b];\mathcal{U})$.*

Proof. We first prove claim (ii). Let therefore $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$ and let $\begin{bmatrix} x_n \\ w_n \end{bmatrix}$ be a sequence in $\mathfrak{V}[a,b]$, which tends to $\begin{bmatrix} x \\ w \end{bmatrix}$. Then every $\begin{bmatrix} x_n \\ w_n \end{bmatrix}$ satisfies

$$\|x_n(t)\|_{\mathcal{X}} + \|w_n\|_{L^p([a,t];\mathcal{W})} \le K_{b-a} \left(\|x_n(a)\|_{\mathcal{X}} + \|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n\|_{L^p([a,t];\mathcal{U})} \right)$$

for all $t \in [a,b]$ by a combination of Lemma 2.2(ii) and Definition 2.7(iii). Letting $n \to \infty$, we obtain (3.9)

According to Lemma 3.4, the operator $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}^p[a,b]$ one-to-one onto $\begin{bmatrix} \mathcal{X} \\ L^p([a,b];\mathcal{U}) \end{bmatrix}$ and $\mathfrak{W}^p[a,\infty)$ one-to-one onto $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}([a,\infty);\mathcal{U}) \end{bmatrix}$. This implies claims (i) and (iii). In particular, $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}_0^p[0,T]$ one-to-one onto $\begin{bmatrix} \{0\} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}$. Thus claim (v) is valid and then claim (iv) follows from Lemma 3.5. □

The following proposition shows that the L^p trajectories of a well-posed s/s system can be extended with great flexibility. This is, together with property (i) of Proposition 3.7, one of the main advantages of using L^p trajectories instead of classical trajectories. Compare the following proposition to Lemma 2.2(iv) and Lemma 2.11, which are the corresponding results for classical trajectories.

Proposition 3.8. *Let $c \in (a,b)$ and let $\Sigma_{s/s} = (\mathfrak{W}^p; \mathcal{X}, \mathcal{W})$ be a well-posed s/s system. Let $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \in \mathfrak{W}^p[a,c]$, $\begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{W}^p[c,b]$ and $\begin{bmatrix} x^3 \\ w^3 \end{bmatrix} \in \mathfrak{W}^p[c,\infty)$.*

Then the following claims are true:

(i) *The concatenation* $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} x^2 \\ w^2 \end{bmatrix}$ *is an element of* $\mathfrak{W}^p[a,b]$ *if and only if* $x^1(c) = x^2(c)$. *Moreover,* $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} x^3 \\ w^3 \end{bmatrix} \in \mathfrak{W}^p[a,\infty)$ *if and only if* $x^1(c) = x^3(c)$.

(ii) *If* $(\mathcal{U},\mathcal{Y})$ *is an admissible i/o pair of* Σ*, then for every* $u \in L^p_{loc}([c,\infty);\mathcal{U})$*, the trajectory* $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix}$ *on* $[a,c]$ *can be extended to a trajectory* $\begin{bmatrix} x \\ w \end{bmatrix}$ *on* $[a,\infty)$ *such that* $\rho_{[c,\infty)}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w = u$.

Proof. Assume that $(\mathcal{U},\mathcal{Y})$ is an admissible i/o pair of Σ.

(i) If $x^1(c) \neq x^2(c)$ then the concatenation $x^1 \bowtie_c x^2$ is discontinuous at c and it cannot be a state trajectory on $[a,b]$ by Definition 3.1. Therefore we now assume that $x^1(c) = x^2(c)$.

According to Proposition 3.7 there exists a unique trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ on $[a,b]$, such that $x(a) = x^1(a)$ and $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w = \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}(w^1 \bowtie_c w^2)$. This trajectory satisfies $x(c) = x^1(c) = x^2(c)$ and $\rho_{[c,b]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w = \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w^2$. Since we have $\rho_{[c,b]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[c,b]$ by Corollary 3.2, we also have $\rho_{[c,b]}\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x^2 \\ w^2 \end{bmatrix}$ by uniqueness of trajectories. This proves that $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{W}^p[a,b]$.

If $x^1(c) = x^3(c)$ then $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} x^3 \\ w^3 \end{bmatrix}$ can be proved to be an element of $\mathfrak{W}^p[a,\infty)$ by considering $\begin{bmatrix} x^2 \\ w^2 \end{bmatrix} := \rho_{[a,b]}\begin{bmatrix} x^3 \\ w^3 \end{bmatrix}$, applying claim (i) for the case $\mathfrak{W}^p[a,b]$ and letting $b \to \infty$.

(ii) For an arbitrary $u \in L^p_{loc}([c,\infty);\mathcal{U})$ we may, by Proposition 3.7, take $\begin{bmatrix} x^3 \\ w^3 \end{bmatrix}$ to be the unique trajectory in $\mathfrak{W}^p[c,\infty)$ which satisfies $x^3(c) = x^1(c)$ and $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w^3 = u$. Then

$$\begin{bmatrix} x \\ w \end{bmatrix} := \begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} x^3 \\ w^3 \end{bmatrix} \in \mathfrak{W}^p[a,\infty)$$

by claim (i), and moreover, $\begin{bmatrix} x \\ w \end{bmatrix}$ is obviously an extension of $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix}$ such that $\rho_{[c,\infty)}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w = u$. □

Property (i) in the preceding proposition means that $x^1(c)$ and $x^2(c)$ contain all the information that is needed to determine whether two L^p trajectories $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix}$ and $\begin{bmatrix} x^2 \\ w^2 \end{bmatrix}$ of Σ can be concatenated at time c or not. This is referred to as "x

splitting the past and the future" or "x having the property of state", see e.g. [PW98, Rem. 4.3.4]. In the space $\mathfrak{V}[a,b]$ of classical trajectories, the state does not split the past and the future.

Proposition 3.9. *Let $-\infty < a < b < \infty$ and let $(V;\mathcal{X},\mathcal{W})$ be an L^p-well-posed s/s node with the space $\mathfrak{W}^p[a,b]$ of trajectories on $[a,b]$. Let $\Sigma_{s/s} = (\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ be the s/s system induced by $(V;\mathcal{X},\mathcal{W})$. Then*

$$\forall -\infty < a < b < \infty : \qquad \mathfrak{W}^p[a,b] = \rho_{[a,b]}\tau^{-a}\mathfrak{W}^p \quad \text{and} \tag{3.10}$$

$$\mathfrak{W}^p = \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix} \,\middle|\, \forall b > 0 : \ \rho_{[0,b]} \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[0,b] \right\}. \tag{3.11}$$

Proof. Corollary 3.2 immediately yields that $\rho_{[a,b]}\tau^{-a}\mathfrak{W}^p \subset \mathfrak{W}^p[a,b]$ for all a and b. We thus need to show that $\rho_{[a,b]}\tau^{-a}\mathfrak{W}^p \supset \mathfrak{W}^p[a,b]$ and that

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}, \ \forall b > 0 : \ \rho_{[0,b]} \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[0,b] \implies \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p. \tag{3.12}$$

In order to prove that $\mathfrak{W}^p[a,b] \subset \rho_{[a,b]}\tau^{-a}\mathfrak{W}^p$, we let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[a,b]$ be arbitrary. By Proposition 3.8(ii) $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ can be extended to some $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \mathfrak{W}^p[a,\infty)$. Then $\tau^a \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \mathfrak{W}^p$ and $\rho_{[a,b]}\tau^{-a}\left(\tau^a \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$.

We now prove (3.12) and therefore assume the left-hand side of the implication. We define $\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} := \mathfrak{T}_0 \begin{bmatrix} x(0) \\ \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w \end{bmatrix} \in \mathfrak{W}^p$, so that, in particular, $\rho_{[0,b]} \begin{bmatrix} x \\ w \end{bmatrix}$ and $\rho_{[0,b]} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix}$ lie in $\mathfrak{W}^p[0,b]$ with $x(0) = \widetilde{x}(0)$ and $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\rho_{[0,b]}w = \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\rho_{[0,b]}\widetilde{w}$ for all $b > 0$. Then, by (3.2), for all $b > 0$:

$$\rho_{[0,b]} \begin{bmatrix} x \\ w \end{bmatrix} = \mathfrak{T}^b_0 \begin{bmatrix} x(0) \\ \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\rho_{[0,b]}w \end{bmatrix} = \rho_{[0,b]} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix}.$$

This implies that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \mathfrak{W}^p$, cf. Definition A.3. □

Note that we cannot always extend trajectories in the backward time direction, because in general there is no guarantee that for every $x_a \in \mathcal{X}$ there is a trajectory $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ on $[a',a]$ such that $\widetilde{x}(a) = x_a$.

Proposition 3.10. *Let $T > 0$ and let $\Sigma_{s/s} = (\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ be a well-posed s/s system. Then*

$$\mathfrak{W}^p = \left\{ \begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_T \tau^{-T} \begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \bowtie_{2T} \tau^{-2T} \begin{bmatrix} x^3 \\ w^3 \end{bmatrix} \bowtie_{3T} \cdots \,\middle|\, \begin{bmatrix} x^n \\ w^n \end{bmatrix} \in \mathfrak{W}^p[0,T], \ x^{n+1}(0) = x^n(T), \ n \ge 1 \right\}. \tag{3.13}$$

Proof. Denote the right-hand side of (3.13) by $\widetilde{\mathfrak{W}^p}$. We first show that $\mathfrak{W}^p \subset \widetilde{\mathfrak{W}^p}$. Corollary 3.2 implies that for all $t \ge 0$:

$$\rho_{[0,T]}\tau^t\mathfrak{W}^p = \rho_{[0,T]}\rho_+\tau^t\mathfrak{W}^p \subset \rho_{[0,T]}\mathfrak{W}^p \subset \mathfrak{W}^p[0,T].$$

For any $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p$ we can thus define the sequence

$$\begin{bmatrix} x^n \\ w^n \end{bmatrix} := \rho_{[0,T]}\tau^{(n-1)T}\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[0,T], \quad n \geq 1,$$

which obviously satisfies

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_T \tau^{-T}\begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \bowtie_{2T} \ldots \quad \text{and} \quad x^{n+1}(0) = x(nT) = x^n(T).$$

In order to prove the inclusion $\widetilde{\mathfrak{W}}^p \subset \mathfrak{W}^p$, we let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \widetilde{\mathfrak{W}}^p$ be arbitrary. An induction argument, which uses Proposition 3.8(i), yields that

$$\rho_{[0,NT]}\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_T \tau^{-T}\begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \bowtie_{2T} \ldots \tau^{-(N-1)T}\begin{bmatrix} x^N \\ w^N \end{bmatrix} \in \mathfrak{W}^p[0,NT]$$

for all integers $N \geq 1$.

For every $b > 0$ we can now choose $N > b/T$ in order to get $NT > b$ and

$$\rho_{[0,b]}\begin{bmatrix} x \\ w \end{bmatrix} = \rho_{[0,b]}\rho_{[0,NT]}\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[0,b]$$

by Corollary 3.2(ii). According to (3.11), this implies that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p$. □

In the following proposition we characterise well-posedness of s/s systems and the respective admissible i/o pairs under the assumption that $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0^p[0,T]$. This condition is necessary for well-posedness, as we showed in Proposition 3.7.

Proposition 3.11. *Let $1 \leq p < \infty$, $-\infty < a < b < \infty$, $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ and let $(V;\mathcal{X},\mathcal{W})$ be a s/s node with trajectories $\mathfrak{W}^p[a,b]$ on $[a,b]$. Assume that $\mathfrak{V}_0[a,b]$ given in (2.11) is dense in $\mathfrak{W}_0^p[a,b]$ given in (3.4). Then the following statements are equivalent:*

- (i) *The s/s node $(V;\mathcal{X},\mathcal{W})$ is L^p well posed with admissible i/o pair $(\mathcal{U},\mathcal{Y})$. This s/s node induces the L^p-well-posed s/s system $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$, where $\mathfrak{W}^p$ is given by (3.13) with $\mathfrak{W}^p[0,T] := \tau^a\mathfrak{W}^p[a,b]$.*
- (ii) *The operator $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}^p[a,b]$ one-to-one onto $\begin{bmatrix} \mathcal{X} \\ L^p([a,b];\mathcal{U}) \end{bmatrix}$.*
- (iii) *The operator $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}_0^p[a,b]$ one-to-one onto $L^p([a,b];\mathcal{U})$ and*

$$\left\{x(a) \mid \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[a,b]\right\} = \mathcal{X}. \tag{3.14}$$

Proof. We only prove the case $a = 0$ and $b = T$. The general case can be reduced to this case using Corollary 3.2(i).

(ii) $\Longrightarrow$ (iii): We proved that $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps the space $\mathfrak{W}_0^p[a,b]$ one-to-one onto $L^p([a,b];\mathcal{U})$ in Proposition 3.7. The space on the left-hand side of (3.14) is the range of the operator $\begin{bmatrix} \delta_0 & 0 \end{bmatrix}\big|_{\mathfrak{W}^p[0,T]}$, which by assumption equals $\mathcal{X}$.

(iii) $\Longrightarrow$ (ii): We first prove injectivity of the operator in (ii). If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[0,T]$ and $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0$, then $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}_0^p[0,T]$ and $\mathcal{P}_\mathcal{U}^\mathcal{Y} w = 0$. Using the injectivity of $\begin{bmatrix} 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \big|_{\mathfrak{W}_0^p[0,T]}$ we then obtain that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = 0$, i.e. that $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \Big|_{\mathfrak{W}^p[0,T]}$ is injective.

Considering surjectivity, we take arbitrary $x_0 \in \mathcal{X}$ and $u_0 \in L^p([0,T];\mathcal{U})$. Condition (3.14) implies that there exists an $\begin{bmatrix} x_x \\ w_x \end{bmatrix} \in \mathfrak{W}^p[0,T]$ with $x(0) = x_0$. By the surjectivity of $\begin{bmatrix} 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \big|_{\mathfrak{W}_0^p[0,T]}$ we can find $\begin{bmatrix} x_u \\ w_u \end{bmatrix} \in \mathfrak{W}^p[0,T]$ such that $x_u(0) = 0$ and $\mathcal{P}_\mathcal{U}^\mathcal{Y} w_u = u_0 - \mathcal{P}_\mathcal{U}^\mathcal{Y} w_x$ in $L^p([0,T];\mathcal{U})$. We then have that $\begin{bmatrix} x \\ w \end{bmatrix} := \begin{bmatrix} x_u + x_x \\ w_u + w_x \end{bmatrix}$ lies in $\mathfrak{W}^p[0,T]$ and, moreover, that

$$\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x(0) \\ \mathcal{P}_\mathcal{U}^\mathcal{Y}(w_u + w_x) \end{bmatrix} = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}.$$

(i) $\Longrightarrow$ (ii): This was established in Lemma 3.4.
(ii) $\Longrightarrow$ (i): We already proved that if condition (ii) holds for some $T > 0$, then condition (iii) holds for the same T. This allows us to make use of Lemma 3.5, (3.6) and (3.14) for that particular T. We now prove that the conditions in Definition 2.7 are satisfied.

We start with condition (i) and, therefore, let $x_0 \in \mathcal{X}$ be arbitrary. By (3.14) there exists a trajectory $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[0,T]$, with $x(0) = x_0$. Let $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}[0,T]$ be a sequence of classical trajectories, which converges to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$. Then $x_n(0)$ lies in the space in Definition 2.7(i) for all n and, moreover, $x_n(0) \to x_0$, since $x_n \to x$ uniformly on $[0,T]$. This proves that condition (i) of Definition 2.7 is satisfied. Condition (ii) is proved by combining the assumption $\overline{\mathfrak{V}_0[0,T]} = \mathfrak{W}_0^p[0,T]$ with Lemma 3.5.

Proceeding to Definition 2.7(iii), we recall that $\begin{bmatrix} x \\ w \end{bmatrix} = \mathfrak{T}_0^T \begin{bmatrix} x_0 \\ u \end{bmatrix}$ by the definition of $\mathfrak{T}_0^T$ is the unique $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[0,T]$, which satisfies $x(0) = x_0$ and $\mathcal{P}_\mathcal{U}^\mathcal{Y} w = u$. Fix $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T]$ arbitrarily, so that $\begin{bmatrix} x \\ w \end{bmatrix} = \mathfrak{T}_0^T \begin{bmatrix} x(0) \\ \mathcal{P}_\mathcal{U}^\mathcal{Y} w \end{bmatrix}$. For any given $t \in [0,T]$ we define

$$\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} := \mathfrak{T}_0^T \begin{bmatrix} x(0) \\ \rho_{[0,T]} \pi_{[0,t]} \mathcal{P}_\mathcal{U}^\mathcal{Y} w \end{bmatrix},$$

thus obtaining that $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] - \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ lies in $\mathfrak{W}_0^p[0,T]$ with $\rho_{[0,t]} \mathcal{P}_\mathcal{U}^\mathcal{Y}(w - \widetilde{w}) = 0$.

From (3.6) we now get $\rho_{[0,t]}\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right]=\rho_{[0,t]}\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$, which implies that:

$$\begin{aligned}\|x(t)\|_{\mathcal{X}}+\|w\|_{L^p([0,t];\mathcal{W})}&=\|\widetilde{x}(t)\|_{\mathcal{X}}+\|\widetilde{w}\|_{L^p([0,t];\mathcal{W})}\\&\le\|\widetilde{x}\|_{C([0,T];\mathcal{X})}+\|\widetilde{w}\|_{L^p([0,T];\mathcal{W})}\\&\le\|\mathfrak{T}_0^T\|\big(\|\widetilde{x}(0)\|_{\mathcal{X}}+\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}\widetilde{w}\|_{L^p([0,T];\mathcal{U})}\big)\\&\le\|\mathfrak{T}_0^T\|\big(\|x(0)\|_{\mathcal{X}}+\|\pi_{[0,t]}\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}w\|_{L^p([0,T];\mathcal{U})}\big)\\&\le\|\mathfrak{T}_0^T\|\big(\|x(0)\|_{\mathcal{X}}+\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}w\|_{L^p([0,t];\mathcal{U})}\big).\end{aligned}$$

We have shown that $(V;\mathcal{X},\mathcal{W})$ induces an L^p-well-posed s/s system $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$. By assumption, $\mathfrak{W}^p[a,b]$ is the space of L^p trajectories on $[a,b]$ generated by V, which implies that $\rho_{[a,b]}\tau^{-a}\mathfrak{W}^p=\mathfrak{W}^p[a,b]$, according to Proposition 3.9. This is equivalent to $\rho_{[0,b-a]}\mathfrak{W}^p=\tau^a\mathfrak{W}^p[a,b]$ and an application of Proposition 3.10 now yields that (3.13) holds. □

Note that conditions (ii) and (iii) of Proposition 3.11 hold for some choice of $-\infty<a<b<\infty$ if and only if they hold for all such a and b. If Σ is known to be well posed, then checking a given i/o pair for admissibility is quite simple, as the following corollary shows.

Corollary 3.12. *Let $-\infty<a<b<\infty$, let $\Sigma_{s/s}=(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ be an L^p-well-posed s/s system and let $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$. Then the following conditions are equivalent:*

(i) *The i/o pair $(\mathcal{U},\mathcal{Y})$ is admissible for the s/s system Σ.*
(ii) *$(\mathcal{U},\mathcal{Y})$ is admissible for* some *well-posed s/s node which generates Σ.*
(iii) *$(\mathcal{U},\mathcal{Y})$ is admissible for* every *well-posed s/s node which generates Σ.*
(iv) *The operator $\begin{bmatrix}0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}\end{bmatrix}$ maps $\mathfrak{W}_0^p[a,b]$ one-to-one onto $L^p([a,b];\mathcal{U})$.*

Proof. (i) $\Longleftrightarrow$ (ii): This is Definition 3.3.
(i) $\Longrightarrow$ (iii): According to Proposition 3.7(iv) $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0^p[0,T]$ for every well-posed s/s node $(V;\mathcal{X},\mathcal{W})$ which generates Σ. By Lemma 3.4 condition (ii) of Proposition 3.11 is satisfied whenever $(\mathcal{U},\mathcal{Y})$ is admissible for Σ. Proposition 3.11(i) then yields that $(\mathcal{U},\mathcal{Y})$ is admissible for $(V;\mathcal{X},\mathcal{W})$.
(iii) $\Longrightarrow$ (ii): This is trivial.
(i) $\Longleftrightarrow$ (iv): Again $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0^p[0,T]$ for any well-posed s/s node which generates Σ. Proposition 3.11(iii) yields that (3.14) is satisfied, because the space in (3.14) does not depend on the i/o pair. Now the equivalence of claims (i) and (iii) in Proposition 3.11 finishes the proof. □

Next we give a theorem which shows that the only example of a well-posed s/s system with external signal space $\mathcal{W}=\{0\}$ is given by a C_0 semigroup on $\mathcal{X}$. In order to formulate and prove this result we first need to recall some basic facts about strongly continuous semigroups.

Definition 3.13. *Let $\mathcal{X}$ be a Banach space. A family $t \to \mathfrak{A}^t$, $t \geq 0$, of bounded linear operators on $\mathcal{X}$ is a* semigroup *on $\mathcal{X}$ if $\mathfrak{A}^0 = 1$ and $\mathfrak{A}^{s+t} = \mathfrak{A}^s\mathfrak{A}^t$ for all $s, t \geq 0$.*

The semigroup is strongly continuous*, or shorter C_0, if $\lim_{t\to 0^+} \mathfrak{A}^t x_0 = x_0$ for all $x_0 \in \mathcal{X}$.*

The generator *$A : \mathcal{X} \supset \mathrm{Dom}\,(A) \to \mathcal{X}$ of $\mathfrak{A}$ is the (in general unbounded) linear operator defined by*

$$Ax_0 := \lim_{t\to 0^+} \frac{1}{t}(\mathfrak{A}^t x_0 - x_0), \tag{3.15}$$

with domain consisting of those $x_0 \in \mathcal{X}$ for which the limit exists.

The generator A of a C_0 semigroup on $\mathcal{X}$ is closed and $\mathrm{Dom}\,(A^n)$ is dense in $\mathcal{X}$ for all integer $n \geq 1$, see e.g. [Paz83, Thm 1.2.7]. Moreover, according to [Paz83, Thm 1.2.6], a C_0 semigroup $\mathfrak{A}$ is uniquely determined by its generator A and we may therefore say that A *generates* $\mathfrak{A}$. The following lemma is a part of [Sta05, Thm 2.5.4(i)].

Lemma 3.14. *Let $\mathfrak{A}$ be a C_0 semigroup on the Banach space $\mathcal{X}$. Then there exists an $\omega_{\mathfrak{A}} \in \mathbb{R} \cup \{-\infty\}$, the* growth bound *of the semigroup $\mathfrak{A}$, such that:*

$$\omega_{\mathfrak{A}} = \lim_{t\to\infty} \frac{\log(\|\mathfrak{A}^t\|)}{t} = \inf_{t>0} \frac{\log(\|\mathfrak{A}^t\|)}{t}.$$

Moreover, for each $\omega > \omega_{\mathfrak{A}}$, we have that $e^{-\omega t}\|\mathfrak{A}^t\| \to 0$ as $t \to \infty$ and there exists some $M \geq 1$, such that

$$e^{\omega_{\mathfrak{A}} t} \leq \|\mathfrak{A}^t\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

Every contraction semigroup $\mathfrak{A}$, i.e., a semigroup such that $\|\mathfrak{A}^t\| \leq 1$ for all $t \geq 0$, has growth bound at most zero:

$$\omega_{\mathfrak{A}} = \lim_{t\to\infty} \frac{\log(\|\mathfrak{A}^t\|)}{t} \leq \lim_{t\to\infty} \frac{\log(1)}{t} = 0, \tag{3.16}$$

because the logarithm function is nondecreasing.

The proof of the next theorem depends on the following fact, which can be proved by combining Theorem 3.2.1(iii) and Theorem 3.8.2(ii) of [Sta05]. Let A generate the C_0 semigroup $\mathfrak{A}$ on the Banach space $\mathcal{X}$. Then for all $x_0 \in \mathrm{Dom}\,(A)$ the initial-value problem $\dot{x}(t) = Ax(t)$, $t \geq 0$, $x(0) = x_0$ has the unique continuously differentiable solution $x(t) = \mathfrak{A}^t x_0$, $t \geq 0$.

Theorem 3.15. *Let $\mathcal{X}$ be a Banach space, let $p \in [1, \infty)$ be arbitrary, and let $V \subset \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\end{smallmatrix}\right]$. Then the following claims are true:*

(i) *If V is the graph*

$$V = \begin{bmatrix} A \\ 1 \end{bmatrix} \mathrm{Dom}\,(A) \tag{3.17}$$

of the generator A of a C_0 semigroup $\mathfrak{A}$ on $\mathcal{X}$, then $(V; \mathcal{X}, \{0\})$ is an L^p-well-posed s/s node for all $1 \leq p < \infty$.

(ii) *Conversely, if* $(V;\mathcal{X},\{0\})$ *is a well-posed s/s node for some* $1 \le p < \infty$*, then* V *is given by* (3.17)*, where* $A : \mathcal{X} \supset \operatorname{Dom}(A) \to \mathcal{X}$ *is a closed operator. The operator* A *can be extended to the generator of a* C_0 *semigroup on* $\mathcal{X}$*.*

(iii) *If* $(V;\mathcal{X},\{0\})$ *is a well-posed s/s node, then it generates the* L^p*-well-posed s/s system* $(\mathfrak{W}^p;\mathcal{X},\{0\})$*, where*

$$\mathfrak{W}^p = \{x \in C(\mathbb{R}^+;\mathcal{X}) \mid x(t) = \mathfrak{A}^t x_0,\ t \ge 0,\ x_0 \in \mathcal{X}\}.$$

Proof. Part 1 (Proof of (i)): Let $T > 0$ be arbitrary. By the discussion after Definition 3.13, the generator of any C_0 semigroup is closed, i.e. V has property (i) of Definition 2.3. From (3.17) we have that $\begin{bmatrix} z_0 \\ x_0 \end{bmatrix} \in V$ if and only if $x_0 \in \operatorname{Dom}(A)$ and $z_0 = Ax_0$. In particular, condition (ii) of Definition 2.7(ii) holds.

For condition (iii), define $x := t \to \mathfrak{A}^t x_0$ for $t \in [0,T]$ and $x_0 \in \operatorname{Dom}(A)$. Then we obtain that $\dot{x}(t) = Ax(t)$ for $t \in [0,T]$, so that x is a classical trajectory of V on $[0,T]$. Moreover, this trajectory satisfies

$$\begin{bmatrix} \dot{x}(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} Ax(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \end{bmatrix}.$$

This proves that $(V;\mathcal{X},\{0\})$ is a s/s node, but we still need to show that it is well posed.

The domain of any C_0 semigroup generator A is dense and thus condition (i) of Definition 2.7 is met. Condition (ii) becomes trivial in the case $\mathcal{U} = \mathcal{W} = \{0\}$. Considering condition (iii), we note that every classical trajectory of V is of the form $x(t) = \mathfrak{A}^t x(0)$, $t \ge 0$. Lemma 3.14 then yields that there exists constants M and $\omega > \max\{\omega_{\mathfrak{A}}, 0\}$ such that:

$$\|x(t)\| = \|\mathfrak{A}^t x(0)\| \le \|\mathfrak{A}^t\|\|x(0)\| \le Me^{\omega t}\|x(0)\| \le Me^{\omega T}\|x(0)\|, \quad t \in [0,T].$$

This shows that the s/s node $(V;\mathcal{X},\{0\})$ is L^p well posed for all $p \in [1,\infty)$, because the only condition, which involves p, becomes trivial.

Part 2 (Proof of (ii) and (iii)): By the definition of a s/s node we immediately obtain that V can be written as the graph (3.17) of a closed operator A, and that there for every $T > 0$ and $x_0 \in \operatorname{Dom}(A)$ exists some $x \in \mathfrak{V}[0,T]$, such that $x(0) = x_0$. Moreover, as $(V;\mathcal{X},\{0\})$ is well posed, we know that $\operatorname{Dom}(A)$ is dense in $\mathcal{X}$ and that there exists some K_T such that $\|x(t)\| \le K_T\|x(0)\|$ for $t \in [0,T]$. The latter implies that $x \in \mathfrak{V}[0,T]$ is uniquely determined by $x(0)$.

The above argument and the fact that every state trajectory is continuous allow us to define the following family of bounded operators from $\operatorname{Dom}(A)$ to $\mathcal{X}$. For $x_0 \in \operatorname{Dom}(A)$ and $t \in [0,T]$ define a family $t \to \mathfrak{A}^t$ of bounded linear operators by $\mathfrak{A}^t x_0 := x(t)$, such that $x \in \mathfrak{V}[0,T]$ and $x(0) = x_0$. The conditions in Definitions 2.3 and 2.7 hold for every $T > 0$ and we may extend the family $t \to \mathfrak{A}^t$ to all of $\mathbb{R}^+$ by choosing an arbitrary $T > t$ for every $t \ge 0$. Every $\mathfrak{A}^t$ can moreover be uniquely extended from $\operatorname{Dom}(A)$ to all of $\mathcal{X}$ by continuity.

Let $x_n \in \mathfrak{V}[0,T]$. From $\|x(t)\| \le K_T\|x(0)\|$, $t \in [0,T]$, we have $x_n(0) \to x(0)$ in $\mathcal{X}$ if and only if $x_n \to x$ uniformly on $[0,T]$. This proves that

$$\mathfrak{W}^p[0,T] = \left\{ x \in C([0,T];\mathcal{X}) \;\middle|\; x(t) = \mathfrak{A}^t x_0,\ t \in [0,T],\ x_0 \in \mathcal{X} \right\}. \tag{3.18}$$

In particular claim (iii) above holds for the family $\mathfrak{A}$ of operators we have defined above. We finish the proof by showing that $\mathfrak{A}$ is a C_0 semigroup.

We have $\mathfrak{A}^0 x_0 = x_0$ for all $x_0 \in \mathcal{X}$ by the definition of $\mathfrak{A}$. Moreover, $\lim_{t\to 0^+} \mathfrak{A}^t x_0 = \lim_{t\to 0^+} x(t) = x_0$, because every state trajectory x on $[0,T]$ is continuous from the right at 0. For the condition $\mathfrak{A}^s\mathfrak{A}^t = \mathfrak{A}^{s+t}$, $s,t \ge 0$, we make the following argument. Let $x_0 \in \mathcal{X}$ and $s,t \ge 0$ be arbitrary. By (3.18) there exists a unique $x \in \mathfrak{W}^p[0,s+t]$ such that $x(0) = x_0$. Then, by Corollary 3.2 in particular $\rho_{[0,t]}x \in \mathfrak{W}^p[0,t]$ and $\tau^t\rho_{[t,s+t]}x \in \mathfrak{W}^p[0,s]$. From the construction of $\mathfrak{A}$ we now get that

$$\forall x_0 \in \mathcal{X}: \quad \mathfrak{A}^{s+t}x_0 = x(s+t) = (\tau^t x)(s) = \mathfrak{A}^s(\tau^t x)(0) = \mathfrak{A}^s x(t) = \mathfrak{A}^s\mathfrak{A}^t x_0.$$

Finally, we for all $x_0 \in \mathrm{Dom}\,(A)$ have

$$\begin{aligned}\lim_{h\to 0^+} \frac{1}{h}(\mathfrak{A}^h - 1)x_0 &= \lim_{h\to 0^+} \frac{1}{h}\big(x(h) - x(0)\big) = \lim_{h\to 0^+} \frac{1}{h}\int_0^h \dot{x}(s)\,\mathrm{d}s \\ &= \lim_{h\to 0^+} \frac{1}{h}\int_0^h (Ax)(s)\,\mathrm{d}s = (Ax)(0) = Ax_0\end{aligned}$$

by standard integration theory and the fact that $\dot{x} = Ax$ is continuous on $[0,T]$. This shows that A satisfies (3.15), i.e. that A is the restriction of the generator of $\mathfrak{A}$ to $\mathrm{Dom}\,(A)$, because by Definition 3.13 the generator is the maximally defined operator that satisfies (3.15). The proof is complete. □

We finish the section with the following question, to which Proposition 3.11 provides only a partial answer. A definite answer will be given in Theorem 6.4.

Remark 3.16. *Let $T > 0$ and $W[0,T]$ be an arbitrary subspace of $\begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$, where $\mathcal{X}$ and $\mathcal{W}$ are Banach spaces. Define $W^+ \subset \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ by*

$$\begin{aligned} W^+ = \Bigg\{ &\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_T \tau^{-T} \begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \bowtie_{2T} \tau^{-2T} \begin{bmatrix} x^3 \\ w^3 \end{bmatrix} \bowtie_{3T} \cdots \;\Bigg| \\ &\begin{bmatrix} x^n \\ w^n \end{bmatrix} \in W[0,T],\ x^{n+1}(0) = x^n(T),\ n \ge 1 \Bigg\}. \end{aligned} \tag{3.19}$$

When is $(W^+;\mathcal{X},\mathcal{W})$ an L^p-well-posed s/s system?

The reason for not using the notations $\mathfrak{W}^p[0,T]$ and $\mathfrak{W}^p$ in Remark 3.16 is that we do not a priori know that they consist of L^p trajectories of some $V \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

4. Input/state/output representations

In this section we first show how well-posed i/s/o systems may be used to represent well-posed s/s systems. Thereafter we proceed by characterising the admissible i/o pairs and giving their associated i/s/o representations.

The theory of well-posed i/s/o systems is due to Salamon, Šmuljan, Weiss, Lax, Phillips and many others. Selected results of these authors are collected in [Sta05, Ch. 4], which we use as our standard reference also in this section.

In the following definition we need the function space $L^p_{c,loc}(\mathbb{R};\mathcal{U})$. See Definition A.3 in the appendix for its definition.

Definition 4.1. *The space $TIC^p_{loc}(\mathcal{U};\mathcal{Y})$ consists of all continuous operators*

$$\mathfrak{D} : L^p_{c,loc}(\mathbb{R};\mathcal{U}) \to L^p_{c,loc}(\mathbb{R};\mathcal{Y}),$$

which for all $u \in L^p_{c,loc}(\mathbb{R};\mathcal{U})$ and $t \in \mathbb{R}$ satisfy $\tau^t\mathfrak{D}u = \mathfrak{D}\tau^t u$ (time invariance) and $\rho_-\mathfrak{D}\pi_+u = 0$ (causality).

If the domain and codomain of $\mathfrak{D} \in TIC^p_{loc}(\mathcal{U},\mathcal{Y})$ are clear from the context, then we sometimes briefly write $\mathfrak{D} \in TIC^p_{loc}$.

Definition 4.2. *Let $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ be Banach spaces. By a* causal, time-invariant and L^p-well-posed input/state/output system (well posed i/s/o system) *on $(\mathcal{X},\mathcal{U},\mathcal{Y})$ we mean a quadruple $\left(\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$, such that:*

(i) *The map $t \to \mathfrak{A}^t$ is a C_0 semigroup on $\mathcal{X}$, cf. Definition 3.13.*
(ii) *The operator $\mathfrak{B} : L^p_c(\mathbb{R}^-;\mathcal{U}) \to \mathcal{X}$ is continuous and it has the property $\mathfrak{A}^t\mathfrak{B}u = \mathfrak{B}\rho_-\tau^t\pi_-u$ for all $u \in L^p_c(\mathbb{R}^-;\mathcal{U})$ and $t \geq 0$.*
(iii) *The continuous operator $\mathfrak{C} : \mathcal{X} \to L^p_{loc}(\mathbb{R}^+;\mathcal{Y})$ satisfies $\mathfrak{C}\mathfrak{A}^t x = \rho_+\tau^t\mathfrak{C}x$ for all $x \in \mathcal{X}$ and $t \geq 0$.*
(iv) *The operator $\mathfrak{D}$ lies in $TIC^p_{loc}(\mathcal{U};\mathcal{Y})$ and it satisfies $\rho_+\mathfrak{D}\pi_-u = \mathfrak{C}\mathfrak{B}u$ for all $u \in L^p_c(\mathbb{R}^-;\mathcal{U})$.*

Condition (ii) of Definition 4.2 means that $\mathfrak{B}$ intertwines the semigroup $\mathfrak{A}$ with the left-shift semigroup $\rho_-\tau\pi_-$ on $L^p_c(\mathbb{R}^-;\mathcal{U})$. Condition (iii) means that $\mathfrak{C}$ intertwines the semigroup $\mathfrak{A}$ with the left-shift semigroup $\rho_+\tau$ on $L^p_{loc}(\mathbb{R}^+;\mathcal{Y})$.

Remark 4.3. *For notational reasons, we usually interpret $\mathfrak{B}$ as an operator defined on $L^p_{c,loc}(\mathbb{R};\mathcal{U})$, still denoting it by the same letter, by defining $\mathfrak{B}u := \mathfrak{B}\rho_-u$ for $u \in L^p_{c,loc}(\mathbb{R};\mathcal{U})$. We also sometimes interpret $\mathfrak{C}$ as an operator with values in $L^p_{c,loc}(\mathbb{R};\mathcal{Y})$ by defining $\mathfrak{C}x := \pi_+\mathfrak{C}x$.*

The following definition is an adaptation of [Sta05, Def. 2.2.7].

Definition 4.4. *Let $-\infty < a < b < \infty$ and $I = [a,b]$ or $I = [a,\infty)$. We call the triple* $\begin{bmatrix} x \\ y \\ u \end{bmatrix} \in \begin{bmatrix} C(I;\mathcal{X}) \\ L^p_{loc}(I;\mathcal{Y}) \\ L^p_{loc}(I;\mathcal{U}) \end{bmatrix}$ *an* L^p trajectory *on I of the L^p-well-posed i/s/o system* $\left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ *if*

$$\begin{aligned} x(t) &= \begin{bmatrix} \mathfrak{A}^{t-a} & \mathfrak{B}\tau^t \end{bmatrix} \begin{bmatrix} x(a) \\ \pi_I u \end{bmatrix} \quad \text{for all } t \in I \text{ and} \\ y &= \rho_I \begin{bmatrix} \tau^{-a}\mathfrak{C} & \mathfrak{D} \end{bmatrix} \begin{bmatrix} x(a) \\ \pi_I u \end{bmatrix} \quad \text{in } L^p_{loc}(I;\mathcal{Y}). \end{aligned} \tag{4.1}$$

By shortly referring to a trajectory we mean an L^p trajectory on $\mathbb{R}^+$.

In the following definition, the equality on the second line of (4.2) should be understood in the sense of (4.1)

Definition 4.5. *Let $\Sigma_{s/s} = (\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ be a well-posed s/s system, which has the admissible i/o pair $(\mathcal{U},\mathcal{Y})$.*

The i/s/o system $\left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ *is an* input/state/output representation (i/s/o representation) *of Σ corresponding to $(\mathcal{U},\mathcal{Y})$ if for some $-\infty < a < b < \infty$:*

$$\begin{aligned} \mathfrak{W}^p[a,b] = \Bigg\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C([a,b];\mathcal{X}) \\ L^p([a,b];\mathcal{W}) \end{bmatrix} \Bigg| \, \forall t \in [a,b] : \\ \begin{bmatrix} x(t) \\ \mathcal{P}^{\mathcal{U}}_{\mathcal{Y}} w \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^{t-a} & \mathfrak{B}\tau^t \\ \rho_{[a,b]}\tau^{-a}\mathfrak{C} & \rho_{[a,b]}\mathfrak{D} \end{bmatrix} \begin{bmatrix} x(a) \\ \pi_{[a,b]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w \end{bmatrix} \Bigg\}. \end{aligned} \tag{4.2}$$

Our next task is to prove that to every admissible i/o pair of a well-posed s/s system there corresponds exactly one i/s/o representation. We split the long proof into a few lemmas for readability.

Lemma 4.6. *Let $T > 0$ and $1 \le p < \infty$, let $\mathcal{X}$ and $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be Banach spaces. Let* $W[0,T] \subset \begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$ *be arbitrary and define W^+ by (3.19).*

Then the following claims are equivalent:

(i) *The space $W[0,T]$ is closed and the operator* $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}$ *maps $W[0,T]$ one-to-one onto* $\begin{bmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}$.

(ii) *The space W^+ is a closed subspace of* $\begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ *and* $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}$ *maps W^+ one-to-one onto* $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}(\mathbb{R}^+;\mathcal{U}) \end{bmatrix}$.

When the equivalent conditions (i) and (ii) hold, $W[0,T] = \rho_{[0,T]}W^+$.

Proof. (i) $\Longrightarrow$ (ii): Let $\begin{bmatrix} x_m \\ w_m \end{bmatrix}$ be a sequence in W^+, which converges to some $\begin{bmatrix} x \\ w \end{bmatrix}$ in $\begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$. Then for all $n \geq 1$:

$$\begin{bmatrix} x_m^n \\ w_m^n \end{bmatrix} := \rho_{[0,T]}\tau^{(n-1)T} \begin{bmatrix} x_m \\ w_m \end{bmatrix} \to \rho_{[0,T]}\tau^{(n-1)T} \begin{bmatrix} x \\ w \end{bmatrix} =: \begin{bmatrix} x^n \\ w^n \end{bmatrix} \text{ as } m \to \infty.$$

By Corollary 3.2, $\begin{bmatrix} x_m^n \\ w_m^n \end{bmatrix}$ all lie in $W[0,T]$, which was assumed to be closed, and therefore $\begin{bmatrix} x^n \\ w^n \end{bmatrix}$ also lies in $W[0,T]$. Moreover,

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_T \tau^{-T} \begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \bowtie_{2T} \tau^{-2T} \begin{bmatrix} x^3 \\ w^3 \end{bmatrix} \bowtie_{3T} \dots \tag{4.3}$$

and by the continuity of x we have $x^n(T) = x(nT) = x^{n+1}(0)$ for all $n \geq 1$. From (3.19) we now get that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W^+$, i.e., that W^+ is closed.

Let $x_0 \in \mathcal{X}$ and $u \in L^p_{loc}(\mathbb{R}^+;\mathcal{U})$ be arbitrary. By assumption $W[0,T]$ there exists a unique $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \in W[0,T]$ such that $x^1(0) = x_0$ and $\mathcal{P}_\mathcal{U}^\mathcal{Y} w = \rho_{[0,T]}u$. Similarly, we for every $n \geq 1$ and $\begin{bmatrix} x^n \\ w^n \end{bmatrix} \in W[0,T]$ may let $\begin{bmatrix} x^{n+1} \\ w^{n+1} \end{bmatrix}$ be the unique element of $W[0,T]$, such that $x^{n+1}(0) = x^n(T)$ and $\mathcal{P}_\mathcal{U}^\mathcal{Y} w^{n+1} = \rho_{[0,T]}\tau^{nT}u$. Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ given in (4.3) lies in W^+, cf. (3.19), $x(0) = x_0$ and $\mathcal{P}_\mathcal{U}^\mathcal{Y} w = u$. This proves that

$$\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} W^+ = \begin{bmatrix} \mathcal{X} \\ L^p_{loc}(\mathbb{R}^+;\mathcal{U}) \end{bmatrix}. \tag{4.4}$$

Moreover, if $x(0) = 0$ and $\mathcal{P}_\mathcal{U}^\mathcal{Y} w = 0$, then an induction argument shows that $\begin{bmatrix} x^n \\ w^n \end{bmatrix} = 0$ for all $n \geq 1$. This means that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = 0$, i.e., that the restriction of $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}$ to W^+ is injective.

(ii) $\Longrightarrow$ (i): Denote $\widetilde{\mathfrak{X}}_0 := \begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}\Big|_{W^+}^{-1}$ and let $x_0 \in \mathcal{X}$ and $u \in L^p([0,T];\mathcal{U})$ be arbitrary. Defining $\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} := \widetilde{\mathfrak{X}}_0 \begin{bmatrix} x_0 \\ \rho_+\pi_{[0,T]}u \end{bmatrix} \in W^+$, we by (3.19) get that $\begin{bmatrix} x \\ w \end{bmatrix} := \rho_{[0,T]} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} \in W[0,T]$. Moreover,

$$\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} \delta_0 & 0 \\ 0 & \rho_{[0,T]}\mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} = \begin{bmatrix} x_0 \\ \rho_{[0,T]}\rho_+\pi_{[0,T]}u \end{bmatrix} = \begin{bmatrix} x_0 \\ u \end{bmatrix}$$

and thus the restriction of $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix}$ to $W[0,T]$ is surjective. We still need to show that this restriction is also injective.

Let $\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} \in W[0,T]$ be arbitrary and define $\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} := \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix}$. By the surjectivity of $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}\Big|_{W[0,T]}$ we can find a sequence of elements $\begin{bmatrix} x^n \\ w^n \end{bmatrix}$ of $W[0,T]$, such that $x^n(0) = x^{n-1}(T)$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w^n = 0$ for all $n \geq 2$. Then $\begin{bmatrix} x \\ w \end{bmatrix}$ given in (4.3) lies in W^+ according to (3.19) and, by construction, $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] = \rho_{[0,T]} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$. In particular, $W[0,T] \subset \rho_{[0,T]} W^+$ and the last claim of this lemma is valid, because (3.19) immediately yields that $W[0,T] \supset \rho_{[0,T]} W^+$. Now, if $\widetilde{x}(0) = 0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w} = 0$, then $x(0) = 0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$ and the injectivity of $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}\Big|_{W^+}$ then implies that $\begin{bmatrix} x \\ w \end{bmatrix} = 0$ and. In particular, $\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} = 0$ and $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}\Big|_{W[0,T]}$ is injective.

In order to show that $W[0,T]$ is closed, we let $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in W[0,T]$ and get

$$\begin{gathered} \begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix} := \rho_{[0,T]} \widetilde{\mathfrak{T}}_0 \begin{bmatrix} x_n(0) \\ \rho_+ \pi_{[0,T]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \end{bmatrix} \in W[0,T] \quad \text{with} \\ \begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix} \left(\begin{bmatrix} x_n \\ w_n \end{bmatrix} - \begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix} \right) = 0, \end{gathered}$$

which implies that $\begin{bmatrix} x_n \\ w_n \end{bmatrix} = \begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix}$. Lemma 3.4 yields that $\widetilde{\mathfrak{T}}_0$ is continuous and, therefore, if $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \to \begin{bmatrix} x \\ w \end{bmatrix}$, then

$$\begin{aligned} \begin{bmatrix} x \\ w \end{bmatrix} &= \lim_{n\to\infty} \begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix} = \lim_{n\to\infty} \rho_{[0,T]} \widetilde{\mathfrak{T}}_0 \begin{bmatrix} x_n(0) \\ \rho_+ \pi_{[0,T]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \end{bmatrix} \\ &= \rho_{[0,T]} \widetilde{\mathfrak{T}}_0 \lim_{n\to\infty} \begin{bmatrix} x_n(0) \\ \rho_+ \pi_{[0,T]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n \end{bmatrix} = \rho_{[0,T]} \widetilde{\mathfrak{T}}_0 \begin{bmatrix} x(0) \\ \rho_+ \pi_{[0,T]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \end{aligned}$$

lies in $W[0,T]$. Thus $W[0,T]$ is closed. □

The following Lemma will be used to prove existence of an i/s/o representation of a well-posed s/s system.

Lemma 4.7. *Let $T > 0$ and $1 \leq p < \infty$, let $\mathcal{X}$ and $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be Banach spaces and assume that $W^+ \subset \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$ satisfies condition (ii) of Lemma 4.6. Furthermore assume that W^+ is invariant under left shift on $\mathbb{R}^+$:*

$$\forall t \geq 0: \quad \rho_+ \tau^t W^+ \subset W^+. \tag{4.5}$$

Then there exists an L^p-well-posed i/s/o system $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$*, such that*

$$\begin{aligned} W^+ = \Bigg\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix} \Bigg| \, \forall t \geq 0 : \\ \begin{bmatrix} x(t) \\ \mathcal{P}^{\mathcal{U}}_{\mathcal{Y}} w \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}\tau^t \\ \mathfrak{C} & \rho_+\mathfrak{D} \end{bmatrix} \begin{bmatrix} x(0) \\ \pi_+\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w \end{bmatrix} \Bigg\}. \end{aligned} \tag{4.6}$$

Proof. We use Theorem 2.2.14 of [Sta05] in order to construct an i/s/o system $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ which satisfies (4.6). Assume therefore (d) – (f) and (4.5).

Part 1 (Definition of $\mathfrak{A}^b_a$*,* $\mathfrak{B}^b_a$*,* $\mathfrak{C}^b_a$ *and* $\mathfrak{D}^b_a$*):* Let $-\infty < a < b < \infty$ be arbitrary and define

$$W[a,\infty) := \tau^{-a}W^+ \quad \text{and} \quad W[a,b] := \rho_{[a,b]}\tau^{-a}W^+. \tag{4.7}$$

Then it follows that for all $c \in \mathbb{R}$:

$$\tau^c W[a,b] = \tau^c\rho_{[a,b]}\tau^{-a}W^+ = \rho_{[a-c,b-c]}\tau^{-(a-c)}W^+ = W[a-c,b-c]. \tag{4.8}$$

Moreover, for all $c \in (a,b)$:

$$\rho_{[a,c]}W[a,b] = \rho_{[a,c]}\rho_{[a,b]}\tau^{-a}W^+ = \rho_{[a,c]}\tau^{-a}W^+ = W[a,c] \tag{4.9}$$

and, using (4.5):

$$\begin{aligned} \rho_{[c,b]}W[a,b] &= \rho_{[c,b]}\rho_{[a,b]}\tau^{-a}W^+ = \rho_{[c,b]}\rho_{[c,\infty)}\tau^{-c}\tau^{c-a}W^+ \\ &= \rho_{[c,b]}\tau^{-c}\rho_+\tau^{c-a}W^+ \subset \rho_{[c,b]}\tau^{-c}W^+ = W[c,b]. \end{aligned} \tag{4.10}$$

By a combination of Lemma 4.6 and Corollary 3.2(i), the operator

$$\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix} = \tau^{-a} \begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix} \tau^a$$

maps $W[a,b]$ one-to-one onto $\begin{bmatrix} \mathcal{X} \\ L^p([a,b];\mathcal{U}) \end{bmatrix}$. Letting $\widetilde{\mathfrak{T}}^b_a := \begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}\Big|_{W[a,b]}$, we get from Lemma 3.4 that both $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} \end{bmatrix}$ and $\widetilde{\mathfrak{T}}^b_a$ are continuous. Thus $\widetilde{\mathfrak{T}}^b_a$ maps any $x_a \in \mathcal{X}$ and $u \in L^p([a,b];\mathcal{U})$ continuously into the unique element $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W[a,b]$, such that $x(a) = x_a$ and $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w = u$.

We now define the quadruple of operator families

$$\begin{aligned} &\begin{bmatrix} \mathfrak{A}^b_a & \mathfrak{B}^b_a \\ \mathfrak{C}^b_a & \mathfrak{D}^b_a \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ L^p_{c,loc}(\mathbb{R};\mathcal{U}) \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ L^p_{c,loc}(\mathbb{R};\mathcal{Y}) \end{bmatrix} \quad \text{by} \\ &\begin{bmatrix} \mathfrak{A}^b_a & \mathfrak{B}^b_a \\ \mathfrak{C}^b_a & \mathfrak{D}^b_a \end{bmatrix} := \begin{bmatrix} \delta_b & 0 \\ 0 & \pi_{[a,b]}\mathcal{P}^{\mathcal{U}}_{\mathcal{Y}} \end{bmatrix} \widetilde{\mathfrak{T}}^b_a \begin{bmatrix} 1 & 0 \\ 0 & \rho_{[a,b]} \end{bmatrix}, \quad a < b, \end{aligned} \tag{4.11}$$

and $\mathfrak{A}^a_a := 1_{\mathcal{X}}$, $\mathfrak{B}^a_a := 0$, $\mathfrak{C}^a_a := 0$ and $\mathfrak{D}^a_a := 0$. These operators inherit continuity from $\widetilde{\mathfrak{T}}^b_a$. We next prove the crucial implication

$$\begin{bmatrix} x \\ w \end{bmatrix} \in W[a,b] \quad \Longrightarrow \quad \begin{bmatrix} x(b) \\ \pi_{[a,b]}\mathcal{P}^{\mathcal{U}}_{\mathcal{Y}} w \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^b_a & \mathfrak{B}^b_a \\ \mathfrak{C}^b_a & \mathfrak{D}^b_a \end{bmatrix} \begin{bmatrix} x(a) \\ \pi_{[a,b]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w \end{bmatrix}. \tag{4.12}$$

Let therefore $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W[a,b]$ be arbitrary, so that $\begin{bmatrix} x \\ w \end{bmatrix} = \widetilde{\mathfrak{T}}_a^b \begin{bmatrix} x(a) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix}$ and

$$\begin{bmatrix} \mathfrak{A}_a^b & \mathfrak{B}_a^b \\ \mathfrak{C}_a^b & \mathfrak{D}_a^b \end{bmatrix} \begin{bmatrix} x(a) \\ \pi_{[a,b]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} = \begin{bmatrix} \delta_b & 0 \\ 0 & \pi_{[a,b]} \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} \end{bmatrix} \widetilde{\mathfrak{T}}_a^b \begin{bmatrix} x(a) \\ \rho_{[a,b]} \pi_{[a,b]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix}$$
$$= \begin{bmatrix} \delta_b & 0 \\ 0 & \pi_{[a,b]} \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} x(b) \\ \pi_{[a,b]} \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix}.$$

Part 2 (Some algebraic properties of $\mathfrak{A}_a^b$, $\mathfrak{B}_a^b$, $\mathfrak{C}_a^b$ and $\mathfrak{D}_a^b$): We now prove that the operators $\mathfrak{A}_a^b$, $\mathfrak{B}_a^b$, $\mathfrak{C}_a^b$ and $\mathfrak{D}_a^b$ have all the algebraic properties which are assumed in [Sta05, Thm 2.2.14]. It is trivial that (4.11) implies the equality

$$\begin{bmatrix} \mathfrak{A}_a^b & \mathfrak{B}_a^b \\ \mathfrak{C}_a^b & \mathfrak{D}_a^b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \pi_{[a,b]} \end{bmatrix} \begin{bmatrix} \mathfrak{A}_a^b & \mathfrak{B}_a^b \\ \mathfrak{C}_a^b & \mathfrak{D}_a^b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \pi_{[a,b]} \end{bmatrix}. \tag{4.13}$$

We proceed by verifying the following time-invariance property:

$$\forall a \le b, c \in \mathbb{R}: \quad \begin{bmatrix} \mathfrak{A}_{a-c}^{b-c} & \mathfrak{B}_{a-c}^{b-c} \\ \mathfrak{C}_{a-c}^{b-c} & \mathfrak{D}_{a-c}^{b-c} \end{bmatrix} = \begin{bmatrix} \mathfrak{A}_a^b & \mathfrak{B}_a^b \tau^{-c} \\ \tau^c \mathfrak{C}_a^b & \tau^c \mathfrak{D}_a^b \tau^{-c} \end{bmatrix}. \tag{4.14}$$

The case $a = b$ is trivial and therefore we assume that $a < b$.

Let $\widetilde{\xi} \in \mathcal{X}$ and $\widetilde{u} \in L^p([a-c, b-c]; \mathcal{U})$ be arbitrary. By part 1 of this proof we can find $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in W[a-c, b-c]$ such that $\widetilde{x}(a-c) = \widetilde{\xi}$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w} = \widetilde{u}$. By (4.8) we then have that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] := \tau^{-c} \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ is an element of $W[a,b]$, and hence by (4.12):

$$\begin{bmatrix} \mathfrak{A}_{a-c}^{b-c} & \mathfrak{B}_{a-c}^{b-c} \\ \mathfrak{C}_{a-c}^{b-c} & \mathfrak{D}_{a-c}^{b-c} \end{bmatrix} \begin{bmatrix} \widetilde{\xi} \\ \pi_{[a-c,b-c]} \widetilde{u} \end{bmatrix} = \begin{bmatrix} \widetilde{x}(b-c) \\ \pi_{[a-c,b-c]} \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} \widetilde{w} \end{bmatrix}$$
$$= \begin{bmatrix} x(b) \\ \tau^c \pi_{[a,b]} \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} = \begin{bmatrix} \mathfrak{A}_a^b & \mathfrak{B}_a^b \\ \tau^c \mathfrak{C}_a^b & \tau^c \mathfrak{D}_a^b \end{bmatrix} \begin{bmatrix} x(a) \\ \pi_{[a,b]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix}$$
$$= \begin{bmatrix} \mathfrak{A}_a^b & \mathfrak{B}_a^b \\ \tau^c \mathfrak{C}_a^b & \tau^c \mathfrak{D}_a^b \end{bmatrix} \begin{bmatrix} \widetilde{x}(a-c) \\ \pi_{[a,b]} \tau^{-c} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w} \end{bmatrix}$$
$$= \begin{bmatrix} \mathfrak{A}_a^b & \mathfrak{B}_a^b \tau^{-c} \\ \tau^c \mathfrak{C}_a^b & \tau^c \mathfrak{D}_a^b \tau^{-c} \end{bmatrix} \begin{bmatrix} \widetilde{\xi} \\ \pi_{[a-c,b-c]} \widetilde{u} \end{bmatrix}.$$

From (4.11) we get $\begin{bmatrix} \mathfrak{B}_{a-c}^{b-c} \\ \mathfrak{D}_{a-c}^{b-c} \end{bmatrix} (\pi_{(-\infty,a-c)} + \pi_{(b-c,\infty)}) = 0$ and

$$\begin{bmatrix} \mathfrak{B}_a^b \\ \mathfrak{D}_a^b \end{bmatrix} \tau^{-c} (\pi_{(-\infty,a-c)} + \pi_{(b-c,\infty)}) = \begin{bmatrix} \mathfrak{B}_a^b \\ \mathfrak{D}_a^b \end{bmatrix} (\pi_{(-\infty,a)} + \pi_{(b,\infty)}) \tau^{-c} = 0.$$

We have proved (4.14), since $\pi_{[a-c,b-c]} + \pi_{(-\infty,a-c)} + \pi_{(b-c,\infty)} = 1$.

We proceed by verifying the following composition identities, valid for all $a \le c \le b$:

$$\begin{aligned} \mathfrak{A}_a^b = \mathfrak{A}_c^b \mathfrak{A}_a^c, \quad \mathfrak{B}_a^b = \mathfrak{A}_c^b \mathfrak{B}_a^c + \mathfrak{B}_c^b, \quad \mathfrak{C}_a^b = \mathfrak{C}_a^c + \mathfrak{C}_c^b \mathfrak{A}_a^c, \\ \text{and} \quad \mathfrak{D}_a^b = \mathfrak{D}_a^c + \mathfrak{C}_c^b \mathfrak{B}_a^c + \mathfrak{D}_c^b. \end{aligned} \tag{4.15}$$

The cases where $a = c$ or $c = b$ are trivial, so we treat only the case $a < c < b$. Let $x_a \in \mathcal{X}$ and $u \in L^p([a,b]; \mathcal{U})$ be arbitrary and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ be the unique element of

$W[a,b]$ such that $x(a)=a$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}w=u$. Then (4.9) yields that $\rho_{[a,c]}\left[\begin{smallmatrix}x\\w\end{smallmatrix}\right]\in W[a,c]$ and by (4.10), $\rho_{[c,b]}\left[\begin{smallmatrix}x\\w\end{smallmatrix}\right]\in W[c,b]$. Now we get from (4.12) that:

$$\begin{bmatrix}x(b)\\ \pi_{[a,b]}\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}w\end{bmatrix}=\begin{bmatrix}\mathfrak{A}_a^b & \mathfrak{B}_a^b\\ \mathfrak{C}_a^b & \mathfrak{D}_a^b\end{bmatrix}\begin{bmatrix}x_a\\ \pi_{[a,b]}u\end{bmatrix},$$
$$\begin{bmatrix}x(c)\\ \pi_{[a,c]}\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}w\end{bmatrix}=\begin{bmatrix}\mathfrak{A}_a^c & \mathfrak{B}_a^c\\ \mathfrak{C}_a^c & \mathfrak{D}_a^c\end{bmatrix}\begin{bmatrix}x_a\\ \pi_{[a,c]}u\end{bmatrix}\quad\text{and}$$
$$\begin{bmatrix}x(b)\\ \pi_{[c,b]}\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}w\end{bmatrix}=\begin{bmatrix}\mathfrak{A}_c^b & \mathfrak{B}_c^b\\ \mathfrak{C}_c^b & \mathfrak{D}_c^b\end{bmatrix}\begin{bmatrix}x(c)\\ \pi_{[c,b]}u\end{bmatrix}.$$

From these identities we eliminate $x(c)$ in order to get (4.15):

$$\begin{aligned}\begin{bmatrix}\mathfrak{A}_a^b & \mathfrak{B}_a^b\\ \mathfrak{C}_a^b & \mathfrak{D}_a^b\end{bmatrix}\begin{bmatrix}x_a\\ \pi_{[a,b]}u\end{bmatrix}&=\begin{bmatrix}x(b)\\ \pi_{[a,b]}\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}w\end{bmatrix}=\begin{bmatrix}x(b)\\ (\pi_{[c,b]}+\pi_{[a,c]})\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}w\end{bmatrix}\\
&=\begin{bmatrix}\mathfrak{A}_c^b & \mathfrak{B}_c^b\\ \mathfrak{C}_c^b & \mathfrak{D}_c^b\end{bmatrix}\begin{bmatrix}x(c)\\ \pi_{[c,b]}u\end{bmatrix}+\begin{bmatrix}0\\ \pi_{[a,c]}\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}w\end{bmatrix}\\
&=\left(\begin{bmatrix}\mathfrak{A}_c^b & \mathfrak{B}_c^b\\ \mathfrak{C}_c^b & \mathfrak{D}_c^b\end{bmatrix}\begin{bmatrix}\mathfrak{A}_a^c & \mathfrak{B}_a^c\pi_{[a,c]}\\ 0 & \pi_{[c,b]}\end{bmatrix}+\begin{bmatrix}0 & 0\\ \mathfrak{C}_a^c & \mathfrak{D}_a^c\pi_{[a,c]}\end{bmatrix}\right)\begin{bmatrix}x_a\\ \pi_{[a,b]}u\end{bmatrix}.\end{aligned}$$

In addition, it is assumed in [Sta05, Theorem 2.2.14] that $\lim_{t\to0+}\mathfrak{A}_0^tx_0=x_0$ for all $x_0\in\mathcal{X}$. Also this condition holds because of the continuity of the state component of a trajectory at zero, cf. (4.11) and $W[0,T]\subset\begin{bmatrix}C([0,T;\mathcal{X})\\ L^p([0,T];\mathcal{W})\end{bmatrix}$:

$$\begin{bmatrix}x\\ w\end{bmatrix}=\widetilde{\mathfrak{T}}_0^T\begin{bmatrix}x_0\\ 0\end{bmatrix}\in W[0,T]\quad\Longrightarrow\quad\lim_{t\to0+}\mathfrak{A}_0^tx_0=\lim_{t\to0+}x(t)=x(0)=x_0.$$

Part 3 (The i/s/o system $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$*)*: As is pointed out in the comment in the proof of [Sta05, Thm 2.2.14], combining (4.13) and (4.15) allows us to apply that theorem to $\begin{bmatrix}\mathfrak{A}_a^b & \mathfrak{B}_a^b\\ \mathfrak{C}_a^b & \mathfrak{D}_a^b\end{bmatrix}$ in the following way. If we define

$$\begin{aligned}&\mathfrak{A}^tx_0=\mathfrak{A}_0^tx_0,\ x_0\in\mathcal{X},t\geq0, && \mathfrak{B}u=\lim_{a\to-\infty}\mathfrak{B}_a^0\pi_-u,\ u\in L_c^p(\mathbb{R}^-;\mathcal{U}),\\
&\mathfrak{C}x_0=\rho_+\lim_{b\to\infty}\mathfrak{C}_0^bx_0,\ x_0\in\mathcal{X}, && \mathfrak{D}u=\lim_{a\to-\infty,b\to\infty}\mathfrak{D}_a^bu,\ u\in L_{c,loc}^p(\mathbb{R};\mathcal{U}),\end{aligned}\tag{4.16}$$

then the operators $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ form an L^p-well-posed i/s/o system by [Sta05, Thm 2.2.14]. In particular, the three limits in (4.16) exist. Moreover, by that same theorem, for all $a\leq0\leq b$, all $x_0\in\mathcal{X}$ and all $u\in L_{c,loc}^p(\mathbb{R};\mathcal{U})$:

$$\begin{aligned}\mathfrak{A}_a^bx_0&=\mathfrak{A}^{b-a}x_0, & \mathfrak{B}_a^0u&=\mathfrak{B}\rho_-\pi_{[a,0]}u,\\
\mathfrak{C}_0^bx_0&=\pi_{[0,b]}\mathfrak{C}x_0\quad\text{and} & \mathfrak{D}_a^bu&=\pi_{[a,b]}\mathfrak{D}\pi_{[a,b]}u.\end{aligned}\tag{4.17}$$

The formulas [Sta05, (2.2.11) and Def. 2.2.6(iii)] corresponding to (4.16) and (4.17), respectively, look slightly different. This is because the convention in [Sta05] is that the domain of $\mathfrak{B}$ is $L_{c,loc}^p(\mathbb{R};\mathcal{U})$ and the codomain of $\mathfrak{C}$ is $L_{c,loc}^p(\mathbb{R};\mathcal{Y})$, cf. Remark 4.3.

Denote the right-hand side of (4.6) by $\widetilde{W}^+$. We first show that $W^+ \subset \widetilde{W}^+$. Let therefore $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W^+$ and $t \geq 0$ be arbitrary and denote $u := \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ and $y := \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w$. It follows from (4.7) that $\rho_{[0,t]} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W[0,t]$. Then (4.12), (4.14), (4.17) and the equality $\rho_- \tau^t \pi_{[0,t]} = \rho_- \tau^t \pi_+$ yield that:

$$\begin{aligned} x(t) &= \mathfrak{A}_0^t x(0) + \mathfrak{B}_0^t \pi_{[0,t]} u = \mathfrak{A}^t x(0) + \mathfrak{B}_{-t}^0 \tau^t \pi_{[0,t]} u \\ &= \mathfrak{A}^t x(0) + \mathfrak{B} \rho_- \pi_{[-t,0]} \tau^t \pi_{[0,t]} u = \mathfrak{A}^t x(0) + \mathfrak{B} \rho_- \tau^t \pi_{[0,t]} u \\ &= \mathfrak{A}^t x(0) + \mathfrak{B} \rho_- \tau^t \pi_+ u. \end{aligned}$$

This shows that the x-component satisfies the last line of (4.6).

We get from (4.12) that the y-component satisfies

$$\pi_{[0,t]} y = \mathfrak{C}_0^t x(0) + \mathfrak{D}_0^t \pi_{[0,b]} u, \quad t \geq 0,$$

and hence, using the equality $\rho_+ \pi_+ = \rho_+$ on $L^p_{c,loc}(\mathbb{R}; \mathcal{Y})$, we obtain that

$$y = \rho_+ \lim_{t \to \infty} \pi_{[0,t]} y = \rho_+ \left(\lim_{t \to \infty} \pi_{[0,t]} \mathfrak{C} x(0) + \pi_{[0,t]} \mathfrak{D} \pi_{[0,t]} u \right) = \mathfrak{C} x(0) + \rho_+ \mathfrak{D} \pi_+ u.$$

This shows that also the w-component satisfies the last line of (4.6), and thus, that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \widetilde{W}^+$.

Now we prove that $\widetilde{W}^+ \subset W^+$. Let $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \widetilde{W}^+$ be arbitrary and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W^+$ be the unique trajectory with $x(0) = \widetilde{x}(0)$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w}$. Then we, by the inclusion $W^+ \subset \widetilde{W}^+$, have that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \widetilde{W}^+$, i.e. that

$$\begin{aligned} \forall t \geq 0 : \quad \begin{bmatrix} x(t) \\ \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} &= \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}\tau^t \\ \mathfrak{C} & \rho_+ \mathfrak{D} \end{bmatrix} \begin{bmatrix} x(0) \\ \pi_+ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}\tau^t \\ \mathfrak{C} & \rho_+ \mathfrak{D} \end{bmatrix} \begin{bmatrix} \widetilde{x}(0) \\ \pi_+ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w} \end{bmatrix} = \begin{bmatrix} \widetilde{x}(t) \\ \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} \widetilde{w} \end{bmatrix}, \end{aligned}$$

which implies that $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W^+$. □

The following lemma yields uniqueness of the i/s/o representation given an admissible i/o pair.

Lemma 4.8. *Let* $-\infty < a < b < \infty$ *and let* $W^+ \subset \begin{bmatrix} C(\mathbb{R}^+; \mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+; \mathcal{W}) \end{bmatrix}$ *and define* $W[a,b] := \rho_{[a,b]} \tau^{-a} W^+$. *The following claims are true:*

(i) *If the well-posed i/s/o system* $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *satisfies* (4.6), *then it also satisfies*

$$\begin{aligned} W[a,b] = \Bigg\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C([a,b]; \mathcal{X}) \\ L^p([a,b]; \mathcal{W}) \end{bmatrix} \Bigg| \; \forall t \in [a,b] : \\ \begin{bmatrix} x(t) \\ \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^{t-a} & \mathfrak{B}\tau^t \\ \rho_{[a,b]} \tau^{-a} \mathfrak{C} & \rho_{[a,b]} \mathfrak{D} \end{bmatrix} \begin{bmatrix} x(a) \\ \pi_{[a,b]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \Bigg\}. \end{aligned} \tag{4.18}$$

(ii) *If* $\begin{bmatrix} \delta_a & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ *maps* $W[a,b]$ *densely into* $\begin{bmatrix} \mathcal{X} \\ L^p([a,b]; \mathcal{U}) \end{bmatrix}$, *then at most one well-posed i/s/o system* $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *satisfies* (4.18).

Proof. First we generally note that for any well-posed i/s/o system $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$:

$$\forall t' \geq t: \quad \mathfrak{B}\tau^t\pi_{[t',\infty)} = \mathfrak{B}\rho_-\pi_{[t'-t,\infty)}\tau^t = 0$$
$$\forall t \in \mathbb{R}: \quad \rho_{(-\infty,t)}\mathfrak{D}\pi_{[t,\infty)} = \rho_{(-\infty,t)}\tau^{-t}\mathfrak{D}\tau^t\pi_{[t,\infty)} = \tau^{-t}\rho_-\mathfrak{D}\pi_+\tau^t = 0$$

and, therefore, we have

$$\begin{aligned}\forall a \leq t \leq b: &\quad \mathfrak{B}\tau^t\pi_{[a,t]}u = \mathfrak{B}\tau^t\pi_{[a,b]}u, \ u \in L^p([a,b];\mathcal{U}),\\ \forall t \geq a: &\quad \mathfrak{B}\tau^t\pi_{[a,t]}u = \mathfrak{B}\tau^t\pi_{[a,\infty)}u, \ u \in L^p_{loc}([a,\infty);\mathcal{U}) \quad \text{and}\\ \forall t \geq 0: &\quad \rho_{[0,t]}\mathfrak{D}\pi_+u = \rho_{[0,t]}\mathfrak{D}\pi_{[0,t]}u, \ u \in L^p_{loc}(\mathbb{R}^+;\mathcal{U}).\end{aligned} \tag{4.19}$$

We now proceed to prove claims (i) and (ii).

(i) We denote the right-hand side of (4.18) by $\widetilde{W}[a,b]$ and use (4.6) to prove that $\widetilde{W}[a,b] = \rho_{[a,b]}\tau^{-a}W^+$.

Let $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \widetilde{W}[a,b]$ be arbitrary and define

$$\begin{bmatrix}\widetilde{x}(t)\\ \widetilde{w}\end{bmatrix} := \begin{bmatrix}\mathfrak{A}^t & \mathfrak{B}\tau^t\\ \mathfrak{C} & \rho_+(1+\mathfrak{D})\end{bmatrix}\begin{bmatrix}x(a)\\ \tau^a\pi_{[a,b]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w\end{bmatrix}, \quad t \geq 0. \tag{4.20}$$

Then $\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right] \in W^+$ by (4.6) and, moreover, $\rho_{[a,b]}\tau^{-a}\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right] = \left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ because:

$$\begin{aligned}\begin{bmatrix}(\tau^{-a}\widetilde{x})(t+a)\\ \rho_{[a,b]}\tau^{-a}\widetilde{w}\end{bmatrix} &= \begin{bmatrix}\mathfrak{A}^t & \mathfrak{B}\tau^t\\ \rho_{[a,b]}\tau^{-a}\mathfrak{C} & \rho_{[a,b]}\tau^{-a}(1+\mathfrak{D})\end{bmatrix}\begin{bmatrix}x(a)\\ \tau^a\pi_{[a,b]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w\end{bmatrix}\\ &= \begin{bmatrix}\mathfrak{A}^{(t+a)-a} & \mathfrak{B}\tau^{t+a}\\ \rho_{[a,b]}\tau^{-a}\mathfrak{C} & \rho_{[a,b]}(1+\mathfrak{D})\end{bmatrix}\begin{bmatrix}x(a)\\ \pi_{[a,b]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w\end{bmatrix}\end{aligned} \tag{4.21}$$

for all $t+a \in [a,b]$ and the second line obviously equals $\left[\begin{smallmatrix}x(t+a)\\ w\end{smallmatrix}\right]$ for all $t+a \in [a,b]$, cf. (4.18).

Conversely, let $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \rho_{[a,b]}\tau^{-a}W^+$. This means that there exists some $\left[\begin{smallmatrix}\widehat{x}\\ \widehat{w}\end{smallmatrix}\right]$ in W^+, such that $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] = \rho_{[a,b]}\tau^{-a}\left[\begin{smallmatrix}\widehat{x}\\ \widehat{w}\end{smallmatrix}\right]$. This $\left[\begin{smallmatrix}\widehat{x}\\ \widehat{w}\end{smallmatrix}\right] \in W^+$ satisfies

$$\forall t \geq 0: \quad \begin{bmatrix}\widehat{x}(t)\\ \widehat{w}\end{bmatrix} = \begin{bmatrix}\mathfrak{A}^t & \mathfrak{B}\tau^t\\ \mathfrak{C} & \rho_+(1+\mathfrak{D})\end{bmatrix}\begin{bmatrix}x(a)\\ \pi_+\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\widehat{w}\end{bmatrix}$$

by (4.6). Using (4.19) and $\tau^a\pi_{[a,b]}w = \pi_{[0,b-a]}\widehat{w}$, we get for all $t \in [0, b-a]$ that:

$$\begin{bmatrix}\mathfrak{B}\tau^t\\ \rho_{[0,b-a]}\rho_+(1+\mathfrak{D})\end{bmatrix}\pi_+\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\widehat{w} = \begin{bmatrix}\mathfrak{B}\tau^t\\ \rho_{[0,b-a]}(1+\mathfrak{D})\end{bmatrix}\pi_+\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\widehat{w}$$
$$\begin{bmatrix}\mathfrak{B}\tau^t\\ \rho_{[0,b-a]}(1+\mathfrak{D})\end{bmatrix}\pi_{[0,b-a]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}\widehat{w} = \begin{bmatrix}\mathfrak{B}\tau^t\\ \rho_{[0,b-a]}\rho_+(1+\mathfrak{D})\end{bmatrix}\tau^a\pi_{[a,b]}\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}w$$

and, therefore, $\left[\begin{smallmatrix}\widehat{x}\\ \widehat{w}\end{smallmatrix}\right]$ coincides with $\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right]$ defined in (4.20) on $[0, b-a]$. Thus (4.21) holds for $t+a \in [a,b]$ with $\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix}\widehat{x}\\ \widehat{w}\end{smallmatrix}\right]$. Comparing this to (4.2) we get that

$$\begin{bmatrix}x\\ w\end{bmatrix} = \rho_{[a,b]}\tau^{-a}\begin{bmatrix}\widehat{x}\\ \widehat{w}\end{bmatrix} \in \widetilde{W}[a,b].$$

(ii) The space $W[a,b]$ determines the space $W[0,b-a]$ uniquely through (4.8). Letting $T := b-a$, we get that $W[0,b-a]$ determines the continuous operators

$$\mathfrak{A}^t, t \in [0,T],\ \mathfrak{B}\tau^T\pi_{[0,T]} = \mathfrak{B}\pi_{[-T,0]}\tau^T,\ \rho_{[0,T]}\mathfrak{C} \text{ and } \rho_{[0,T]}\mathfrak{D}\pi_{[0,T]} \tag{4.22}$$

on dense subspaces of their domains through (4.18). Therefore the operators in (4.22) are uniquely determined by $W[a,b]$. Furthermore, [Sta05, Lemma 2.4.3] yields that this information uniquely determines the well-posed i/s/o system $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ through the equalities:

$$\begin{aligned}
\mathfrak{A}^t &= \mathfrak{A}^{nT}\mathfrak{A}^{t-nT},\ n \in \mathbb{Z}^+,\ t \in [nT,(n+1)T],\\
\mathfrak{B} &= \sum_{n=0}^{\infty} \mathfrak{A}^{nT}\mathfrak{B}\rho_-\pi_{[-T,0]}\tau^{-nT},\\
\mathfrak{C} &= \rho_+\sum_{n=0}^{\infty} \tau^{-nT}\pi_{[0,T]}\mathfrak{C}\mathfrak{A}^{nT} \text{ and}\\
\mathfrak{D} &= \sum_{n=-\infty}^{\infty} \tau^{-nT}(\pi_+\mathfrak{C}\mathfrak{B}\pi_{[-T,0]} + \pi_{[0,T]}\mathfrak{D}\pi_{[0,T]})\tau^{nT}.
\end{aligned}$$

□

We now arrive at one of the main results of this paper.

Theorem 4.9. *Assume that $\Sigma_{s/s} = (\mathfrak{W}^p; \mathcal{X}, \mathcal{W})$ is a well-posed s/s system that has the admissible i/o pair $(\mathcal{U},\mathcal{Y})$. Then Σ has a unique i/s/o representation $\left(\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]; \mathcal{X},\mathcal{U},\mathcal{Y}\right)$ corresponding to this i/o pair. This i/s/o representation satisfies* (4.6) *with $W^+ := \mathfrak{W}^p$.*

Proof. Proposition 3.10 yields that $\mathfrak{W}^p$ is given by (3.13) and Corollary 3.2(iii) yields that (4.5) holds. Lemma 4.6(i) holds by Definition 3.1 and Proposition 3.7(i). The well-posed i/s/o system $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ that we constructed in the proof of Lemma 4.7 satisfies (4.6). By Proposition 3.7(i) and Lemma 4.8(ii), $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B}\\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ is the unique i/s/o system that satisfies (4.2). □

In Theorem 6.4 below we prove the converse direction of Theorem 4.9, i.e. that every well-posed i/s/o system on $(\mathcal{X},\mathcal{U},\mathcal{U})$ generates a unique well-posed s/s system $\left(W^+; \mathcal{X}, \left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right]\right)$ through (4.6).

In the sequel we need the concept of flow inversion and we now give an adaptation of the version, which was presented in [Sta05].

Definition 4.10. *Let $\mathcal{X}$ and $\mathcal{W}$ be Banach spaces, where $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, $\mathcal{U} = \mathcal{U}_1 \dotplus \mathcal{U}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \dotplus \mathcal{Y}_2$. Let Σ be an L^p-well-posed s/s system on $(\mathcal{X},\mathcal{W})$, for which the i/o pair $(\mathcal{U},\mathcal{Y}) = \left(\begin{bmatrix}\mathcal{U}_1\\ \mathcal{U}_2\end{bmatrix}, \begin{bmatrix}\mathcal{Y}_1\\ \mathcal{Y}_2\end{bmatrix}\right)$ is admissible, and let the corresponding* *i/s/o representation be given by:*

$$\Sigma_{i/s/o} = \left(\begin{bmatrix} \mathfrak{A} & \mathfrak{B}_1 & \mathfrak{B}_2 \\ \mathfrak{C}_1 & \mathfrak{D}_{11} & \mathfrak{D}_{12} \\ \mathfrak{C}_2 & \mathfrak{D}_{21} & \mathfrak{D}_{22} \end{bmatrix}; \mathcal{X}, \begin{bmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \end{bmatrix}, \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix} \right), \tag{4.23}$$

where, for instance, $\mathfrak{D}_{12}$ *is the restriction of* $\mathcal{P}_{\mathcal{Y}_1}^{\mathcal{Y}_2}\mathfrak{D}$ *to* $L^p_{c,loc}(\mathbb{R};\mathcal{U}_2)$.

The i/s/o representation $\Sigma_{i/s/o}$ *is* partially flow invertible *with respect to the change* $\mathcal{U}_2 \leftrightarrow \mathcal{Y}_2$ *if* $\left(\begin{bmatrix} \mathcal{U}_1 \\ \mathcal{Y}_2 \end{bmatrix}, \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{U}_2 \end{bmatrix} \right)$ *is an admissible i/o pair of* Σ*. In that case, the i/s/o representation* $\Sigma_{i/s/o}^{\curvearrowleft}$ *of* Σ*, which corresponds to the admissible i/o pair* $\left(\begin{bmatrix} \mathcal{U}_1 \\ \mathcal{Y}_2 \end{bmatrix}, \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{U}_2 \end{bmatrix} \right)$*, is called the* partial flow inverse *of* $\Sigma_{i/s/o}$.

If $\mathcal{U}_1 = \{0\}$ *and* $\mathcal{Y}_1 = \{0\}$ *then we say that the flow inversion is* full.

By definition, flow inversion of an i/s/o representation results in another i/s/o representation of the same s/s system. The core idea of the s/s approach is that the external signals can be split into inputs and outputs in various ways without changing the *system* itself. The i/s/o representations change under flow inversion, but the relationships between all signals is preserved, and since we here define a s/s system through its trajectories this means that the system itself is preserved.

The following proposition gives useful characterisations of flow invertibility.

Proposition 4.11. *With* $-\infty < a < b < \infty$ *and the same set-up as in Definition 4.10, the following conditions are equivalent:*

(i) *The i/s/o system* $\Sigma_{i/s/o}$ *is partially flow invertible with respect to* $\mathcal{U}_2 \leftrightarrow \mathcal{Y}_2$.
(ii) *The operator* $\mathfrak{D}_{22}$ *has an inverse in* $TIC^p_{loc}(\mathcal{Y}_2;\mathcal{U}_2)$.
(iii) *The operator* $\rho_{[a,b]}\mathfrak{D}_{22}\pi_{[a,b]}$ *maps* $L^p([a,b];\mathcal{U}_2)$ *one-to-one onto* $L^p([a,b];\mathcal{Y}_2)$.

If the above equivalent conditions hold, then $\Sigma_{i/s/o}^{\curvearrowleft}$ *is given by*

$$\begin{aligned} \begin{bmatrix} \mathfrak{A}^{\curvearrowleft} & \mathfrak{B}_1^{\curvearrowleft}\tau & \mathfrak{B}_2^{\curvearrowleft}\tau \\ \mathfrak{C}_1^{\curvearrowleft} & \mathfrak{D}_{11}^{\curvearrowleft} & \mathfrak{D}_{12}^{\curvearrowleft} \\ \mathfrak{C}_2^{\curvearrowleft} & \mathfrak{D}_{21}^{\curvearrowleft} & \mathfrak{D}_{22}^{\curvearrowleft} \end{bmatrix} &= \begin{bmatrix} \mathfrak{A} & \mathfrak{B}_1\tau & \mathfrak{B}_2\tau \\ \mathfrak{C}_1 & \mathfrak{D}_{11} & \mathfrak{D}_{12} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathfrak{C}_2 & \mathfrak{D}_{21} & \mathfrak{D}_{22} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & -\mathfrak{B}_2\tau \\ 0 & 1 & -\mathfrak{D}_{12} \\ 0 & 0 & \mathfrak{D}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A} & \mathfrak{B}_1\tau & 0 \\ \mathfrak{C}_1 & \mathfrak{D}_{11} & 0 \\ -\mathfrak{C}_2 & -\mathfrak{D}_{21} & 1 \end{bmatrix}. \end{aligned} \tag{4.24}$$

Proof. See [Sta05, Thms 6.3.5 and 6.6.1 and Cor. 6.6.3]. □

Note that condition (iii) of Proposition 4.11 holds for *some* a and b if and only if it holds for *all* a and b, because condition (i) of the proposition is independent of a and b.

In order to prove the final theorem of this section we need to do the following small trick. Let $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} \mathfrak{A}_1 & \mathfrak{B}_1 \\ \mathfrak{C}_1 & \mathfrak{D}_1 \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}_1, \mathcal{Y}_1\right)$ be an i/s/o representation of an L^p-well-posed s/s system. Embed Σ into a larger system Σ_{ext}, whose input and output spaces are both $\mathcal{W}$, by setting

$$\begin{bmatrix} x(t) \\ w \end{bmatrix} = \begin{bmatrix} \mathfrak{A}_1^t & \mathfrak{B}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \tau^t \\ \mathfrak{C}_1 & \rho_+ (1 + \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}) \end{bmatrix} \begin{bmatrix} x(0) \\ \pi_+ \widetilde{w} \end{bmatrix}, \quad t \geq 0. \tag{4.25}$$

The system Σ_{ext} is illustrated in Figure 1.

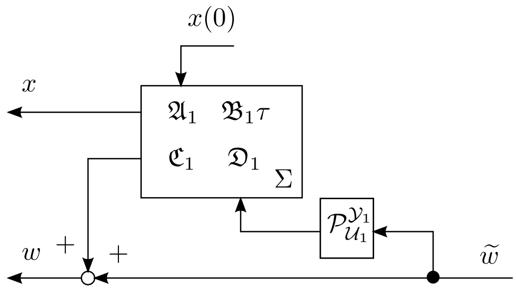

FIGURE 1. An i/s/o representation of the extended system Σ_{ext}, which has state trajectory x, input $\widetilde{w}$ and output w. The full flow inverse of Σ_{ext} is obtained by simply reversing the direction of the two signals at the bottom.

Partial flow inversion of Σ_{ext} will be the main tool in our proof of Theorem 4.13, which can be considered to be the main result of this section. First, however, we need to to take a closer look at Σ_{ext}.

Lemma 4.12. *The system Σ_{ext} defined in* (4.25) *has the following five properties:*

(i) *The i/s/o system Σ_{ext} is L^p well posed.*

(ii) *Every trajectory $\left[\begin{smallmatrix} x \\ w \\ \widetilde{w} \end{smallmatrix}\right]$ of Σ_{ext} satisfies $\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \widetilde{w} = \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w$.*

(iii) *The triple $\begin{bmatrix} x \\ \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1} w \\ \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w \end{bmatrix}$ is a trajectory of Σ if and only if $\begin{bmatrix} x \\ w \\ \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w \end{bmatrix}$ is a trajectory of Σ_{ext}.*

(iv) *For any $T > 0$ we have that*

$$\left(\rho_{[0,T]} (1 + \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}) \pi_{[0,T]}\right)^{-1} = \rho_{[0,T]} (1 - \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}) \pi_{[0,T]}, \tag{4.26}$$

where both operators are bounded on $L^p([0,T]; \mathcal{W})$.

(v) *The system Σ_{ext} is (fully) flow invertible in the sense that w can be chosen as input and $\widetilde{w}$ as output. The corresponding i/s/o representation is*

$$\begin{bmatrix} x(t) \\ \widetilde{w} \end{bmatrix} = \begin{bmatrix} \mathfrak{A}_1^t & \mathfrak{B}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \tau^t \\ -\mathfrak{C}_1 & \rho_+(1 - \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}) \end{bmatrix} \begin{bmatrix} x(0) \\ \pi_+ w \end{bmatrix}, \quad t \geq 0. \tag{4.27}$$

Proof. (i) We prove that Σ_{ext} has the properties listed in Definition 4.2 by using the corresponding properties of Σ, which we assumed to be well posed. The semigroup $\mathfrak{A}_1$ and the state/output map $\mathfrak{C}_1$ are the same in both systems and thus Σ_{ext} has properties (i) and (iii) of Definition 4.2.

In proving properties (ii) and (iv) we need the fact that the almost everywhere pointwise projection $\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}$ is time invariant and static, so that e.g. $\tau^t \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} = \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \tau^t$ and $\pi_- \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} = \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \pi_-$. We obtain that

$$\begin{aligned} \mathfrak{A}_1^t \mathfrak{B}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} &= \mathfrak{B}_1 \rho_- \tau^t \pi_- \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} = \mathfrak{B}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \rho_- \tau^t \pi_- \quad \text{and} \\ \rho_+(1 + \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}) \pi_- &= \rho_+ \mathfrak{D}_1 \pi_- \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} = \mathfrak{C}_1 \mathfrak{B}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}. \end{aligned} \tag{4.28}$$

(ii) From (4.25) and the fact that $\mathfrak{C}_1 x(0) + \rho_+ \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \pi_+ \widetilde{w}$ lies in $L^p_{loc}(\mathbb{R}^+; \mathcal{Y}_1)$, we immediately get

$$\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w = \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} (\mathfrak{C}_1 x(0) + \widetilde{w} + \rho_+ \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \pi_+ \widetilde{w}) = \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \widetilde{w}.$$

(iii) The triple $\begin{bmatrix} x \\ w \\ \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w \end{bmatrix}$ is by Definition 4.4 a trajectory of Σ_{ext} if and only if (4.25) holds with $\widetilde{w} = \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w$, which is true if and only if

$$\begin{bmatrix} x(t) \\ \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1} w \\ \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w \end{bmatrix} = \begin{bmatrix} \mathfrak{A}_1^t & \mathfrak{B}_1 \tau^t \\ \mathfrak{C}_1 & \rho_+ \mathfrak{D}_1 \\ 0 & \rho_+ \end{bmatrix} \begin{bmatrix} x(0) \\ \pi_+ \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w \end{bmatrix}, \quad t \geq 0.$$

This is obviously equivalent to $\begin{bmatrix} x \\ \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1} w \\ \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w \end{bmatrix}$ being a trajectory of Σ.

(iv) This claim follows from the computation

$$\begin{aligned} \rho_{[0,T]} (1 \pm \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}) \pi_{[0,T]} \rho_{[0,T]} (1 \mp \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}) \pi_{[0,T]} & \\ = (1 \pm \rho_{[0,T]} \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \pi_{[0,T]}) (1 \mp \rho_{[0,T]} \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \pi_{[0,T]}) & \\ = 1 - (\rho_{[0,T]} \mathfrak{D}_1 \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \pi_{[0,T]})^2 = 1, & \end{aligned}$$

because $\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \pi_{[0,T]} \rho_{[0,T]} \mathfrak{D}_1 = 0$. (Here 1 on the first line stands for the identity operator in $L^p_{c,loc}(\mathbb{R}; \mathcal{W})$ and on the other lines 1 stands for the identity in $L^p([0,T]; \mathcal{W})$.)

(v) By claim (iv) and Proposition 4.11 we have that Σ_{ext} is flow invertible (with the flow inverse being a well-posed i/s/o system). Using claim (ii) of this lemma one sees that (4.27) is the flow inverse of (4.25):

$$\begin{aligned} w &= \mathfrak{C}_1 x(0) + \rho_+(1+\mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1})\pi_+\widetilde{w} \implies \\ \widetilde{w} &= w - \mathfrak{C}_1 x(0) - \rho_+\mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\pi_+\widetilde{w} \\ &= -\mathfrak{C}_1 x(0) + \rho_+(1-\mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1})\pi_+ w \end{aligned}$$

and

$$x(t) = \mathfrak{A}_1^t x(0) + \mathfrak{B}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\tau^t\pi_+\widetilde{w} = \mathfrak{A}_1^t x(0) + \mathfrak{B}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\tau^t\pi_+ w.$$

□

We now present a theorem, which characterises the admissible i/o pairs of a well-posed s/s system and gives the corresponding i/s/o representations.

Theorem 4.13. *Let Σ be an L^p-well-posed s/s system with admissible i/o pair $(\mathcal{U}_1,\mathcal{Y}_1)$ and corresponding i/s/o representation $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix}\mathfrak{A}_1 & \mathfrak{B}_1\\ \mathfrak{C}_1 & \mathfrak{D}_1\end{smallmatrix}\right];\mathcal{X},\mathcal{U}_1,\mathcal{Y}_1\right)$.*

Then the i/o pair $(\mathcal{U}_2,\mathcal{Y}_2)$ is admissible for Σ if and only if

$$\left(\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1)\right)^{-1} = \left(\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}\big|_{\mathcal{U}_1} + \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}\big|_{\mathcal{Y}_1}\mathfrak{D}_1\right)^{-1} \in TIC_{loc}^p(\mathcal{U}_2;\mathcal{U}_1), \tag{4.29}$$

or equivalently, if and only if

$$\left(\mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} - \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1}\right)\big|_{\mathcal{Y}_2}^{-1} = \left(\mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{Y}_2} - \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1}\big|_{\mathcal{Y}_2}\right)^{-1} \in TIC_{loc}^p(\mathcal{Y}_1;\mathcal{Y}_2). \tag{4.30}$$

If the i/o pair $(\mathcal{U}_2,\mathcal{Y}_2)$ is admissible for Σ, then the corresponding i/s/o representation $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix}\mathfrak{A}_2 & \mathfrak{B}_2\\ \mathfrak{C}_2 & \mathfrak{D}_2\end{smallmatrix}\right];\mathcal{X},\mathcal{U}_2,\mathcal{Y}_2\right)$ of Σ is given by (for all $t \ge 0$):

$$\begin{aligned} \begin{bmatrix} \mathfrak{A}_2^t & \mathfrak{B}_2\tau^t \\ \mathfrak{C}_2 & \mathfrak{D}_2 \end{bmatrix} &= \begin{bmatrix} \mathfrak{A}_1^t & \mathfrak{B}_1\tau^t \\ \mathcal{P}_{\mathcal{Y}_2}^{\mathcal{U}_2}\mathfrak{C}_1 & \mathcal{P}_{\mathcal{Y}_2}^{\mathcal{U}_2}(1+\mathfrak{D}_1) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}\mathfrak{C}_1 & \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1) \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -\mathfrak{B}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{Y}_2}\tau^t \\ 0 & \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1}\big|_{\mathcal{Y}_2} - \mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{Y}_2} \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{A}_1^t & \mathfrak{B}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{U}_2}\tau^t \\ \mathfrak{C}_1 & \mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{U}_2} - \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1}\big|_{\mathcal{U}_2} \end{bmatrix}. \end{aligned} \tag{4.31}$$

Proof. Let Σ_{ext} be the i/s/o system in (4.25) and write

$$\begin{bmatrix} \widetilde{y} \\ \widetilde{u} \end{bmatrix} := \begin{bmatrix} \mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1} \\ \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \end{bmatrix}\widetilde{w} \quad\text{and}\quad \begin{bmatrix} y \\ u \end{bmatrix} := \begin{bmatrix} \mathcal{P}_{\mathcal{Y}_2}^{\mathcal{U}_2} \\ \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix} w.$$

Note that we use different decompositions of $\mathcal{W}$ for $\widetilde{w}$ and w. With respect to these decompositions, (4.25) splits into

$$\begin{bmatrix} x(t) \\ \begin{bmatrix} y \\ u \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathfrak{A}_1^t & \begin{bmatrix} 0 & \mathfrak{B}_1 \end{bmatrix}\tau^t \\ \begin{bmatrix} \mathcal{P}_{\mathcal{Y}_2}^{\mathcal{U}_2} \\ \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix}\mathfrak{C}_1 & \rho_+\begin{bmatrix} \mathcal{P}_{\mathcal{Y}_2}^{\mathcal{U}_2} \\ \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix}\begin{bmatrix} 1|_{\mathcal{Y}_1} & (1+\mathfrak{D}_1) \end{bmatrix} \end{bmatrix} \begin{bmatrix} x(0) \\ \pi_+\begin{bmatrix} \widetilde{y} \\ \widetilde{u} \end{bmatrix} \end{bmatrix} \tag{4.32}$$

and (4.27) splits in a similar way into

$$\begin{bmatrix} x(t) \\ \begin{bmatrix} \widetilde{u} \\ \widetilde{y} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathfrak{A}_1^t & \mathfrak{B}_1 \begin{bmatrix} \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{U}_2} & \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{Y}_2} \end{bmatrix} \tau^t \\ -\begin{bmatrix} 0 \\ \mathfrak{C}_1 \end{bmatrix} & \rho_+ \begin{bmatrix} \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{U}_2} & \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{Y}_2} \\ (\mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1} - \mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1})\big|_{\mathcal{U}_2} & (\mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1} - \mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1})\big|_{\mathcal{Y}_2} \end{bmatrix} \end{bmatrix} \times \begin{bmatrix} x(0) \\ \pi_+ \begin{bmatrix} u \\ y \end{bmatrix} \end{bmatrix}. \tag{4.33}$$

We swapped places of u and y between (4.32) and (4.33) in order to be able to apply (4.24) directly to these formulas.

Corollary 3.12 yields that $(\mathcal{U}_2, \mathcal{Y}_2)$ is an admissible i/o pair of Σ if and only if $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix}$ maps $\mathfrak{W}_0^p[0,T]$ one-to-one onto $L^p([0,T];\mathcal{U}_2)$. We prove that the latter condition is equivalent to condition (4.29) using Proposition 4.11, which says that (4.29) is equivalent to the statement that $\rho_{[0,T]}\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1)\pi_{[0,T]}$ maps $L^p([0,T];\mathcal{U}_1)$ one-to-one onto $L^p([0,T];\mathcal{U}_2)$. Proposition 4.11 says that bijectivity of this operator is equivalent to (4.29).

From (3.10) and (3.4) we get that

$$\mathfrak{W}_0^p[0,T] = \left\{ \rho_{[0,T]} \begin{bmatrix} x \\ w \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p,\ x(0)=0 \right\}.$$

Lemma 4.12(iii), (4.19) and (4.25) then yield that

$$\mathfrak{W}_0^p[0,T] = \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \;\middle|\; \begin{bmatrix} x(t) \\ w \end{bmatrix} = \begin{bmatrix} \mathfrak{B}\tau^t \\ \rho_{[0,T]}(1+\mathfrak{D}_1) \end{bmatrix} \pi_{[0,T]}\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} w,\ t\in[0,T] \right\} \tag{4.34}$$

with $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1} \end{bmatrix} \mathfrak{W}_0^p[0,T] = L^p([0,T];\mathcal{U}_1)$. Therefore,

$$\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix} \mathfrak{W}_0^p[0,T] = \rho_{[0,T]}\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1)\pi_{[0,T]}L^p([0,T];\mathcal{U}_1)$$

and it is obvious that $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix}$ maps $\mathfrak{W}_0^p[0,T]$ onto $L^p([0,T];\mathcal{U}_2)$ if and only if $\rho_{[0,T]}\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1)\pi_{[0,T]}$ maps $L^p([0,T];\mathcal{U}_1)$ onto $L^p([0,T];\mathcal{U}_2)$.

From (4.34) we also get that

$$u_1 \in L^p([0,T];\mathcal{U}_1),\ w = \rho_{[0,T]}(1+\mathfrak{D}_1)\pi_{[0,T]}u_1 \iff \exists x : \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}_0^p[0,T].$$

If $\rho_{[0,T]}\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1)\pi_{[0,T]}$ is injective, then $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = 0$ and $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p[0,T]$ thus imply that $\rho_{[0,T]}\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1)\pi_{[0,T]}\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}w = 0$, which implies that $\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}w = 0$. Proposition 3.7(ii) then says that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = 0$, i.e., $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix}\big|_{\mathfrak{W}_0^p[0,T]}$ is injective. A similar argument shows that injectivity of $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2} \end{bmatrix}\big|_{\mathfrak{W}_0^p[0,T]}$ implies that $\rho_{[0,T]}\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1)\pi_{[0,T]}$ is injective. This proves that $(\mathcal{U}_2, \mathcal{Y}_2)$ is admissible for Σ if and only if (4.29) holds.

In order to prove the equivalence of (4.29) and (4.30) we first note that

$$\begin{aligned}&\left(\rho_{[0,T]}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\mathcal{P}_{\mathcal{Y}_2}^{\mathcal{U}_2}\\ \mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}\end{bmatrix}\begin{bmatrix}1|_{\mathcal{Y}_1} & (1+\mathfrak{D}_1)\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\pi_{[0,T]}\right)^{-1}\\ &\quad=\rho_{[0,T]}\begin{bmatrix}\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{U}_2} & \mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1}\big|_{\mathcal{Y}_2}\\ (\mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1}-\mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1})\big|_{\mathcal{U}_2} & (\mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1}-\mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1})\big|_{\mathcal{Y}_2}\end{bmatrix}\pi_{[0,T]}\end{aligned} \tag{4.35}$$

in $L^p([0,T];\mathcal{W})$, which can be checked by direct multiplication. All the operators in (4.35) are bounded maps between Banach spaces. We make the following argument using [Sta05, Lemma A.4.2](iii). If the top-left corner of the first operator matrix, i.e, $\mathcal{P}_{\mathcal{U}_2}^{\mathcal{Y}_2}(1+\mathfrak{D}_1)$ is invertible, then the lower-right corner of the inverse, i.e. of $(\mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1}-\mathfrak{D}_1\mathcal{P}_{\mathcal{U}_1}^{\mathcal{Y}_1})\big|_{\mathcal{Y}_2}$, is also invertible and vice versa. We have now proved that (4.29) and (4.30) are equivalent.

The proof of the first line of (4.31) is now a simple application of the first line of (4.24) to (4.32), while the second line of (4.31) is proved using the second line of (4.24) and (4.33). One needs to set $\widetilde{y}=\mathcal{P}_{\mathcal{Y}_1}^{\mathcal{U}_1}\widetilde{w}=0$, because we consider trajectories of Σ, cf. Lemma 4.12(iii). Moreover, computing $\widetilde{u}$ is unnecessary when determining the trajectories $\left[\begin{smallmatrix}x\\u+y\end{smallmatrix}\right]$ of Σ. □

In applications, a system is usually given in terms of the subspace V. In the rest of the paper we therefore focus on obtaining necessary and sufficient conditions on V for this space to be a generating subspace of a s/s system. In order to proceed in this direction we need some results on i/s/o systems, which we develop next.

5. Input/state/output systems and their associated system nodes

In this section we recall the notion of an i/s/o-system node and study its connection to the i/s/o system from which it is derived. For more details on the following definitions, see e.g. [Sta05, pp. 122–123] or [Paz83].

Let A be a closed and densely defined operator on the Banach space $\mathcal{X}$. The *resolvent set* $\operatorname{Res}(A)$ of A is the set of $\lambda\in\mathbb{C}$ such that $\lambda-A$ maps $\operatorname{Dom}(A)$ one-to-one onto $\mathcal{X}$. By the closed graph theorem, $(\alpha-A)^{-1}$ is a bounded operator on $\mathcal{X}$ for all $\alpha\in\operatorname{Res}(A)$. Fix $\alpha\in\operatorname{Res}(A)$ and define $\mathcal{X}_1:=\operatorname{Dom}(A)$ with the norm $\|x\|_1:=\|(\alpha-A)x\|_{\mathcal{X}}$. Denote by $\mathcal{X}_{-1}$ the completion of $\mathcal{X}$ with respect to the norm $\|x\|_{-1}=\|(\alpha-A)^{-1}x\|_{\mathcal{X}}$. This norm is weaker than the norm $\|\cdot\|_{\mathcal{X}}$, because $\|x\|_{-1}\le\|(\alpha-A)^{-1}\|\|x\|_{\mathcal{X}}$ for all $x\in\mathcal{X}$.

The operator $\alpha-A$ maps $\mathcal{X}_1$ isomorphically onto $\mathcal{X}$. The operator A can also be considered as a continuous operator, which maps the dense subspace $\mathcal{X}_1$ of $\mathcal{X}$ into $\mathcal{X}_{-1}$ and we denote the unique continuous extension of A to an operator $\mathcal{X}\to\mathcal{X}_{-1}$ by $A|_{\mathcal{X}}$. Then for any $\alpha\in\operatorname{Res}(A)$ the operator $\alpha-A|_{\mathcal{X}}$ maps $\mathcal{X}$ isomorphically onto $\mathcal{X}_{-1}$ and $(\alpha-A|_{\mathcal{X}})^{-1}$ is the continuous extension of $(\alpha-A)^{-1}$ to $\mathcal{X}_{-1}$.

The spaces $\mathcal{X}_1$ and $\mathcal{X}_{-1}$, which we defined above, satisfy $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_{-1}$ with dense and continuous embeddings. This construction is sometimes referred to as "rigging". Different choices of $\alpha \in \operatorname{Res}(A)$ give the same triple $(\mathcal{X}_1, \mathcal{X}, \mathcal{X}_{-1})$ of spaces. The respective norms on the spaces depend on α, but the norms are nevertheless equivalent to the each other. The norm on $\mathcal{X}_1$ is equivalent to the graph norm of A. If $\mathcal{X}$ is a Hilbert space, then, so are $\mathcal{X}_1$ and $\mathcal{X}_{-1}$.

We denote $\mathbb{C}^+_\alpha := \{\lambda \in \mathbb{C} \mid \Re\lambda > \alpha\}$, for $\alpha \in \mathbb{R}$, and abbreviate $\mathbb{C}^+ := \mathbb{C}^+_0$. Let A generate a C_0 semigroup $\mathfrak{A}$ with growth bound $\omega_{\mathfrak{A}}$ on some Banach space $\mathcal{X}$, cf. Lemma 3.14. Then [Sta05, Thm 3.2.9] says that $\mathbb{C}^+_{\omega_{\mathfrak{A}}} \subset \operatorname{Res}(A)$, so that the resolvent set is non-empty. Moreover, the domain of every C_0-semigroup generator is dense, according to [Paz83, Thm 1.2.7]. Thus the following definition of an i/s/o-system node is one of the many versions equivalent to [Sta05, Def. 4.7.2]. See also [SW02].

Definition 5.1. *By an* input/state/output-system node (i/s/o-system node) *on the triple* $(\mathcal{X},\mathcal{U},\mathcal{Y})$ *of Banach spaces we mean a linear operator*

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \operatorname{Dom}(S) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

with domain $\operatorname{Dom}(S)$*, which has the following properties:*

(i) *The operator S is closed.*

(ii) *The operator $A\colon \operatorname{Dom}(A) \to \mathcal{X}$, which is defined by*

$$Ax = A\&B \begin{bmatrix} x \\ 0 \end{bmatrix} \quad \text{on } \operatorname{Dom}(A) = \left\{ x \in \mathcal{X} \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in \operatorname{Dom}(S) \right\},$$

generates a C_0 semigroup on $\mathcal{X}$.

(iii) *The operator $A\&B$ can be extended to an operator $\begin{bmatrix} A|_{\mathcal{X}} & B \end{bmatrix}$, which maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ continuously into $\mathcal{X}_{-1}$.*

(iv) *The domain of S satisfies the condition*

$$\operatorname{Dom}(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \mid A|_{\mathcal{X}} x + Bu \in \mathcal{X} \right\}. \tag{5.1}$$

We now show how to construct an i/s/o-system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ from a given i/s/o system $\left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]; \mathcal{X},\mathcal{U},\mathcal{Y}\right)$. Let therefore $\mathfrak{A}$ have growth bound $\omega_{\mathfrak{A}}$, choose $\alpha > \omega_{\mathfrak{A}}$ and define the function e_α by $e_\alpha(t) := e^{\alpha t}$ for $t \in \mathbb{R}$. We call the generator A of $\mathfrak{A}$ the *main operator* of the system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. Define the *control operator* $B : \mathcal{U} \to \mathcal{X}_{-1}$ by

$$Bu := (\alpha - A|_{\mathcal{X}})\mathfrak{B}(e_\alpha u), \quad u \in \mathcal{U}.$$

In [Sta05, Lemma 4.4.1] it is shown that $\mathfrak{C}x$ is continuous for all $x \in \operatorname{Dom}(A)$. Thus we may define the *observation operator* $C : \mathcal{X}_1 \to \mathcal{Y}$ by $Cx := (\mathfrak{C}x)(0)$.

For all $u \in \mathcal{U}$ and $\lambda \in \mathbb{C}^+_{\omega_{\mathfrak{A}}}$ there exists a $y \in \mathcal{Y}$, such that $\mathfrak{D}(e_\lambda u) = e_\lambda y$ almost everywhere, according to [Sta05, Lemma 4.5.3]. We define the *transfer function* $\widehat{\mathfrak{D}}(\lambda)$ for $\lambda \in \mathbb{C}^+_{\omega_{\mathfrak{A}}}$ and $u \in \mathcal{U}$ as the map $\widehat{\mathfrak{D}}(\lambda)u := y$, which satisfies $\mathfrak{D}(e_\lambda u) = e_\lambda y$ almost everywhere. Then [Sta05, Lemma 4.5.3] says that $\widehat{\mathfrak{D}}(\lambda)$ is a bounded linear operator from $\mathcal{U}$ to $\mathcal{Y}$, i.e. $\widehat{\mathfrak{D}} : \mathbb{C}^+_{\omega_{\mathfrak{A}}} \to \mathcal{L}(\mathcal{U};\mathcal{Y})$.

Lemma 5.2. *With A, B, C and $\widehat{\mathfrak{D}}$ given above, let* $\mathrm{Dom}\,(S)$ *be given by* (5.1) *and define*

$$A\&B := \begin{bmatrix} A|_{\mathcal{X}} & B \end{bmatrix}\big|_{\mathrm{Dom}(S)} \quad \text{and}$$

$$C\&D\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] := C\big(x - (\alpha - A|_{\mathcal{X}})^{-1}Bu\big) + \widehat{\mathfrak{D}}(\alpha)u, \quad \mathrm{Dom}\,(C\&D) = \mathrm{Dom}\,(S).$$

Then $C\&D$ does not depend on $\alpha \in \mathbb{C}^+_{\omega_{\mathfrak{A}}}$ and the operator $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an i/s/o-system node on $(\mathcal{X},\mathcal{U},\mathcal{Y})$.

Moreover, for any $1 \le p < \infty$ the norm

$$\left\|\begin{bmatrix} x \\ u \end{bmatrix}\right\|_{\mathrm{Dom}(S)} := \left(\left\|A\&B\begin{bmatrix} x \\ u \end{bmatrix}\right\|^p_{\mathcal{X}} + \|x\|^p_{\mathcal{X}} + \|u\|^p_{\mathcal{U}}\right)^{1/p} \tag{5.2}$$

makes $\mathrm{Dom}\,(S)$ *a Banach space. If $p = 2$ and $\mathcal{X}$ and $\mathcal{U}$ are both Hilbert spaces, then this norm makes* $\mathrm{Dom}\,(S)$ *a Hilbert space.*

The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ maps $\mathrm{Dom}\,(S)$ *equipped with the norm in* (5.2) *continuously into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$.*

For every $\lambda \in \mathbb{C}^+_{\omega_{\mathfrak{A}}}$, the operator $\begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix}$ maps $\mathcal{U}$ into $\mathrm{Dom}\,(S)$ *and the transfer function $\widehat{\mathfrak{D}}$ is given by*

$$\widehat{\mathfrak{D}}(\lambda) = C\&D\begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix}. \tag{5.3}$$

Proof. The definition of B is from [Sta05, Thm 4.2.1], while C is from [Sta05, Thm 4.4.2]. The transfer function $\widehat{\mathfrak{D}}$ is given in [Sta05, Def. 4.6.1]. The i/s/o-system node S is constructed in [Sta05, Def. 4.6.4] and, according to [Sta05, Thm 4.6.7], the operator $C\&D$ is independent of α as long as $\Re\alpha > \omega_{\mathfrak{A}}$. The operator S has all the properties in Definition 5.1, as proved in [Sta05, Prop. 4.7.1].

The completeness of $\mathrm{Dom}\,(S)$ with respect to the norm (5.2) is proven in [Sta05, Lemma 4.3.10] for the case $p = 2$. Using Lemma 2.6 we may extend the result to any $p \in [1,\infty)$. Continuity of S now follows from the assumed closedness of S.

For the last claim, [Sta05, Lemma 4.7.3] yields that $\begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix}$ maps $\mathcal{U}$ into $\mathrm{Dom}\,(S)$ for every $\lambda \in \mathrm{Res}\,(A)$. The formula (5.3) is given in [Sta05, Thm 4.6.7]. □

From now on, we always assume that $\mathrm{Dom}\,(S)$ has the norm in (5.2). We proceed by giving an example of an i/s/o-system node. The example is an expansion of [Sta02a, Ex. 4.8].

Example 5.3. *Let A generate a contraction semigroup $\mathfrak{A}$ on the Hilbert space $\mathcal{X}$. Then A is maximally dissipative, i.e. $\Re\langle Ax, x\rangle \le 0$ for all $x \in \mathrm{Dom}\,(A)$ and $\mathbb{C}^+ \subset \mathrm{Res}\,(A)$, according to the Lumer-Phillips Theorem, see e.g.* [Paz83, Thm 3.9 and 4.3]. *In the most interesting case the operator A is closed but unbounded.*

The linear operator

$$S := \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \\ -A|_{\mathcal{X}} & -A|_{\mathcal{X}} \end{bmatrix}\Bigg|_{\mathrm{Dom}(S)}$$

with domain

$$\mathrm{Dom}\,(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix} \mid x + u \in \mathrm{Dom}\,(A) \right\}$$

is an i/s/o-system node:

(i) *We prove that S inherits closedness from A. If $\begin{bmatrix} x_n \\ u_n \end{bmatrix} \in \mathrm{Dom}\,(S)$ converges to some $\begin{bmatrix} x \\ u \end{bmatrix}$ in $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix}$ and $S\begin{bmatrix} x_n \\ u_n \end{bmatrix}$ tends to some $\begin{bmatrix} z \\ y \end{bmatrix}$ in $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix}$, then $x_n + u_n \in \mathrm{Dom}\,(A)$ and*

$$S\begin{bmatrix} x_n \\ u_n \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} A|_{\mathcal{X}}(x_n + u_n) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} A(x_n + u_n) \to \begin{bmatrix} z \\ -z \end{bmatrix}.$$

This implies that $x_n + u_n \to x + u$ and that $A(x_n + u_n) \to z$. By the closedness of A we then have that $x + u \in \mathrm{Dom}\,(A)$ and $z = A(x+u)$, which implies that $\begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{Dom}\,(S)$ and $\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} z \\ -z \end{bmatrix} = S\begin{bmatrix} x \\ u \end{bmatrix}$. We have proved that S is closed.

(ii) *From the definition of $\mathrm{Dom}\,(S)$ we have that $\begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathrm{Dom}\,(S)$ if and only if $x \in \mathrm{Dom}\,(A)$, in which case $S\begin{bmatrix} x \\ 0 \end{bmatrix} = A|_{\mathcal{X}}x = Ax$. The operator A by assumption generates a C_0 semigroup on $\mathcal{X}$.*

(iii) *Let $\alpha \in \mathrm{Res}\,(A)$ be the constant used in the rigging construction described at the beginning of this section, so that $(\alpha - A|_{\mathcal{X}})^{-1}$ is a bounded operator on $\mathcal{X}_{-1}$. By definition, $A\&B$ is the restriction of $\begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \end{bmatrix}$ to $\mathrm{Dom}\,(S)$ and the norm of $\begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \end{bmatrix}$ as an operator from $\begin{bmatrix} \mathcal{X} \\ \mathcal{X} \end{bmatrix}$ to $\mathcal{X}_{-1}$ is finite, because $\|x + u\| \le M \left\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\|$ for some $M \ge 1$ and*

$$\begin{aligned} \left\| \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{-1} &= \left\| (\alpha - A|_{\mathcal{X}})^{-1} \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{\mathcal{X}} \\ &= \|(\alpha - A|_{\mathcal{X}})^{-1} A|_{\mathcal{X}}(x+u)\|_{\mathcal{X}} = \|(\alpha(\alpha - A|_{\mathcal{X}})^{-1} - 1)(x+u)\|_{\mathcal{X}} \\ &\le M\left(|\alpha|\,\|(\alpha - A|_{\mathcal{X}})^{-1}\| + 1\right) \left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right]}, \end{aligned}$$

cf. Lemma 2.6.

(iv) *Recall that* $1 - A$ *is a bijection from* $\mathrm{Dom}\,(A)$ *to* $\mathcal{X}$*, since* A *is maximally dissipative by assumption. This implies that*

$$z \in \mathrm{Dom}\,(A) \Longleftrightarrow (1-A)z \in \mathcal{X} \Longleftrightarrow (1-A|_{\mathcal{X}})z \in \mathcal{X} \Longleftrightarrow A|_{\mathcal{X}} z \in \mathcal{X},$$

because $(1-A) = (1-A|_{\mathcal{X}})\big|_{\mathrm{Dom}(A)}$ *and* $z \in \mathcal{X}$*. Thus*

$$\begin{aligned}\mathrm{Dom}\,(S) &= \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \mid x + u \in \mathrm{Dom}\,(A) \right\} \\ &= \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \middle| \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{X} \right\}.\end{aligned}$$

We are done proving that S *is a system node.*

Combining [Sta05, Thms 4.7.11 and 4.7.13], we see that the following definition of well-posedness of an i/s/o-system node is consistent with [Sta05], although the input signal, state trajectory and output signal of an i/s/o-system node are defined slightly differently in [Sta05, Def. 4.7.10].

Definition 5.4. *Let* I *be a subinterval of* $\mathbb{R}$ *and let* $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ *be an i/s/o system on* $(\mathcal{X},\mathcal{U},\mathcal{Y})$ *with i/s/o-system node* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *constructed in Lemma 5.2.*

The triple $\begin{bmatrix} x \\ y \\ u \end{bmatrix} \in \begin{bmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{Y}) \\ C(I;\mathcal{U}) \end{bmatrix}$ *is a* classical trajectory *of* $\begin{bmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix}$ *on* I *if for all* $t \in I$ *we have*

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \quad \text{and} \quad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

with one-sided derivatives at any end points of I*.*

Let now $1 \le p < \infty$*. The i/s/o-system node* S *is* L^p well posed *if there exist* $T > 0$ *and* $K_T > 0$*, such that every classical trajectory of* S *on* $[0,T]$ *satisfies*

$$\|x(t)\|_{\mathcal{X}} + \|y\|_{L^p([0,t];\mathcal{Y})} \le K_T \left(\|x(0)\|_{\mathcal{X}} + \|u\|_{L^p([0,t];\mathcal{U})}\right) \tag{5.4}$$

for all $t \in [0,T]$*.*

We remark that there exist some $T > 0$ and $K_T > 0$ such that (5.4) holds if and only if there for every $T > 0$ exists a $K_T > 0$ such that (5.4) holds. The proof is very similar to the proof of Lemma 2.10(iii).

We now study well-posedness of the system node in Example 5.3.

Example 5.5. *In Example 5.3, if* A *is bounded, then* S *is* L^p *well posed for all finite* $p \ge 1$*, as was shown in* [Sta05, Prop. 2.3.1]*. We now prove that if* A *is unbounded, then* S *is* L^p *ill-posed for all* p*.*

In (3.16) *we proved that the growth bound of any contraction semigroup is at most zero and* (5.3) *then yields that the transfer function of S for at least all $\lambda \in \mathbb{C}^+$ is given by*

$$\widehat{\mathfrak{D}}(\lambda) = C\&D \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1 \end{bmatrix} = \begin{bmatrix} -A|_{\mathcal{X}} & -A|_{\mathcal{X}} \end{bmatrix} \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}A|_{\mathcal{X}} \\ 1 \end{bmatrix}$$
$$= -A|_{\mathcal{X}}\big((\lambda - A|_{\mathcal{X}})^{-1}A|_{\mathcal{X}} + 1\big) = -A|_{\mathcal{X}}\lambda(\lambda - A|_{\mathcal{X}})^{-1}\big|_{\mathcal{X}} = -A\lambda(\lambda - A)^{-1}.$$

For any $u \in \operatorname{Dom}(A)$ we then have $\lim_{\lambda\to\infty} \widehat{\mathfrak{D}}(\lambda)u = -Au$, according to [Sta05, Thm 3.2.9(iii)]. *This shows that $\widehat{\mathfrak{D}}$ cannot be bounded on any right half-plane and, therefore,* [Sta05, Lemma 4.6.2] *yields that S is L^p ill posed for every $1 \le p < \infty$.*

In Example 6.8 below we show that the ill-posed i/s/o-system node S of Example 5.3 can still be modelled as a well-posed s/s system.

Lemma 5.6. *Let I be a subinterval of $\mathbb{R}$ and let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be any continuous linear operator from $\operatorname{Dom}(S) \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$. Assume that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C^n(I; \operatorname{Dom}(S))$ for some $n \in \mathbb{Z}^+$. Then*

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \in C^n\left(I; \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}\right) \tag{5.5}$$

and for all $0 \le k \le n$ we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \begin{bmatrix} x \\ u \end{bmatrix} \tag{5.6}$$

everywhere on I, with one-sided derivatives at any end points of I.

Proof. The proof uses only the definition of the derivative, the continuity of S and induction over k. □

The rather technical lemma that we now present connects classical and generalised state trajectories of i/s/o systems. See Definition A.3 in the appendix for a definition of the space $W^{1,p}_{loc}(I;\mathcal{U})$.

Lemma 5.7. *Let $I = [a, b]$ or $I = [a,\infty)$ and assume that $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ is an L^p-well-posed i/s/o-system on $(\mathcal{X},\mathcal{U},\mathcal{Y})$ with i/s/o-system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$.*

(i) *For all $x_a \in \mathcal{X}$ and $u \in L^p_{loc}(I;\mathcal{U})$ the function*

$$x(t) = \mathfrak{A}^{t-a}x_a + \mathfrak{B}\tau^t\pi_I u, \quad t \in I \tag{5.7}$$

is the unique solution in $C(I;\mathcal{X}) \cap W^{1,p}_{loc}(I;\mathcal{X}_{-1})$ of the equation

$$x(a) = x_a \text{ and } \dot{x}(t) = A|_{\mathcal{X}}x(t) + Bu(t) \text{ in } \mathcal{X}_{-1} \text{ a.e. in } I, \tag{5.8}$$

where the derivative $\dot{x}$ of x is taken in the distribution sense, i.e., for all $t \in I$, $x(t) = \int_a^t \dot{x}(s)\,\mathrm{d}s$ in $\mathcal{X}_{-1}$.

(ii) *Assume that* (5.8) *holds with* $x \in W^{1,p}_{loc}(I;\mathcal{X}_{-1})$ *and* $u \in L^p_{loc}(I;\mathcal{U})$. *Then* $x \in C^1(I;\mathcal{X})$ *and* $u \in C(I;\mathcal{U})$ *if and only if* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(I;\mathrm{Dom}\,(S))$. *In this case* $\dot{x}(t) = A\&B \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ *in* $\mathcal{X}$ *for all* $t \in I$, *with one-sided derivatives at the end point(s) of* I.

(iii) *Assume that* (5.7) *holds. If* $u \in W^{1,p}_{loc}(I;\mathcal{U})$ *and* $\left[\begin{matrix} x_a \\ u(a) \end{matrix}\right] \in \mathrm{Dom}\,(S)$ *then* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(I;\mathrm{Dom}\,(S))$.

Proof. (i) It is well known that $L^p_{loc}(I;\mathcal{U}) \subset L^1_{loc}(I;\mathcal{U})$ for all $1 \le p < \infty$. Thus, if $I = [a,\infty)$ then it suffices to combine Definition 3.8.1 and Theorem 4.3.1 of [Sta05] in order to prove claim (i).

In the case $I = [a,b]$ we first note that $\pi_{[a,\infty)}\pi_I = \pi_I$, which implies that

$$\widetilde{x}(t) := \mathfrak{A}^{t-a}x_a + \mathfrak{B}\tau^t\pi_I u, \quad t \in [a,\infty) \tag{5.9}$$

is the unique solution in $C([a,\infty);\mathcal{X}) \cap W^{1,p}_{loc}([a,\infty);\mathcal{X}_{-1})$ of the initial-value problem

$$\widetilde{x}(a) = x_a \quad \textit{and} \quad \dot{\widetilde{x}}(t) = A|_{\mathcal{X}}\widetilde{x}(t) + B(\pi_I u)(t) \text{ in } \mathcal{X}_{-1} \tag{5.10}$$

for almost all $t \in [a,\infty)$, by claim (i) of this lemma for the case $I = [a,\infty)$.

Comparing (5.7) and (5.9) we see that $x = \rho_{[a,b]}\widetilde{x}$ and (5.8) is then obtained as a special case of (5.10), i.e., the function x in (5.7) solves (5.8). Replacing the interval $[s,\infty)$ by the interval $[a,b]$ in the proof of [Sta05, Thm 3.8.2(ii)], we see that the equation (5.8) has only one solution x in $W^{1,p}_{loc}([a,b];\mathcal{X}_{-1}) \cap C([a,b];\mathcal{X})$, namely the function x in (5.7).

(ii) Assume first that $x \in C^1(I;\mathcal{X})$, $u \in C(I;\mathcal{U})$ and that (5.8) holds. Then $\dot{x} \in C(I;\mathcal{X}_{-1})$, because the norm on $\mathcal{X}_{-1}$ is weaker than the norm on $\mathcal{X}$. Moreover, $\left[\begin{matrix} A|_{\mathcal{X}} & B \end{matrix}\right]$ maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ continuously into $\mathcal{X}_{-1}$, by Definition 5.1 of an i/s/o-system node, and thus also the function $t \to A|_{\mathcal{X}}x(t) + Bu(t)$ lies in $C(I;\mathcal{X}_{-1})$. This implies that actually $\dot{x}(t) = A|_{\mathcal{X}}x(t) + Bu(t)$ in $\mathcal{X}_{-1}$ for all $t \in I$, instead of only for almost all t.

The assumption $x \in C^1(I;\mathcal{X})$ also implies that $\dot{x}(t) = A|_{\mathcal{X}}x(t) + Bu(t)$ lies in $\mathcal{X}$ instead of only in $\mathcal{X}_{-1}$. This implies that $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathrm{Dom}\,(S)$ and that $\dot{x}(t) = A\&B \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ for all $t \in I$. Recalling that the norm in $\mathrm{Dom}\,(S)$ is

$$\begin{aligned} \left\| \left[\begin{matrix} x(t) \\ u(t) \end{matrix}\right] \right\|^p_{\mathrm{Dom}(S)} &= \left\| A\&B \left[\begin{matrix} x(t) \\ u(t) \end{matrix}\right] \right\|^p_{\mathcal{X}} + \|x(t)\|^p_{\mathcal{X}} + \|u(t)\|^p_{\mathcal{U}} \\ &= \|\dot{x}(t)\|^p_{\mathcal{X}} + \|x(t)\|^p_{\mathcal{X}} + \|u(t)\|^p_{\mathcal{U}}, \end{aligned} \tag{5.11}$$

we get that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(I;\mathrm{Dom}\,(S))$, cf. the proof of Lemma 2.2(i).

Now, conversely assume that (5.8) holds with $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(I;\mathrm{Dom}\,(S))$. From (5.11) we get that $x \in C(I;\mathcal{X})$ and $u \in C(I;\mathcal{U})$. By Definition 5.1,

$\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \operatorname{Dom}(S)$ implies that $A|_{\mathcal{X}} x(t) + Bu(t)$ lies in $\mathcal{X}$ and that

$$A|_{\mathcal{X}} x(t) + Bu(t) = A\&B \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

From (5.11) it immediately follows that $A\&B$ maps $\operatorname{Dom}(S)$ continuously into $\mathcal{X}$ and Lemma 5.6 then yields that $A\&B \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(I;\mathcal{X})$.

Equation (5.8) implies that $\dot{x} = A\&B \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ in $\mathcal{X}_{-1}$ almost everywhere in I, i.e., that

$$x(t) - x(s) = \int_s^t A\&B \begin{bmatrix} x(v) \\ u(v) \end{bmatrix} \mathrm{d}v, \quad s, t \in I.$$

Dividing this identify by $t-s$ and letting $t-s$ tend to zero, we for all $t \in I$ get that $\dot{x}(t) = A\&B \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$, with one-sided derivatives at the end point(s) of I, due to the continuity of the function $A\&B \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ on I. In particular $\dot{x} \in C(I;\mathcal{X})$.

(iii) Now assume that $u \in W^{1,p}_{loc}(I;\mathcal{U})$ and $\begin{bmatrix} x_a \\ u(a) \end{bmatrix} \in \operatorname{Dom}(S)$. In the case $I = [a,b]$ we start by extending u to a function (which we still denote by u) in $W^{1,p}_{loc}([a,\infty);\mathcal{U})$ by setting $u(t) = u(b)$ for all $t > b$. Define $\widetilde{x}$ by (5.9) with $I = [a,\infty)$. Combining (4.19) and [Sta05, Thm 4.3.7] we get that the function $\widetilde{x}$ lies in $C^1([a,\infty);\mathcal{X})$. We again have $x = \rho_I \widetilde{x}$ and this function obviously lies in $C^1(I;\mathcal{X})$.

We finally note that $W^{1,p}_{loc}(I;\mathcal{U}) \subset C(I;\mathcal{U}) \subset L^p_{loc}(I;\mathcal{U})$ and, combining claims (i) and (ii) of this lemma, we get that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ lies in $C(I;\operatorname{Dom}(S))$. □

As the following theorem shows, every classical trajectory of an i/s/o-system is also an L^p trajectory of the same i/s/o system. The converse is also true in the sense that every L^p trajectory of an i/s/o system, which has the necessary smoothness, is actually classical.

Theorem 5.8. *Let $I = [a,b]$ or $I = [a,\infty)$, let $\left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ be an L^p-well-posed i/s/o system and let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be the i/s/o-system node in Lemma 5.2.*

(i) *Assume that $x \in C^1(I;\mathcal{X})$, $u \in C(I;\mathcal{U})$ and $y \in L^p_{loc}(I;\mathcal{Y})$ satisfy*

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^{t-a} & \mathfrak{B}\tau^t \\ \rho_I \tau^{-a}\mathfrak{C} & \rho_I \mathfrak{D} \end{bmatrix} \begin{bmatrix} x(a) \\ \pi_I u \end{bmatrix} \quad \text{for all } t \in I. \tag{5.12}$$

Then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(I;\operatorname{Dom}(S))$ and $\dot{x}(t) = A\&B \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ for all $t \in I$. Moreover, y coincides with the continuous function $C\&D \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$, $t \in I$, almost everywhere.

(ii) *If $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(I;\operatorname{Dom}(S))$ and*

$$\forall t \in I: \quad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \tag{5.13}$$

then $x \in C^1(I;\mathcal{X})$, $u \in C(I;\mathcal{U})$, $y \in C(I;\mathcal{Y})$ and (5.12) holds.

Proof. Lemma 5.7 yields that the first line of (5.12) holds with $x \in C^1(I;\mathcal{X})$ and $u \in C(I;\mathcal{U})$ if and only if the first line of (5.13) holds with $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C(I;\mathrm{Dom}\,(S))$. Now assume that these conditions hold and define

$$\begin{bmatrix} x_1(t) \\ u_1(t) \end{bmatrix} := \int_a^t \begin{bmatrix} x(v) \\ u(v) \end{bmatrix} \mathrm{d}v, \quad t \in I.$$

Denote the output y given in (5.12) by $\widehat{y}$. An application of [Sta05, Thm 4.6.12] yields that $\widehat{y}$ coincides almost everywhere on I with the function

$$\widetilde{y}(t) := \frac{\mathrm{d}}{\mathrm{d}t} C\&D \begin{bmatrix} x_1(t) \\ u_1(t) \end{bmatrix}, \quad t \in I.$$

Moreover, $\begin{bmatrix} x_1 \\ u_1 \end{bmatrix}$ obviously lies in $C^1(\mathbb{R}^+;\mathrm{Dom}\,(S))$ and applying the second lines of (5.5) and (5.6) with $k = n = 1$ we obtain that $\widetilde{y}$ is continuous on I and $\widetilde{y}(t) = C\&D \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ for all $t \in I$. Thus $\widetilde{y}$ coincides with y given in (5.13) on I. This proves that the functions y given on second lines of (5.12) and (5.13) are equal almost everywhere, and that the latter is continuous on I. □

It is now time to return to s/s systems. In the next section we define maximality of a s/s node and show that maximality gives some quite useful extra structure to well-posed s/s nodes.

6. Maximal s/s nodes

In this section we prove the existence of a unique maximal generating s/s node for any given well-posed state/signal system. We derive an expression for this maximal s/s node in terms of i/s/o-system nodes. The results in this section also provide us with some tools for proving that a given subspace V generates a well-posed s/s-system.

In the next definition we denote the space of classical trajectories on $[a,b]$ of the s/s node $(V_{max};\mathcal{X},\mathcal{W})$ by

$$\mathfrak{V}_{max}[a,b] := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{bmatrix} \;\middle|\; \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in C([a,b];V_{max}) \right\}. \tag{6.1}$$

Definition 6.1. *The s/s node* $(V_{max};\mathcal{X},\mathcal{W})$ *is a* maximal generating state/signal node *of a well-posed s/s system* $\Sigma_{s/s} = (\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ *if the following two conditions hold:*

(i) *The s/s node* $(V_{max};\mathcal{X},\mathcal{W})$ *generates* Σ, *i.e.,* $\overline{\mathfrak{V}_{max}[0,T]} = \rho_{[0,T]}\mathfrak{W}^p$ *for some* $T > 0$, *where the bar denotes closure in* $\begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$.

(ii) *The generating subspace* V_{max} *is a maximal one among the generating subspaces, i.e.,* $V \subset V_{max}$ *for all s/s nodes* $(V;\mathcal{X},\mathcal{W})$ *that generate* Σ.

We have the following immediate observation.

Lemma 6.2. *If a maximal generating subspace V_{max} of $\Sigma_{s/s} = (\mathfrak{W}^p; \mathcal{X}, \mathcal{W})$ exists then it is unique. Every space $\mathfrak{V}[0,T]$ of classical trajectories of Σ is then contained in $\mathfrak{V}_{max}[0,T]$.*

Proof. If V_1 and V_2 are both maximal and $\overline{\mathfrak{V}_1[0,T]} = \overline{\mathfrak{V}_2[0,T]} = \rho_{[0,T]}\mathfrak{W}^p$, then by definition we have both $V_1 \subset V_2$ and $V_2 \subset V_1$, which implies that $V_1 = V_2$.

The second claim follows from comparing Definition 2.1 to (6.1), taking into account that $V \subset V_{max}$. □

As we shall see later, every well-posed s/s system has a maximal generating s/s node $(V_{max}; \mathcal{X}, \mathcal{W})$, where V_{max} can be defined e.g. as in the following preliminary lemma.

Lemma 6.3. *Let $1 \le p < \infty$ and $T > 0$. Assume that $\left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ is an L^p-well-posed i/s/o system with system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ given in Lemma 5.2. Define*

$$V_{max} := \begin{bmatrix} A\&B \\ \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} \\ C\&D \\ \begin{bmatrix} 0 & 1_{\mathcal{U}} \end{bmatrix} \end{bmatrix} \mathrm{Dom}\,(S). \tag{6.2}$$

Then the image of the space

$$\mathfrak{V}_{max,0}[0,T] := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}_{max}[0,T] \;\middle|\; \begin{bmatrix} x(0) \\ w(0) \end{bmatrix} = 0 \right\} \tag{6.3}$$

under $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ is dense in $L^p([0,T];\mathcal{U})$.

Moreover, $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ *maps $\overline{\mathfrak{V}_{max}[0,T]}$ one-to-one onto* $\begin{bmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}$.

Proof. Part 1 ($\left[\begin{smallmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{smallmatrix}\right]$ maps $\overline{\mathfrak{V}_{max}[0,T]}$ onto $\left[\begin{smallmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{smallmatrix}\right]$): We first recall that the space

$$C_0^1([0,T];\mathcal{U}) := \left\{ u \in C^1([0,T];\mathcal{U}) \mid u(0) = 0 \right\}.$$

is dense in $L^p([0,T];\mathcal{U})$ for all $1 \le p < \infty$. Moreover, the domain of A is dense in $\mathcal{X}$ by Definition 5.1. Thus, for all $\begin{bmatrix} x_0 \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}$ we can find a sequence $\begin{bmatrix} \xi_n \\ u_n \end{bmatrix}$ in $\begin{bmatrix} \mathrm{Dom}\,(A) \\ C_0^1([0,T];\mathcal{U}) \end{bmatrix}$ which tends to $\begin{bmatrix} x_0 \\ u \end{bmatrix}$ in $\begin{bmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}$. For every element of this sequence we have that

$$\begin{bmatrix} \xi_n \\ u_n(0) \end{bmatrix} \in \begin{bmatrix} \mathrm{Dom}\,(A) \\ \{0\} \end{bmatrix} \subset \mathrm{Dom}\,(S).$$

Defining

$$x_n(t) := \mathfrak{A}^t \xi_n + \mathfrak{B}\tau^t \pi_{[0,T]} u_n, \quad t \in [0,T], \tag{6.4}$$

we thus get from Lemma 5.7 that $\begin{bmatrix} x_n \\ u_n \end{bmatrix} \in C([0,T];\mathrm{Dom}\,(S))$. We can then define the functions y_n by

$$y_n(t) := C\&D \begin{bmatrix} x_n(t) \\ u_n(t) \end{bmatrix}, \quad t \in [0,T]. \tag{6.5}$$

Lemma 5.6 yields that y_n is continuous on $[0,T]$ and combining Lemma 5.7 with Theorem 5.8 we now get that

$$\forall t \in [0,T]: \quad \begin{bmatrix} x_n(t) \\ y_n \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}\tau^t \\ \rho_{[0,T]}\mathfrak{C} & \rho_{[0,T]}\mathfrak{D} \end{bmatrix} \begin{bmatrix} \xi_n \\ \pi_{[0,T]}u_n \end{bmatrix}. \tag{6.6}$$

The continuity of $\mathfrak{C}$ and $\mathfrak{D}$ implies that

$$y_n \to \rho_{[0,T]}\mathfrak{C}x_0 + \rho_{[0,T]}\mathfrak{D}\pi_{[0,T]}u =: y, \; n \to \infty.$$

We now show that x_n converges uniformly to the function

$$x(t) := \mathfrak{A}^t x_0 + \mathfrak{B}\tau^t\pi_{[0,T]}u, \quad t \in [0,T], \quad \text{as } n \to \infty.$$

Noting that $\mathfrak{B}\pi_{[-T,0]}$ is a continuous, and hence bounded, operator from $L^p([-T,0];\mathcal{U})$ to $\mathcal{X}$ we for all t in $[0,T]$ get that

$$\begin{aligned} \|x_n(t) - x(t)\|_\mathcal{X} &= \|\mathfrak{A}^t(\xi_n - x_0) + \mathfrak{B}\tau^t\pi_{[0,t]}(u_n - u)\|_\mathcal{X} \\ &\le \|\mathfrak{A}^t\|\|\xi_n - x_0\|_\mathcal{X} + \|\mathfrak{B}\pi_{[-T,0]}\tau^t\pi_{[0,t]}(u_n - u)\|_\mathcal{X} \\ &\le e^{2\omega_\mathfrak{A}T}\|\xi_n - x_0\|_\mathcal{X} + \|\mathfrak{B}\pi_{[-T,0]}\|\|\tau^t\pi_{[0,t]}(u_n - u)\|_{L^p([-T,0];\mathcal{U})} \\ &\le e^{2\omega_\mathfrak{A}T}\|\xi_n - x_0\|_\mathcal{X} + \|\mathfrak{B}\pi_{[-T,0]}\|\|u_n - u\|_{L^p([0,T];\mathcal{U})}, \end{aligned} \tag{6.7}$$

cf. (4.19) and Lemma 3.14. The last line of (6.7) tends to 0 as $n \to \infty$ and the convergence does not depend on t, which implies that the convergence is uniform in t. We have shown that $\left[\begin{smallmatrix} x \\ u+y \end{smallmatrix}\right] \in \overline{\mathfrak{V}_{max}[0,T]}$ with

$$\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \begin{bmatrix} x \\ u+y \end{bmatrix} = \begin{bmatrix} x_0 \\ u \end{bmatrix}.$$

Part 2 (The other two claims): We first prove that $\begin{bmatrix} 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \mathfrak{V}_{max,0}[0,T]$ is dense in $L^p([0,T];\mathcal{U})$. Let $u \in L^p([0,T];\mathcal{U})$ be arbitrary, let $u_n \in C_0^1([0,T];\mathcal{U})$ tend to u in $L^p([0,T];\mathcal{U})$ and take $\xi_n = 0$ for all n. As in part 1 of this proof, define x_n by (6.4) and y_n by (6.5), so that $\begin{bmatrix} x_n \\ u_n + y_n \end{bmatrix} \in \mathfrak{V}_{max}[0,T]$. Moreover, $u_n(0) = 0$, since we took u_n from $C_0^1([0,T];\mathcal{U})$, $x_n(0) = \xi_n = 0$ and $y_n(0) = C\&D \begin{bmatrix} x_n(0) \\ u_n(0) \end{bmatrix} = 0$. We have proved that $\begin{bmatrix} x_n \\ u_n + y_n \end{bmatrix} \in \mathfrak{V}_{max,0}[0,T]$ with $\begin{bmatrix} 0 & \mathcal{P}_\mathcal{U}^\mathcal{Y} \end{bmatrix} \begin{bmatrix} x_n \\ u_n + y_n \end{bmatrix} = u_n \to u$.

We finally show that the restriction of $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ to $\overline{\mathfrak{V}_{max}[0,T]}$ is injective. Let $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}_{max}[0,T]$, let x_n tend to x uniformly and let $w_n \to w$ in $L^p([0,T];\mathcal{W})$. Assume that $x(0) = 0$ and that $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$. Define $u_n := \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n$ and $y_n := \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w_n$. Then $\dot{x}_n$, x_n, u_n and y_n are all continuous and by (6.2) we have that

$$\begin{bmatrix} \dot{x}_n(t) \\ y_n(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x_n(t) \\ u_n(t) \end{bmatrix}, \quad t \in [0,T].$$

Theorem 5.8 yields that (6.6) holds. Arguing as in part 1 of this proof we then get that

$$x_n(t) \to \mathfrak{A}^t 0 + \mathfrak{B}\tau^t \pi_{[0,T]} 0 = 0, \quad t \in [0,T]$$

and $y = \lim y_n = 0$. This implies that $x(t) = 0$ for all $t \in [0,T]$ and by assumption we have $u = \lim u_n = 0$. This shows that $\left[\begin{smallmatrix} x \\ u+y \end{smallmatrix}\right] = 0$, i.e., that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \overline{\mathfrak{V}_{max}[0,T]}$ and $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0$ imply that $\begin{bmatrix} x \\ w \end{bmatrix} = 0$. □

The following theorem is the main result of this section. In the formulation of the theorem we have two Banach spaces $\mathcal{U}$ and $\mathcal{Y}$, which we identify with the subspaces $\begin{bmatrix} \{0\} \\ \mathcal{U} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{Y} \\ \{0\} \end{bmatrix}$ of $\begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$, respectively. In this way the Cartesian product $\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ is identified with the direct sum $\mathcal{U} \dotplus \mathcal{Y}$, cf. the discussion after Definition 2.5.

Theorem 6.4. *Make the same assumptions as in Lemma 6.3 and let V_{max} be given by* (6.2)*. Then $\left(V_{max}; \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]\right)$ is a maximal L^p-well-posed s/s node. The i/o pair $(\mathcal{U},\mathcal{Y})$ is admissible for the s/s system Σ generated by V_{max} and the corresponding i/s/o representation is $\Sigma_{i/s/o} = (\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$.*

Proof. We first fix $T > 0$ arbitrarily.

Part 1 ($(V_{max}; \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$ is an L^p-well-posed s/s node with i/o pair $(\mathcal{U},\mathcal{Y})$): We first check that V_{max} satisfies the conditions of Definition 2.3. The space V_{max} is closed, because it is essentially the graph of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, which is a closed operator by Definition 5.1. Furthermore, $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V_{max}$ implies that $z = A\&B \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = 0$.

For an arbitrary $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V_{max}$, define $u_0 := \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_0$ and let u be the constant function $u(t) := u_0$ for $t \in [0,T]$. This function u obviously lies in $W^{1,p}_{loc}([0,T];\mathcal{U})$. By (6.2) we moreover have that $\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \mathrm{Dom}\,(S)$, and defining $x(t) := \mathfrak{A}^t x_0 + \mathfrak{B}\tau^t \pi_{[0,T]} u$, $t \in [0,T]$, we obtain from Lemma 5.7(iii) that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ lies in $C([0,T];\mathrm{Dom}\,(S))$. Claims (i) and (ii) of Lemma 5.7 then yield that $x \in C^1([0,T];\mathcal{X})$, $u \in C([0,T];\mathcal{X})$ and $\dot{x}(t) = A\&B \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ for all $t \in [0,T]$.

We now define $y(t) := C\&D \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$, $t \in [0,T]$, and thus get that

$$\begin{bmatrix} \dot x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \in [0,T]. \tag{6.8}$$

Moreover, denoting $w := \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ we get that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}_{max}[0,T]$ with

$$\begin{bmatrix} \dot x(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} A\&B \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ C\&D \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}.$$

This proves that $(V_{max}; \mathcal{X}, \mathcal{W})$ is a s/s node.

Regarding the well-posedness of $(V_{max}; \mathcal{X}, \mathcal{W})$, we note that $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps the space

$$\mathfrak{W}_0^p[0,T] = \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \overline{\mathfrak{V}_{max}[0,T]} \,\middle|\, x(0) = 0 \right\}$$

one-to-one onto $L^p([0,T];\mathcal{U})$ by Lemma 6.3. Then Lemma 3.5 in combination with Lemma 6.3 yields that $\mathfrak{V}_{max,0}[0,T]$ given in (6.3) is dense in $\mathfrak{W}_0^p[0,T]$. Thus the conditions in Proposition 3.11(ii) are satisfied. Now Proposition 3.11(i) says that $(V_{max}; \mathcal{X}, \mathcal{W})$ is L^p well posed with the admissible i/o pair $(\mathcal{U}, \mathcal{Y})$.

Part 2 ($\Sigma_{i/s/o} = (\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y})$): In part one of this proof we showed that the i/s/o system $\left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ induces some L^p-well-posed s/s system $\Sigma_{s/s} = (\mathfrak{W}^p; \mathcal{X}, \mathcal{W})$, which satisfies $\overline{\mathfrak{V}_{max}[0,T]} = \rho_{[0,T]}\mathfrak{W}^p$ and has the admissible i/o pair $(\mathcal{U}, \mathcal{Y})$.

Denote the space of all $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$ that satisfy (4.1) with $y = \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w$, $u = \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ and $I = [0,T]$ by $W[0,T]$. We now prove that $W[0,T]$ is a closed subspace of $\begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$. First note that

$$\left\{ \begin{bmatrix} x(0) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ w \end{bmatrix} \in W[0,T] \right\} = \begin{bmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}.$$

If $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in W[0,T]$ and $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \to \begin{bmatrix} x \\ w \end{bmatrix}$, then $x_n(0) \to x(0)$ in $\mathcal{X}$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n$ tends to $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ in $L^p([0,T];\mathcal{U})$. By the argument in part 1 of the proof of Lemma 6.3 we then have that

$$\begin{aligned} \forall t \in [0,T]: \quad & x(t) = \mathfrak{A}^t x(0) + \mathfrak{B}\tau^t \pi_{[0,T]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \quad \text{and} \\ & \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w = \rho_{[0,T]} \mathfrak{C} x(0) + \rho_{[0,T]} \mathfrak{D} \pi_{[0,T]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w, \end{aligned}$$

which implies that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W[0,T]$, i.e., that $W[0,T]$ is closed.

Definition 4.5 says that we only need to show that $W[0,T] = \rho_{[0,T]}\mathfrak{W}^p$ in order to prove that $\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right]$ is the i/s/o representation of Σ with respect to $(\mathcal{U},\mathcal{Y})$. According to Theorem 5.8 we have that

$$W[0,T] \cap \begin{bmatrix} C^1([0,T];\mathcal{X}) \\ C([0,T];\mathcal{W}) \end{bmatrix} = \mathfrak{V}_{max}[0,T].$$

By part 1 of the proof of Lemma 6.3, for every $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in W[0,T]$ we can find a sequence $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}_{max}[0,T]$, such that $x_n \to x$ uniformly and $w_n \to w$ in $L^p([0,T];\mathcal{W})$. This proves that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \overline{\mathfrak{V}_{max}[0,T]}$ and, therefore, that

$$\mathfrak{V}_{max}[0,T] \subset W[0,T] \subset \overline{\mathfrak{V}_{max}[0,T]}.$$

Since $W[0,T]$ is closed, this implies that

$$W[0,T] = \overline{\mathfrak{V}_{max}[0,T]} = \rho_{[0,T]}\mathfrak{W}^p$$

and we are done proving that $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix}\mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D}\end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$.

Part 3 ($V \subset V_{max}$ for any generating subspace V of Σ): Let V generate Σ and let $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V$ be arbitrary. Due to (2.6) we can find a classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ of V such that $\begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$. Denoting $u := \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$ and $y := \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w$ we obtain from $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T] \subset \rho_{[0,T]}\mathfrak{W}^p$ that

$$\forall t \in [0,T]: \quad \begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathfrak{A}^t & \mathfrak{B}\tau^t \\ \rho_{[0,T]}\mathfrak{C} & \rho_{[0,T]}\mathfrak{D} \end{bmatrix} \begin{bmatrix} x(0) \\ \pi_{[0,T]}u \end{bmatrix}, \tag{6.9}$$

by part 2 of this proof. Moreover $\dot{x}$, x, u and y are continuous on $[0,T]$ for any classical trajectory, and thus $\begin{bmatrix} \dot{x}(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(0) \\ u(0) \end{bmatrix}$ according to Theorem 5.8. We have established that $\begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \in V_{max}$ and, therefore, that $V \subset V_{max}$. □

Part 2 of the proof of Theorem 6.4 yields that the maximal space $\mathfrak{V}_{max}[a,b]$ of classical trajectories of a well-posed s/s system $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ satisfies

$$\mathfrak{V}_{max}[a,b] = \rho_{[a,b]}\tau^{-a}\mathfrak{W}^p \cap \begin{bmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{bmatrix}.$$

Using Lemma 2.4 we can then recover V_{max} from $\rho_{[a,b]}\mathfrak{W}^p$ through

$$V_{max} = \left\{ \begin{bmatrix} \dot{x}(a) \\ x(a) \\ w(a) \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ w \end{bmatrix} \in \rho_{[a,b]}\tau^{-a}\mathfrak{W}^p \cap \begin{bmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{bmatrix} \right\}.$$

Proposition 6.5. *Every L^p-well-posed s/s system has a unique maximal generating s/s node. This maximal s/s node is L^p well posed.*

Proof. By Theorem 4.9 every well-posed s/s system has some i/s/o representation $\left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$. Theorem 6.4 then says that this s/s system has a well-posed maximal generating s/s node. According to Lemma 6.2 this maximal generating s/s node is unique. □

We now answer the question in Remark 3.16.

Theorem 6.6. *Let $T > 0$ and $1 \le p < \infty$, and let $\mathcal{X}$ and $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be Banach spaces. Let $W[0,T] \subset \begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$ and $W^+ \subset \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$.*

Then the following conditions are equivalent:

(i) *The triple $(W^+;\mathcal{X},\mathcal{W})$ is an L^p-well-posed s/s system, which has the admissible i/o pair $(\mathcal{U},\mathcal{Y})$, and $W[0,T] = \rho_{[0,T]}W^+$.*

(ii) *The following four conditions all hold:*

 (a) *The space $W[0,T]$ is a closed subspace of $\begin{bmatrix} C([0,T];\mathcal{X}) \\ L^p([0,T];\mathcal{W}) \end{bmatrix}$.*

 (b) *The operator $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps the space $W[0,T]$ one-to-one onto the space $\begin{bmatrix} \mathcal{X} \\ L^p([0,T];\mathcal{U}) \end{bmatrix}$.*

 (c) *The space W^+ satisfies* (3.19), *i.e.,*
$$W^+ = \left(W[0,T] \bowtie_T \tau^{-T}W[0,T] \bowtie_{2T} \dots\right) \cap \begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}.$$

 (d) *The space W^+ satisfies $\rho_+\tau^tW^+ \subset W^+$ for all $t \ge 0$.*

(iii) *The following four conditions all hold:*

 (e) *The space W^+ is a closed subspace of $\begin{bmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{loc}(\mathbb{R}^+;\mathcal{W}) \end{bmatrix}$.*

 (f) *The operator $\begin{bmatrix} \delta_0 & 0 \\ 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps W^+ one-to-one onto $\begin{bmatrix} \mathcal{X} \\ L^p_{loc}(\mathbb{R}^+;\mathcal{U}) \end{bmatrix}$.*

 (g) *We have that $W[0,T] = \rho_{[0,T]}W^+$.*

 (h) *Condition (d) above holds.*

Proof. (i) $\Longrightarrow$ (ii): The necessity of conditions (a), (b) and (d) was shown in the proof of Theorem 4.9. The necessity of (c) follows from Propositions 3.9 and 3.10.
(ii) $\Longrightarrow$ (iii): Lemma 4.6 yields that (a) – (c) imply (e) – (g).
(iii) $\Longrightarrow$ (i): According to Lemma 4.7, (e), (f) and (h) imply the existence of a well-posed i/s/o system $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ that satisfies (4.6). Theorem 6.4 then yields that $\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]$ induces a well-posed s/s system $\Sigma_{s/s} = (\widetilde{\mathfrak{W}}^p;\mathcal{X},\mathcal{W})$ which has i/s/o representation $\Sigma_{i/s/o} = \left(\left[\begin{smallmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$. Applying Theorem 4.9 to Σ we get that (4.6) holds also with W^+ replaced by $\widetilde{\mathfrak{W}}^p$ and thus $W^+ = \widetilde{\mathfrak{W}}^p$. □

One can apply Theorem 6.6 to a space W' of trajectories on any interval $[a,b]$ or $[a,\infty)$, where $-\infty < a < b < \infty$, by considering $\tau^a W'$, which is a space of trajectories on $[0, b-a]$ or $\mathbb{R}^+$, respectively.

We now give a direct characterisation of the subspaces V of $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$, which are graphs of i/s/o-system nodes in the sense of (6.2). Let therefore $\mathcal{X}$ and $\mathcal{W}$ be Banach spaces and let $V \subset \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$. Define the subspace V_y of V by

$$V_y := \left\{ \begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \;\middle|\; \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0 \right\} = V \cap \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix}. \tag{6.10}$$

If V is the graph of an i/s/o-system node S in the sense of (6.2), then $\left[\begin{smallmatrix}z\\ 0\\ y\end{smallmatrix}\right] \in V_y$ implies that $z, y = 0$, and we may define the operators $A : \mathcal{X} \supset \operatorname{Dom}(A) \to \mathcal{X}$ and $C : \operatorname{Dom}(A) \to \mathcal{Y}$ on $\operatorname{Dom}(A) := \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} V_y \subset \mathcal{X}$ by

$$\forall x \in \operatorname{Dom}(A) : \begin{bmatrix} Ax \\ Cx \end{bmatrix} := \begin{bmatrix} z \\ y \end{bmatrix}, \text{ such that } \begin{bmatrix} z \\ x \\ y \end{bmatrix} \in V_y. \tag{6.11}$$

Proposition 6.7. *Let $\mathcal{X}$ and $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be Banach spaces. Then the following claims are valid:*

(i) *The space V considered in (6.10) is given by (6.2) for some (not necessarily well-posed) i/s/o-system node $S = \left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right] : \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \supset \operatorname{Dom}(S) \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ if and only if the following four conditions are met:*
   - (a) *The subspace V is closed.*
   - (b) *The subspace $V_z := \left[\begin{smallmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}\end{smallmatrix}\right] V$ is closed.*
   - (c) *The operators A and C are well-defined by (6.11) and A generates a C_0 semigroup on $\mathcal{X}$.*
   - (d) *For all $u \in \mathcal{U}$ there exists an $\left[\begin{smallmatrix}z\\ x\\ w\end{smallmatrix}\right] \in V$ such that $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = u$.*

(ii) *If, in addition to (a) – (d), Condition (iii) of Definition 2.7 is satisfied, then S is L^p-well posed, and then V is the maximal generating subspace of an L^p-well-posed s/s system, which has has the admissible i/o pair $(\mathcal{U}, \mathcal{Y})$.*

(iii) *In particular, if the following two extra conditions are satisfied, then condition (iii) in Definition 2.7 is met:*
   - (e) *For all $u \in \mathcal{U}$ there exists an $\left[\begin{smallmatrix}z\\ 0\\ w\end{smallmatrix}\right] \in V$ with $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = u$.*
   - (f) *The operator C given in (6.11) is bounded.*

   Condition (e) obviously implies condition (d).

Proof. (i) We begin with the implication ($\Longleftarrow$). Condition (c) implies that V is the graph of some operator $S := \left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]$ in the sense of (6.2) and that:

$$\begin{bmatrix} A \\ C \end{bmatrix} x = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} \quad \text{for all}$$
$$x \in \mathrm{Dom}\,(A) = \left\{ x \in \mathcal{X} \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathrm{Dom}\,(S) \right\}.$$

Since A generates a C_0 semigroup on $\mathcal{X}$, it follows, in particular, that A has a nonempty resolvent set and a dense domain. Condition (a) is equivalent to closedness of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ and condition (b) is equivalent to closedness of $A\&B$. Condition (d) is equivalent to the statement that for all $u \in \mathcal{U}$ there exists an x such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{Dom}\,(S)$. From [Sta05, Def. 4.7.2 and Lemma 4.7.7] we obtain that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an i/s/o-system node.

Regarding the implication ($\Longrightarrow$), if V is the graph of an i/s/o-system node, then by the above, V has all properties (a) – (d).

(ii) If (2.12) holds, then S is well posed by Definition 5.4. By Theorem 6.4, any i/s/o-system node generates a well-posed s/s system, which has the admissible i/o pair $(\mathcal{U}, \mathcal{Y})$.

(iii) Condition (e) means that $\left[\begin{smallmatrix} \{0\} \\ \mathcal{U} \end{smallmatrix}\right] \subset \mathrm{Dom}\,(S)$, which implies that $\mathrm{Dom}\,(S)$ decomposes into $\left[\begin{smallmatrix} \mathrm{Dom}(A) \\ \mathcal{U} \end{smallmatrix}\right]$ and that S accordingly splits into $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Closedness of $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ follows from closedness of S and by the closed graph theorem, $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ is bounded in this case. If also C is bounded (condition (f)), then S is an L^p-well-posed i/s/o-system node, for $1 \le p < \infty$ according to [Sta05, Prop. 2.3.1].

□

We remark that conditions (e) and (f) of Proposition 6.7 are sufficient for well-posedness, as stated in the proposition. However, they are far from necessary unless $\mathcal{X}$ is finite-dimensional. Passive systems form a very important class of systems which are well posed in the s/s setting. These systems will not, in general, satisfy the conditions (e) and (f) of Proposition 6.7. A proper definition and a more elaborate treatment of passive systems will be presented elsewhere.

We now conclude the paper with two examples. The first example shows that the ill-posed i/s/o-system node of Example 5.3 induces a well posed s/s system.

Example 6.8. *With the same set-up as in Example 5.3, let* $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right]$, $\mathcal{U} := \left[\begin{smallmatrix} \{0\} \\ \mathcal{X} \end{smallmatrix}\right]$, $\mathcal{Y} := \left[\begin{smallmatrix} \mathcal{X} \\ \{0\} \end{smallmatrix}\right]$. *Following Theorem 6.4 we define the subspace* $V \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ *by:*

$$V := \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \\ 1 & 0 \\ \begin{bmatrix} -A|_{\mathcal{X}} \\ 0 \end{bmatrix} & \begin{bmatrix} -A|_{\mathcal{X}} \\ 1 \end{bmatrix} \end{bmatrix} \mathrm{Dom}\,(S),$$

where $\mathrm{Dom}\,(S) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right] \;\middle|\; x + u \in \mathrm{Dom}\,(A) \right\}$.

With respect to the presumptive i/o-pair $\left(\left[\begin{smallmatrix}\{0\}\\ \mathcal{X}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{X}\\ \{0\}\end{smallmatrix}\right]\right)$*, the space* V *is essentially the graph of the i/s/o-system node* S*. The main point of this example is to show that* V *indeed generates a well-posed s/s system* Σ *on* $(\mathcal{X}, \mathcal{W})$*, in spite of the fact that* S *is an ill-posed i/s/o-system node. The ill-posedness of* S *is due to the fact that Definition 2.7(iii) is not satisfied and thus* $\left(\left[\begin{smallmatrix}\{0\}\\ \mathcal{X}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{X}\\ \{0\}\end{smallmatrix}\right]\right)$ *cannot be an* admissible *i/o pair of* Σ.

In order to obtain an admissible i/o pair of V *we replace the original decomposition* $\mathcal{W} = \left[\begin{smallmatrix}\{0\}\\ \mathcal{X}\end{smallmatrix}\right] \dotplus \left[\begin{smallmatrix}\mathcal{X}\\ \{0\}\end{smallmatrix}\right]$ *by a new decomposition* $\mathcal{W} = \mathcal{U}' \dotplus \mathcal{Y}'$*, where* $\mathcal{U}' = \left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]\mathcal{X}$ *and* $\mathcal{Y}' = \left[\begin{smallmatrix}-1\\ 1\end{smallmatrix}\right]\mathcal{X}$*. Then* $\mathcal{P}_{\mathcal{U}'}^{\mathcal{Y}'} = \frac{1}{2}\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 1\end{smallmatrix}\right]$ *and* $\mathcal{P}_{\mathcal{Y}'}^{\mathcal{U}'} = \frac{1}{2}\left[\begin{smallmatrix}-1\\ 1\end{smallmatrix}\right]\left[\begin{smallmatrix}-1 & 1\end{smallmatrix}\right]$*. Identifying* $\begin{bmatrix}\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w\\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\end{bmatrix} = w = \begin{bmatrix}\mathcal{P}_{\mathcal{Y}'}^{\mathcal{U}'} w\\ \mathcal{P}_{\mathcal{U}'}^{\mathcal{Y}'} w\end{bmatrix}$ *as in* (2.9)*, we obtain that the space* V *is identified with* V' *given by:*

$$V' = \left\{\begin{bmatrix} z\\ x\\ y'\\ u'\end{bmatrix} \in V\right\} = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \mathcal{P}_{\mathcal{Y}'}^{\mathcal{U}'}\\ 0 & 0 & \mathcal{P}_{\mathcal{U}'}^{\mathcal{Y}'}\end{bmatrix}\begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}}\\ 1 & 0\\ \begin{bmatrix}-A|_{\mathcal{X}}\\ 0\end{bmatrix} & \begin{bmatrix}-A|_{\mathcal{X}}\\ 1\end{bmatrix}\end{bmatrix}\operatorname{Dom}(S).$$

Carrying out the multiplication on the right-hand side, we get that

$$V' = \begin{bmatrix} \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}}\end{bmatrix}\\ \begin{bmatrix} 1 & 0\end{bmatrix}\\ \frac{1}{2}\begin{bmatrix}-1\\ 1\end{bmatrix}\begin{bmatrix} A|_{\mathcal{X}} & 1 + A|_{\mathcal{X}}\end{bmatrix}\\ \frac{1}{2}\begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix} -A|_{\mathcal{X}} & 1 - A|_{\mathcal{X}}\end{bmatrix}\end{bmatrix}\left\{\begin{bmatrix} x\\ u\end{bmatrix} \;\middle|\; x + u \in \operatorname{Dom}(A)\right\}. \tag{6.12}$$

We now check that V' *has properties (a)–(c), (e) and (f) listed in Proposition 6.7.*

Condition (a) is trivially satisfied, because V' *is isomorphic to the graph of an i/s/o-system node. For condition (b) we recall from Example 5.3 that we have* $1 \in \operatorname{Res}(A)$*, because* A *is assumed to generate a contraction semigroup. Then* $1 \in \operatorname{Res}(A|_{\mathcal{X}})$ *and* $\begin{bmatrix} 1 & 0\\ -A|_{\mathcal{X}} & 1 - A|_{\mathcal{X}}\end{bmatrix}\operatorname{Dom}(S) = \begin{bmatrix}\mathcal{X}\\ \mathcal{X}\end{bmatrix}$*. Taking into account that* $(1 - A|_{\mathcal{X}})^{-1}A|_{\mathcal{X}} = A(1 - A)^{-1}$ *and that*

$$\begin{bmatrix} 1 & 0\\ -A|_{\mathcal{X}} & 1 - A|_{\mathcal{X}}\end{bmatrix}\Bigg|_{\operatorname{Dom}(S)}^{-1} = \begin{bmatrix} 1 & 0\\ A(1-A)^{-1} & (1-A)^{-1}\end{bmatrix},$$

we obtain that:

$$V'_z = \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}}\\ 1 & 0\\ -A|_{\mathcal{X}} & 1 - A|_{\mathcal{X}}\end{bmatrix}\operatorname{Dom}(S) = \begin{bmatrix} A(1-A)^{-1}\begin{bmatrix} 1 & 1\end{bmatrix}\\ \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}\end{bmatrix}\begin{bmatrix}\mathcal{X}\\ \mathcal{X}\end{bmatrix}.$$

This space is obviously closed, because it is the graph of a bounded operator with closed domain.

Condition (e) also holds. This is because $\left[\begin{smallmatrix}0\\u\end{smallmatrix}\right] \in \mathrm{Dom}\,(S)$ *if and only if* u *lies in* $\mathrm{Dom}\,(A)$, *see equation* (6.12), *which then yields that:*

$$\left\{ u' \in \mathcal{U} \mid \begin{bmatrix} z \\ 0 \\ y' \\ u' \end{bmatrix} \in V' \right\} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} (1 - A|_{\mathcal{X}}) \mathrm{Dom}\,(A) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathcal{X} = \mathcal{U}'.$$

We now finally turn our attention to the conditions (c) and (f). Note that $u' = 0$ *if and only if* $\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right] \in \mathcal{N}\left(\begin{bmatrix} -A|_{\mathcal{X}} & 1 - A|_{\mathcal{X}} \end{bmatrix}\right) = \begin{bmatrix} 1 \\ A(1-A)^{-1} \end{bmatrix} \mathcal{X}$. *Thus we obtain*

$$\begin{aligned} V'_y &= \begin{bmatrix} \begin{bmatrix} A|_{\mathcal{X}} & A|_{\mathcal{X}} \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} A|_{\mathcal{X}} & 1 + A|_{\mathcal{X}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ A(1-A)^{-1} \end{bmatrix} \mathcal{X} \\ &= \begin{bmatrix} A(1-A)^{-1} \\ 1 \\ \begin{bmatrix} -1 \\ 1 \end{bmatrix} A(1-A)^{-1} \end{bmatrix} \mathcal{X} =: \begin{bmatrix} A' \\ 1 \\ C' \end{bmatrix} \mathcal{X}. \end{aligned}$$

Both A' *and* C' *are bounded, and so condition (f) is met. Moreover, condition (c) is met, because* A' *generates the uniformly continuous group* $(\mathfrak{A}')^t = e^{A't}$ *on* $\mathcal{X}$, *according to* [Sta05, Example 3.1.2].

Thus V' *generates an* L^p*-well-posed s/s system, which has the admissible i/o pair* $\left(\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\mathcal{X}, \left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]\mathcal{X}\right)$ *by Proposition 6.7. Recalling that we identify* $V = V'$ *finishes the proof of our claim.*

The technique we used in the above example amounts to the replacement of the original impedance representation of $(V; \mathcal{X}, \mathcal{W})$ by a scattering representation, which is always L^2 well posed. See [Kur09] for details.

The next example, which includes PID controllers, shows that the systems in [KS07] are well-posed s/s systems, although they are not well posed in the i/s/o sense. We refer the reader to [ÅH95] for more information on PID controllers.

Example 6.9. *Let* $\mathcal{X} = \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{X}_1 \end{bmatrix}$, $\mathcal{U}$ *and* $\mathcal{Y}$ *be Banach spaces and assume that* A_1 *generates a* C_0 *semigroup* $\mathfrak{A}_1$ *on* $\mathcal{X}_1$. *Let* $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$, $\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \end{bmatrix}$, u *and* y *be continuous on* $\mathbb{R}^+$. *Consider the system*

$$\begin{bmatrix} x_0(t) \\ \dot{x}_1(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & B_0 \\ 0 & A_1 & B_1 \\ C_0 & C_1 & D_1 \end{bmatrix} \begin{bmatrix} \dot{x}_0(t) \\ x_1(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad \begin{bmatrix} x_0(0) \\ x_1(0) \end{bmatrix} \text{ given,} \tag{6.13}$$

where B_i, C_i, D_1 *are bounded and* B_0, C_0 *have closed range. In this example we study in which case* (6.13) *determines the space of classical trajectories of a well-posed s/s system* Σ *on* $\left(\begin{bmatrix}\mathcal{X}_0\\ \mathcal{X}_1\end{bmatrix}, \begin{bmatrix}\mathcal{Y}\\ \mathcal{U}\end{bmatrix}\right)$.

First we observe that $x_0(0) = B_0 u(0)$ *and thus, if* B_0 *does not have dense range, then we may not choose the starting state* $\begin{bmatrix}x_0(0)\\ x_1(0)\end{bmatrix}$ *densely in* $\begin{bmatrix}\mathcal{X}_0\\ \mathcal{X}_1\end{bmatrix}$ *and thus* Σ *is ill posed, because condition (i) of Definition 2.7 is violated. If* C_0 *is not injective, then* $x_0(0) = 0$, $x_1(0) = 0$, $w(0) = 0$ *does not imply that* $\dot{x}_0(0) = 0$ *and thus* Σ *is ill posed, by condition (ii) of Definition 2.3. From now on we thus assume that* B_0 *is surjective and* C_0 *is injective with closed range.*

Moreover, if $x_0(0) = 0$ *then* $u(0) \in \mathcal{N}(B_0) := \mathcal{U}_1$ *and it seems reasonable that* $\mathcal{U}_0 := \mathcal{U} \ominus \mathcal{U}_1$ *is not part of any input space. On the other hand,* $x_0(0) = 0$, $x_1(0) = 0$ *and* $u(0) = 0$ *only imply that* $y \in \mathrm{Ran}\,(C_0)$, *which hints at* $\mathcal{Y}_1 := \mathrm{Ran}\,(C_0)$ *being part of an input space. Then* $\mathcal{Y}_0 := \mathcal{Y} \ominus \mathcal{Y}_1$ *could be part of an output space. In accordance with these splittings of* $\mathcal{U}$ *and* $\mathcal{Y}$, *the equation* (6.13) *splits into:*

$$\begin{bmatrix} x_0(t)\\ \dot{x}_1(t)\\ y_1(t)\\ y_0(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & B_{00}\\ 0 & A_1 & B_{11} & B_{10}\\ C_{10} & C_{11} & D_{11} & D_{10}\\ 0 & C_{01} & D_{01} & D_{00}\end{bmatrix}\begin{bmatrix}\dot{x}_0(t)\\ x_1(t)\\ u_1(t)\\ u_0(t)\end{bmatrix}, \tag{6.14}$$

where B_{00} *and* C_{10} *are bijective. Thus* B_{00} *and* C_{10} *have bounded inverses by the closed graph theorem.*

Let $1 \le p < \infty$ *be arbitrary. We will now use Proposition 6.7 to show that*

$$V := \left\{\begin{bmatrix}\dot{x}_0\\ \dot{x}_1\\ x_0\\ x_1\\ y_0\\ u_0\\ y_1\\ u_1\end{bmatrix}\right\} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & A_1 & B_{11} & B_{10}\\ 0 & 0 & 0 & B_{00}\\ 0 & 1 & 0 & 0\\ 0 & C_{01} & D_{01} & D_{00}\\ 0 & 0 & 0 & 1\\ C_{10} & C_{11} & D_{11} & D_{10}\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\mathcal{X}_0\\ \mathrm{Dom}\,(A_1)\\ \mathcal{U}_1\\ \mathcal{U}_0\end{bmatrix} \subset \begin{bmatrix}\mathcal{X}_0\\ \mathcal{X}_1\\ \mathcal{X}_0\\ \mathcal{X}_1\\ \mathcal{Y}_0\\ \mathcal{U}_0\\ \mathcal{Y}_1\\ \mathcal{U}_1\end{bmatrix}$$

generates an L^p*-well-posed s/s system with admissible i/o pair* $\left(\begin{bmatrix}\mathcal{Y}_1\\ \mathcal{U}_1\end{bmatrix}, \begin{bmatrix}\mathcal{Y}_0\\ \mathcal{U}_0\end{bmatrix}\right)$.

One may generally show that if H *is a closed operator and* K *is a bounded operator with closed domain, then* $\begin{bmatrix} H & K\end{bmatrix}$ *with domain* $\begin{bmatrix}\mathrm{Dom}\,(H)\\ \mathrm{Dom}\,(K)\end{bmatrix}$ *is closed. If moreover* $\mathrm{Dom}\,(H) \subset \mathrm{Dom}\,(K)$, *then* $\left[\begin{smallmatrix}H\\ K\end{smallmatrix}\right]$ *with domain* $\mathrm{Dom}\,(H)$ *is also closed. This immediately gives that* V *has properties (a) and (b) given in Proposition 6.7 because* V *is a trivial permutation of the graph of* $\begin{bmatrix} 0 & 0 & 0 & B_{00}\\ 0 & A_1 & B_{11} & B_{10}\\ C_{10} & C_{11} & D_{11} & D_{10}\\ 0 & C_{01} & D_{01} & D_{00}\end{bmatrix}$ *and*

V_z is essentially the graph of $\begin{bmatrix} 0 & A_1 & B_{11} & B_{10}B_{00}^{-1} \\ 0 & C_{01} & D_{01} & D_{00}B_{00}^{-1} \end{bmatrix}$*, where A_1 is closed and the rest of the operators are bounded with closed domains. For condition (e) we obtain that*

$$\left\{ \begin{bmatrix} y_1 \\ u_1 \end{bmatrix} \mid \exists z_0, z_1, y_0, u_0 : \begin{bmatrix} z_0 \\ z_1 \\ 0 \\ 0 \\ y_0 \\ u_0 \\ y_1 \\ u_1 \end{bmatrix} \in V \right\} \supset \begin{bmatrix} C_{10} & D_{11} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{X}_0 \\ \mathcal{U}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{U}_1 \end{bmatrix},$$

by the surjectivity of C_{10}.

We still need to check conditions (c) and (f). We have $\begin{bmatrix} z_0 \\ z_1 \\ x_0 \\ x_1 \\ y_0 \\ u_0 \\ 0 \\ 0 \end{bmatrix} \in V$ *if and only if:*

$$\begin{bmatrix} z_0 \\ z_1 \\ x_0 \\ x_1 \\ y_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_1 & B_{10} \\ 0 & 0 & B_{00} \\ 0 & 1 & 0 \\ 0 & C_{01} & D_{00} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ x_1 \\ u_0 \end{bmatrix} \text{ and } \begin{bmatrix} z_0 \\ x_1 \\ u_0 \end{bmatrix} \in \mathcal{N}\left(\begin{bmatrix} C_{10} & C_{11} & D_{10} \end{bmatrix}\right).$$

Due to the invertibility of C_{10} we may write the null space as

$$\begin{bmatrix} -C_{10}^{-1}C_{11} & -C_{10}^{-1}D_{10} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Dom}\,(A_1) \\ \mathcal{U}_0 \end{bmatrix}.$$

After some straightforward computations, which use the fact that $\mathcal{U}_0 = B_{00}^{-1}\mathcal{X}_0$, we obtain:

$$V_y = \begin{bmatrix} -C_{10}^{-1}D_{10}B_{00}^{-1} & -C_{10}^{-1}C_{11} \\ B_{10}B_{00}^{-1} & A_1 \\ 1 & 0 \\ 0 & 1 \\ D_{00}B_{00}^{-1} & C_{01} \\ B_{00}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{X}_0 \\ \mathrm{Dom}\,(A_1) \end{bmatrix}.$$

The operator $A' := \begin{bmatrix} -C_{10}^{-1}D_{10}B_{00}^{-1} & -C_{10}^{-1}C_{11} \\ B_{10}B_{00}^{-1} & A_1 \end{bmatrix}$ *is a bounded perturbation of the operator* $\begin{bmatrix} 0 & 0 \\ 0 & A_1 \end{bmatrix}$*, which generates the C_0 semigroup* $t \to \begin{bmatrix} 1 & 0 \\ 0 & \mathfrak{A}_1^t \end{bmatrix}$ *on*

$\begin{bmatrix} \mathcal{X}_0 \\ \mathcal{X}_1 \end{bmatrix}$. *From* [Kat95, Thm IX.2.1] *we know that* A' *generates a* C_0 *semigroup on* $\begin{bmatrix} \mathcal{X}_0 \\ \mathcal{X}_1 \end{bmatrix}$. *The operator* $\begin{bmatrix} D_{00}B_{00}^{-1} & C_{01} \\ B_{00}^{-1} & 0 \end{bmatrix}$ *is bounded and thus conditions (c) and (f) of Proposition 6.7 are also met.*

Concluding the example, we assumed that A_1 *generates a* C_0 *semigroup on* $\mathcal{X}_1$ *and the operators* B_i, C_i, D_1 *are bounded, where* B_0 *and* C_0 *in addition have closed range. Under these assumptions we showed that* (6.13) *determines a s/s system which is* L^p*-well-posed for all* $1 \le p < \infty$ *if and only if* B_0 *is surjective and* C_0 *is injective.*

As we shall see in [Kur09], one can replace the boundedness conditions on the involved operators by other conditions related to passivity and still obtain a well-posed s/s system.

7. Conclusions

We have introduced the new class of continuous-time L^p-well-posed linear s/s systems. The definition of this class is based on the idea of equal treatment of inputs and outputs, which is inherent to network theory. We have presented the most important basic properties of these s/s systems and showed how to work with them, mainly using their trajectories. We also indicated some advantages of our approach. One of the central notions in the paper is the i/s/o representation, from which we have derived an explicit expression for the maximal generating subspace of any given well-posed s/s system.

We will return elsewhere with a study of passive s/s systems as an extension of Example 6.8. All passive s/s systems are L^2 well posed in the sense of the current article and these systems have a rich additional structure. Interconnection of s/s systems in the spirit of [KZvdSB08] is also a main point of interest, which still remains to be explored.

Appendix A. Background

This appendix provides notation and some general background for the paper.

Definition A.1. *Let* I, I', I_f *and* I_g *be subsets of* $\mathbb{R}$ *and let* $\mathcal{U}$ *be a Banach space.*

(i) *The vector space of functions defined everywhere on* I *with values in* $\mathcal{U}$ *is denoted by* $\mathcal{U}^I$.

(ii) *For* $f \in \mathcal{U}^I$ *and* $a \in I$ *we define the* point-evaluation operator δ_a *through* $\delta_a f := f(a)$.

(iii) *For all* $t \in \mathbb{R}$ *we define the* shift operator τ^t, *which maps functions in* $\mathcal{U}^I$ *into functions in* $\mathcal{U}^{I-t}$, *by* $(\tau^t f)(v) = f(v+t)$ *for* $f \in \mathcal{U}^I$ *and* $v + t \in I$. *If* $t > 0$ *then* τ^t *is a left shift by the amount* t.

(iv) *The operator* $\pi_I : \mathcal{U}^I \to \mathcal{U}^{\mathbb{R}}$ *is defined by*

$$(\pi_I f)(v) := \begin{cases} f(v), & v \in I \\ 0, & v \in \mathbb{R} \setminus I \end{cases}.$$

We briefly write $\pi_+ := \pi_{[0,\infty)}$ *and* $\pi_- := \pi_{(-\infty,0)}$.

(v) *For* $I' \supset I$, *the* restriction operator $\rho_I : \mathcal{U}^{I'} \to \mathcal{U}^I$ *is given by*

$$(\rho_I f)(v) = f(v),\ v \in I, \quad \text{i.e.} \quad \rho_I f = f|_I, \quad f \in \mathcal{U}^{I'}.$$

We abbreviate $\rho_+ := \rho_{[0,\infty)}$ *and* $\rho_- := \rho_{(-\infty,0)}$.

(vi) *For* $f \in \mathcal{U}^{I_f}$, $g \in \mathcal{U}^{I_g}$ *and* $c \in \mathbb{R}$ *we define the* concatenation $f \bowtie_c g$ *of* f *and* g *at* c *as the function*

$$(f \bowtie_c g)(v) = \begin{cases} f(v), & t < c \ (,\ t \in I_f) \\ g(v), & t \geq c \ (,\ t \in I_g) \end{cases}.$$

We note that $\tau^0 = 1$ and that for all $s, t \in \mathbb{R}$ we have $\tau^s \tau^t = \tau^{s+t}$. Thus, the shift operators $t \to \tau^t$ form a group on $\mathcal{U}^{\mathbb{R}}$. If $s, t \geq 0$ then $\rho_+ \tau^s \rho_+ \tau^t = \rho_+ \tau^{s+t}$ and $\rho_- \tau^s \pi_- \rho_- \tau^t \pi_- = \rho_- \tau^{s+t} \pi_-$, i.e. $\rho_+ \tau$ is a semigroup on $\mathcal{U}^{\mathbb{R}^+}$ and $\rho_- \tau \pi_-$ is a semigroup on $\mathcal{U}^{\mathbb{R}^-}$.

The following spaces of continuous functions are used frequently.

Definition A.2. *Let* $\mathcal{U}$ *be a Banach space and let* $-\infty < a < b < \infty$.

(i) *The space of continuous* $\mathcal{U}$*-valued functions defined on* $[a, b]$ *is denoted by* $C([a, b]; \mathcal{U})$. *This space is equipped with the supremum norm*

$$\|f\|_{C([a,b];\mathcal{U})} := \sup_{t \in [a,b]} \|f(t)\|_{\mathcal{U}}.$$

(ii) *The space of all* $\mathcal{U}$*-valued functions defined on* $[a, b]$ *with* $n \in \mathbb{Z}^+$ *continuous derivatives is denoted by* $C^n([a, b]; \mathcal{U})$ *and equipped with the norm*

$$\|f\|_{C^n([a,b];\mathcal{U})} := \sum_{k=0}^{n} \|f^{(k)}\|_{C([a,b];\mathcal{U})}. \tag{A.1}$$

(iii) *The space of* $\mathcal{U}$*-valued functions defined on* $[a, \infty)$ *with* $n \in \mathbb{Z}^+$ *continuous derivatives is denoted by* $C^n([a, \infty); \mathcal{U})$. *This space is equipped with the compact-open topology induced by the family*

$$\|f\|_b := \|\rho_{[a,b]} f\|_{C^n([a,b];\mathcal{U})}$$

of seminorms, which is indexed by $b > a$. *By writing* $C([a, \infty); \mathcal{U})$ *we mean* $C^0([a, \infty); \mathcal{U})$.

The space $C^n([a, b]; \mathcal{U})$ is a Banach space and $C^n([a, \infty); \mathcal{U})$ is a Fréchet space for all $n \in \mathbb{Z}^+$. Convergence to zero of a sequence f_n in a Fréchet spaces means that $\|f_n\|_b \to 0$ for all $b > a$.

Definition A.3. *Let $\mathcal{U}$ be a Banach space and let $I = [a, b]$ or $I = [a, \infty)$.*

(i) *By $L^p(I;\mathcal{U})$ we denote the space of all $\mathcal{U}$-valued Lebesgue-measurable functions f defined on I, such that*

$$\|f\|_{L^p(I;\mathcal{U})} := \left(\int_I \|f(v)\|_{\mathcal{U}}^p \, \mathrm{d}v \right)^{1/p} < \infty. \tag{A.2}$$

(ii) *The space $L^p_{loc}(I;\mathcal{U})$ consists of all Lebesgue-measurable functions, which map I into $\mathcal{U}$, such that $\rho_{[a,b]} f \in L^p([a, b];\mathcal{U})$ for all bounded subintervals $[a, b]$ of I. A family of seminorms on $L^p_{loc}([a, \infty);\mathcal{U})$, which is indexed by $b > a$, is given by*

$$\|f\|_b := \|\rho_{[a,b]} f\|_{L^p([a,b];\mathcal{U})}.$$

(iii) *By $W^{1,p}_{loc}(I;\mathcal{U})$ we denote the space of such $f \in L^p_{loc}(I;\mathcal{U})$, for which there exists some $\dot{f} \in L^p_{loc}(I;\mathcal{U})$ that satisfies:*

$$\forall a, b \in I : \quad f(b) - f(a) = \int_a^b \dot{f}(s) \, \mathrm{d}s.$$

(iv) *The subspace $L^p_c(\mathbb{R}^-;\mathcal{U})$ of $L^p(\mathbb{R}^-;\mathcal{U})$ consists of all functions with bounded support.*

(v) *A function $f \in L^p_{loc}(\mathbb{R};\mathcal{U})$ lies in $L^p_{c,loc}(\mathbb{R};\mathcal{U})$ if $\rho_- f \in L^p_c(\mathbb{R}^-;\mathcal{U})$.*

The functions in L^p_c have compact support, hence the choice of the notation L^p_c. The elements of $L^p_{c,loc}(\mathbb{R};\mathcal{U})$ can equivalently be thought of as being functions in $L^p_{loc}(\mathbb{R};\mathcal{U})$ with support bounded to the left. This means that $f \in L^p_{c,loc}(\mathbb{R};\mathcal{U})$ if and only if there exists a $t \in \mathbb{R}^+$ such that $\rho_+ \tau^{-t} f \in L^p_{loc}(\mathbb{R}^+;\mathcal{U})$.

The space $L^p(I;\mathcal{U})$ is a Banach space for $p \in [1, \infty)$. For finite intervals $[a, b]$, the spaces $L^p_{loc}([a, b];\mathcal{U})$ and $L^p([a, b];\mathcal{U})$ coincide. The spaces $L^p_{loc}([a, \infty);\mathcal{U})$ and $W^{1,p}_{loc}([a, \infty);\mathcal{U})$ are Fréchet spaces, whereas $L^p_c(\mathbb{R}^-;\mathcal{U})$ is not a Fréchet space. In $L^p_c(\mathbb{R}^-;\mathcal{U})$, $f_n \to 0$ if there exists some $s \in \mathbb{R}$ such that $\operatorname{supp}(f_n) \subset [s, 0]$ for all n and $\|f_n\|_{L^p(\mathbb{R}^-;\mathcal{U})} \to 0$.

The operators τ, π, ρ and $\bowtie$ of Definition A.1 have obvious extensions to the L^p spaces in Definition A.3. Moreover, we may also apply the pointwise-projection operator $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}}$ to a function, which belongs to an L^p-type space, by setting $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w = u$ if and only if $\mathcal{P}^{\mathcal{Y}}_{\mathcal{U}} w(t) = u(t)$ almost everywhere. We often apply some of these operators to such a space of functions, meaning e.g.

$$\rho_+ \tau^t L^p_{loc}(\mathbb{R}^+;\mathcal{U}) = \left\{ \rho_+ \tau^t f \mid f \in L^p_{loc}(\mathbb{R}^+;\mathcal{U}) \right\},$$

for some $t \geq 0$.

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