Connections between classical and generalised trajectories of a state/signal system

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Abstract. In our earlier article "Well-posed state/signal systems in continuous time", we originally defined the notion of a trajectory of a state/signal system by means of a generating subspace. However, it was left as an open problem whether the generating subspace is uniquely determined by a given family of all generalised trajectories of a well-posed state/signal system. In this article we give a positive answer to this question and show how this insight simplifies some formulations in the theory of well-posed state/signal systems.

The main contribution of this article is an explicit convolution scheme for constructing classical trajectories approximating an arbitrary generalised trajectory. We apply this scheme by studying relationships between classical and generalised trajectories of continuous-time state/signal systems under very weak assumptions. Among others, we show that there exists a space of classical trajectories that is invariant under differentiation and dense in the space of generalised trajectories.

Some of our results generalise known results for strongly continuous semigroups and input/state/output systems, but we make no use of decompositions of the signal space into an input space and an output space, and in particular, none of our results depend on well-posedness.

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1. Introduction

In our earlier article [4], "Well-posed state/signal systems in continuous time", we defined the notion of a trajectory of a state/signal system by means of a generating subspace V. However, it was left as an open problem whether the generating subspace is uniquely determined by a given family of all generalised trajectories of a well-posed state/signal system. In this article we supplement [4] by proving that a generating subspace V is uniquely determined by its space of all generalised trajectories. We indicate some important implications for the theory of well-posed state/signal systems presented in [4]. In order to prove the uniqueness of V, we introduce a universal convolution scheme for approximating generalised trajectories by classical trajectories. This approximation scheme turns out to be useful in many connections, and therefore it is of independent interest.

We now introduce the reader to this article by recalling the basic definitions of state/signal systems from [4]. We throughout let $\mathcal{X}$ and $\mathcal{W}$ denote two Banach spaces with the same scalar field, which we denote by $\mathbb{F}$, and we call these spaces the *state space* and *signal space*, respectively. Various standard spaces of $\mathcal{X}$- and $\mathcal{W}$-valued functions defined on subintervals of $\mathbb{R}$ are needed in this article, and we refer the reader to the appendix of [4] for the definitions of these spaces.

Another convention in this article is to denote all closed intervals by $[a, b]$ and all open intervals by (a, b), where we allow $a = -\infty$ and $b = \infty$, but not $a = b$, which we write as $-\infty \le a < b \le \infty$. By writing e.g. $[a, b]$ we mean the interval $[a, \infty)$ if $b = \infty$.

Definition 1.1. Let $[a, b]$ be a closed interval of positive and possibly infinite length: $-\infty \le a < b \le \infty$. Let V be a closed subspace of the product space $\mathfrak{K} := \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{bmatrix}$ which we equip with the product norm and call the *node space*.

(i) The space $\mathfrak{V}[a, b]$ of *classical trajectories* generated by V on $[a, b]$ consists of all pairs $\begin{bmatrix} x \\ w \end{bmatrix} \in \begin{bmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{bmatrix}$, such that $\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for all $t \in (a, b)$, where $\dot{x} := \frac{\mathrm{d}}{\mathrm{d}t} x$. The most important classical trajectory space is $\mathfrak{V} := \mathfrak{V}[0, \infty)$.

(ii) Let $1 \le p < \infty$. The space $\mathfrak{W}^p[a, b]$ of *generalised trajectories* generated by V is the closure of $\mathfrak{V}[a, b]$ in $\begin{bmatrix} C([a,b];\mathcal{X}) \\ L^p_{\mathrm{loc}}([a,b];\mathcal{W}) \end{bmatrix}$.

By this we mean that that $\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[a, b]$ if and only if there exists a sequence $\begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}[a, b]$ of classical trajectories, such that on all compact subintervals of $[a, b]$, $x_n \to x$ uniformly and $w_n \to w$ in L^p as $n \to \infty$. We abbreviate $\mathfrak{W}^p := \mathfrak{W}^p[0, \infty)$.

(iii) We say that V *generates* $\mathfrak{W}^p$ as described above. More generally, we call every closed $V' \subset \mathfrak{K}$, whose space $\mathfrak{V}'$ of classical trajectories is dense in $\mathfrak{W}^p$, a *generating subspace* for $\mathfrak{W}^p$.

(iv) The spaces $\mathfrak{V}_0[a, b]$ and $\mathfrak{W}^p_0[a, b]$ of *externally generated* classical and generalised trajectories, respectively, are given by

$$\begin{aligned} \mathfrak{V}_0[a, b] &:= \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[a, b] \;\middle|\; \begin{bmatrix} x(a) \\ w(a) \end{bmatrix} = 0 \right\} \quad \text{and} \\ \mathfrak{W}^p_0[a, b] &:= \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[a, b] \;\middle|\; x(a) = 0 \right\}, \end{aligned} \tag{1.1}$$

where we in the case $a = -\infty$ interpret $\begin{bmatrix} x(-\infty) \\ w(-\infty) \end{bmatrix} := \lim_{t \to -\infty} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$.

Assume that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{smallmatrix}\right]$ and that $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V$ for all $t \in (a,b)$. Then $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V$ also for $t = a$ and $t = b$ when we consider the appropriate one-sided derivatives. In the rest of the article we make the standing assumption that $\mathfrak{V}$ and $\mathfrak{W}^p$ denote the spaces of classical and generalised trajectories corresponding to some generating subspace V, where V may or may not be known a priori.

In order to obtain a meaningful theory, we need to assume that the generating subspace has some additional structure. In this connection it is natural to impose the following two conditions, and they turn out to be sufficient for most of our considerations in this article.

Definition 1.2. Recall that $\mathcal{X}$ and $\mathcal{W}$ are Banach spaces, and let V generate the classical trajectories $\mathfrak{V}$ on $\mathbb{R}^+$. The triple $(V;\mathcal{X},\mathcal{W})$ is a *state/signal node (shortly s/s node)* if V has the following properties in addition to being closed:

(i) The space V has the property $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Longrightarrow z = 0$.

(ii) Every element of V can be chosen as the starting data of some classical trajectory of uniformly positive length:

$$\exists T > 0 : \forall \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V : \exists \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T] : \quad \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}.$$

Letting $\mathfrak{W}^p$ be the closure of $\mathfrak{V}$ in $\left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ as above, we call the triple $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ the *state/signal system (s/s system)* generated by $(V;\mathcal{X},\mathcal{W})$.

Remark 1.3. In Definition 1.1, a generating subspace is only assumed to be closed. In particular, it is not assumed to have properties (i) or (ii) of Definition 1.2.

It was proved in Lemmas 2.2 and 2.4 of [4] that condition (ii) of Definition 1.2 holds if and only if for all $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ there exists a $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$, such that $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$. A direct consequence of this condition and Definition 1.1(i) is that V is uniquely determined by $\mathfrak{V}$ through

$$V = \left\{ \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V} \right\}. \tag{1.2}$$

In fact, (1.2) remains true if we replace $\mathfrak{V}$ by $\mathfrak{V}[a,b]$ for any $-\infty < a < b \le \infty$.

We need to introduce the *shift operator* τ, which is defined by

$$(\tau^c x)(t) = x(t+c), \quad c \in \mathbb{R},\ t + c \in \operatorname{dom}(x).$$

(If $c > 0$ then τ^c is a left shift.) Moreover, by ρ we denote the operator which *restricts the domains* of functions:

$$(\rho_{[a,b]}x)(t) = x(t) \quad \text{for} \quad t \in [a,b], \quad \text{where} \quad [a,b] \subset \operatorname{dom}(x),$$

and we abbreviate $\rho_+ := \rho_{[0,\infty)}$. Finally, the *concatenation operator* at $c \in \mathbb{R}$ is denoted by $\bowtie_c$:

$$(x^1 \bowtie_c x^2)(t) = \begin{cases} x^1(t), & t \in (-\infty, c) \cap \operatorname{dom}\left(x^1\right), \\ x^2(t), & t \in [c, \infty) \cap \operatorname{dom}\left(x^2\right). \end{cases}$$

The following useful consequences of Definition 1.1 were given in [4, pp. 324–326 and Cor. 3.2].

Lemma 1.4. *The following claims all hold for $-\infty \le a < b \le \infty$:*

(i) *A pair $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ lies in $\mathfrak{V}[a,b]$ if and only if $\left[\begin{smallmatrix} \dot{x} \\ x \\ w \end{smallmatrix}\right] \in C([a,b];V)$.*
(ii) *For all $c \in \mathbb{R}$ we have the* time-invariance property $\tau^c \mathfrak{V}[a,b] = \mathfrak{V}[a-c, b-c]$.
(iii) *For all $[a', b'] \subset [a,b]$ we have $\rho_{[a',b']}\mathfrak{V}[a,b] \subset \mathfrak{V}[a',b']$, with equality if $a' = a$.*
(iv) *Let $c \in (a,b)$, $\left[\begin{smallmatrix} x^1 \\ w^1 \end{smallmatrix}\right] \in \mathfrak{V}[a,c]$ and $\left[\begin{smallmatrix} x^2 \\ w^2 \end{smallmatrix}\right] \in \mathfrak{V}[c,b]$. Then*

$$\begin{bmatrix} x^1 \\ w^1 \end{bmatrix} \bowtie_c \begin{bmatrix} x^2 \\ w^2 \end{bmatrix} \in \mathfrak{V}[a,b]$$

if and only if $\dot{x}^1(c) = \dot{x}^2(c)$, $x^1(c) = x^2(c)$, and $w^1(c) = w^2(c)$.

If V has property (i) in Definition 1.2 then $x^1(c) = x^2(c)$ and $w^1(c) = w^2(c)$ imply that $\dot{x}^1(c) = \dot{x}^2(c)$.

Claims (ii) and (iii) hold also for generalised trajectories if we replace $\mathfrak{V}$ by $\mathfrak{W}^p$.

We now recall the concept of L^p-well-posedness of a s/s node from [4] in order to add context to the discussion in this article. We will not base any of our results on well-posedness.

Definition 1.5. Two closed subspaces $\mathcal{U}$ and $\mathcal{Y}$ of a Banach space $\mathcal{W}$ form a *direct-sum decomposition* if $\mathcal{U}$ and $\mathcal{Y}$ are closed subspaces of $\mathcal{W}$ and $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, i.e., every vector in $\mathcal{W}$ can be written as the sum of unique elements $u \in \mathcal{U}$ and $y \in \mathcal{Y}$.

The corresponding (bounded) *projection* onto $\mathcal{U}$ along $\mathcal{Y}$ is denoted $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}$ and the complementary projection is $\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}$. By this we mean that if $w = u + y$, where $u \in \mathcal{U}$ and $y \in \mathcal{Y}$, then $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = u$ and $\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w = (1 - \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}})w = y$.

We apply the projection $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}$ to a function w pointwisely (almost) everywhere, i.e. $(\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w)(t) := \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(t)$ for (almost) all $t \in \operatorname{dom}(w)$.

Definition 1.6. The s/s *node* $(V; \mathcal{X}, \mathcal{W})$ is *L^p-well-posed*, where $1 \le p < \infty$, if there exists a $T > 0$ and a direct sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$, such that $\mathfrak{V}[0,T]$ satisfies the following conditions:

(i) The space $\left\{ x(0) \;\middle|\; \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T] \right\}$ is dense in $\mathcal{X}$.
(ii) The space $\left\{ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \;\middle|\; \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}_0[0,T] \right\}$ is dense in $L^p([0,T];\mathcal{U})$.
(iii) There exists a $K_T > 0$, such that all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ satisfy

$$\|x(t)\|_{\mathcal{X}} + \|\rho_{[0,t]} w\|_{L^p([0,t];\mathcal{W})} \le K_T \left(\|x(0)\|_{\mathcal{X}} + \|\rho_{[0,t]} \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w\|_{L^p([0,t];\mathcal{U})} \right) \tag{1.3}$$

for all $t \in [0,T]$.

In this case we call $(\mathcal{U},\mathcal{Y})$ an *L^p-admissible input/output space pair (admissible i/o pair)* of the *s/s node* $(V;\mathcal{X},\mathcal{W})$. Here we regard $\mathcal{U}$ as an input space and $\mathcal{Y}$ as an output space.

A s/s *system* $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ is L^p-well-posed with admissible i/o pair $(\mathcal{U},\mathcal{Y})$ if $(\mathcal{U},\mathcal{Y})$ is L^p-admissible for at least one s/s node that generates $\mathfrak{W}^p$.

In order to prove that a s/s system is well-posed directly using Definition 1.6, one needs to find a generating s/s node together with an L^p-admissible i/o pair, which may not be a straightforward task. In [4, Sect. 6], we proved that every L^p-well-posed s/s system $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ *has one unique maximal* s/s node $(V_{\max};\mathcal{X},\mathcal{W})$, which is given by

$$V_{\max} := \left\{ \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \,\middle|\, \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p \cap \begin{bmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{bmatrix} \right\}. \tag{1.4}$$

This simplifies the use of Definition 1.6, because we only need to test i/o pairs for admissibility on $V_{\max}$. (In [4] we also developed other tools for showing well-posedness, such as Prop. 3.11 and Thm 6.6.)

In Section 3 of the present article, we prove that $\mathfrak{W}^p$ determines its generating subspace V uniquely as $V = V_{\max}$ if V has property (ii) in Definition 1.2, which is a much weaker assumption than L^p-well-posedness. This insight leads to a better understanding of continuous-time s/s systems theory and simplifications of some of the formulations in [4], and we point out the most important of these simplifications out as well. The main tool used in the uniqueness proof, and throughout this whole article, is the approximation scheme developed in Section 2.

In Section 4 we present a brief treatment of the technical assumption that $\mathfrak{V}_0[0,T]$ is dense in $\mathfrak{W}_0^p[0,T]$ for some $T > 0$. We often used this assumption in [4], but we never explained it properly there. We apply the results of Section 4 in Section 5 by studying when derivatives and primitives of trajectories are trajectories themselves.

The results in this article are true input/output-free state/signal results in the sense that neither the statements of the results nor the proofs utilise any decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of the signal space into an input space $\mathcal{U}$ and an output space $\mathcal{Y}$.

2. Approximation of generalised trajectories

Our first objective is to construct a sequence of classical approximations of an arbitrary generalised trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$. This is done in Theorem 2.2, which is the main result of this article.

Definition 2.1. For our usual Banach state space $\mathcal{X}$ we define the following function spaces:

(i) The space $BUC(\mathbb{R};\mathcal{X})$ consists of bounded and uniformly continuous functions $\mathbb{R} \to \mathcal{X}$ and we equip it with the supremum norm.

(ii) The space $BUC^1(\mathbb{R};\mathcal{X})$ consists of those continuously differentiable functions in $BUC(\mathbb{R};\mathcal{X})$ whose derivatives also lie in $BUC(\mathbb{R};\mathcal{X})$.
(iii) By $L^1(\mathbb{R})$ we mean $L^1(\mathbb{R};\mathbb{F})$, where $\mathbb{F}$ is the scalar field of $\mathcal{X}$.
(iv) By $BV(\mathbb{R})$ we denote the space of functions of bounded variation defined on $\mathbb{R}$ with values in $\mathbb{F}$; see [2, p. 92] for the definition of total variation.

We make the following construction: we let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$ for $-\infty \le a < b \le \infty$ and $p \in [1,\infty)$, and extend $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to a function pair $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ by setting

$$\widetilde{x}(t) := \begin{cases} x(t), & t \in [a,b] \\ x(a), & t < a \\ x(b), & t > b \end{cases} \quad \text{and} \quad \widetilde{w}(t) := \begin{cases} w(t), & t \in [a,b] \\ 0, & t \notin [a,b] \end{cases} \quad \text{a.e.} \tag{2.1}$$

(If a or b is infinite, then there is no need to extend $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in that direction.) If both a and b are finite, then obviously $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} BUC([a,b];\mathcal{X}) \\ L^p([a,b];\mathcal{W}) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^p(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$. In this article we denote $\mathbb{Z}^+ := \{1,2,3,\ldots\}$, and moreover, we interpret e.g. $[a,\infty - c] = [a,\infty)$ for all finite c.

For every $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ and $r_n \in L^1(\mathbb{R})$ with support contained in the finite interval $[-S_-/n, S_+/n]$, where $n \in \mathbb{Z}^+$, the *convolution*

$$\left(\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} * r_n \right)(t) := \int_{-S_-/n}^{S_+/n} \begin{bmatrix} \widetilde{x}(t-s) \\ \widetilde{w}(t-s) \end{bmatrix} r_n(s)\,\mathrm{d}s, \quad t \in \mathbb{R}, \tag{2.2}$$

of $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ and r_n is well-defined. In the sequel we often denote $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] := \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] * r_n$.

Theorem 2.2. *Let $-\infty \le a < b \le \infty$, let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C([a,b];\mathcal{X}) \\ L^p_{\mathrm{loc}}([a,b];\mathcal{X}) \end{smallmatrix}\right]$, and let $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ be the extension of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ defined in* (2.1)*. Let $r \in L^1(\mathbb{R})$ be supported in $[-S_-, S_+]$, for some finite S_+, $S_- > 0$, and assume that $\int_{-S_-}^{S_+} r(s)\,\mathrm{d}s = 1$.*

Then the following claims are valid for the sequence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] := \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] * r_n$ of functions, where $r_n(t) := nr(nt)$ for $t \in \mathbb{R}$ and $n \in \mathbb{Z}^+$:

(i) *The sequence $\rho_{[a,b]} \left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ converges to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C([a,b];\mathcal{X}) \\ L^p_{\mathrm{loc}}([a,b];\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$.*
(ii) *If $r \in BV(\mathbb{R})$ then $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R};\mathcal{X}) \\ C(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ for all $n \in \mathbb{Z}^+$.*
(iii) *Assume that $r \in BV(\mathbb{R})$ and that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is a generalised trajectory generated by the closed subspace V of $\mathfrak{K}$ on $[a,b]$: $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$.*

Then, for all $n > (S_- + S_+)/(b-a)$, the restriction of $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ to $[a + S_+/n, b - S_-/n]$ is a classical trajectory generated by V:

$$\rho_{[a+S_+/n,\, b-S_-/n]} \begin{bmatrix} x_n \\ w_n \end{bmatrix} \in \mathfrak{V}[a + S_+/n, b - S_-/n]. \tag{2.3}$$

Proof. The proofs of (i)–(iii) are slightly simpler in the case where the interval $[a,b]$ is finite, because the convolution results that we need for this case are well-known and readily found in the literature. Therefore we first treat this case and return to the case where $a = -\infty$ and/or $b = +\infty$ at the end of the proof. The relevant results are often stated and proved only for the case where the functions $\widetilde{x}$ and $\widetilde{w}$ take their values in a finite-dimensional space, but they can in fact be established for functions with values in a Banach space with trivial modifications of the proofs for the finite-dimensional case.

(i) In the case where the interval $[a,b]$ is finite, $\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right] \in \left[\begin{smallmatrix}BUC(\mathbb{R};\mathcal{X})\\ L^p(\mathbb{R};\mathcal{W})\end{smallmatrix}\right]$, and therefore $\left[\begin{smallmatrix}x_n\\ w_n\end{smallmatrix}\right] \in \left[\begin{smallmatrix}BUC(\mathbb{R};\mathcal{X})\\ L^p(\mathbb{R};\mathcal{W})\end{smallmatrix}\right]$ according to (the Banach-space-valued version of) [2, Thm 2.2.2(i)]. Moreover, $\left[\begin{smallmatrix}x_n\\ w_n\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right]$ in $\left[\begin{smallmatrix}BUC(\mathbb{R};\mathcal{X})\\ L^p(\mathbb{R};\mathcal{W})\end{smallmatrix}\right]$ by [2, Lem. 2.7.4(i,ii)], with $\epsilon := 1/n$, and thus $\rho_{[a,b]}\left[\begin{smallmatrix}x_n\\ w_n\end{smallmatrix}\right] \to \rho_{[a,b]}\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right] = \left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$.

(ii) We can redefine r on a set of measure zero in order to make it left-continuous; see [5, Cor. 2 on p. 121]. Then each of the functions r_n are of bounded variation, equal to n times the total variation of r, they are left-continuous, and supported in $[-S_-/n, S_+/n]$ with $\int_{-S_-/n}^{S_+/n} r_n(s)\,\mathrm{d}s = 1$.

We denote the Borel measure induced by the distribution derivative of r_n by μ_n (cf. [6, Ex. 13d on p. 157]):

$$\mu_n([a,b)) := r_n(b) - r_n(a), \quad -\infty \le a \le b \le \infty, \tag{2.4}$$

so that $\mu_n((-\infty,t)) := r_n(t)$ for all $t \in \mathbb{R}$, by the finite support of r_n. It is easy to see that r_n and μ_n have the same (finite) total variation; see [2, p. 92].

Since the interval $[a,b]$ is finite, we have $\widetilde{x} \in BUC(\mathbb{R};\mathcal{X})$ and $\widetilde{w} \in L^p(\mathbb{R};\mathcal{W})$. By [2, Thm 3.7.1(ii)], $x_n = \widetilde{x} * r_n$ and $w_n = \widetilde{w} * r_n$ are locally absolutely continuous, and for almost all $t \in \mathbb{R}$:

$$\begin{bmatrix}\dot{x}_n(t)\\ \dot{w}_n(t)\end{bmatrix} = \left(\begin{bmatrix}\widetilde{x}\\ \widetilde{w}\end{bmatrix} * \mu_n\right)(t) := \int_{\mathbb{R}} \begin{bmatrix}\widetilde{x}(t-s)\\ \widetilde{w}(t-s)\end{bmatrix} \mu_n(\,\mathrm{d}s). \tag{2.5}$$

The definition of convolution with a measure is from [2, Def. 3.2.1]. By [2, Thm 3.6.1(iii)] and the fact that $\widetilde{x} \in BUC(\mathbb{R};\mathcal{X})$, also $\dot{x}_n = \widetilde{x} * \mu_n \in BUC(\mathbb{R};\mathcal{X})$. Thus $x_n \in C^1(\mathbb{R};\mathcal{X})$ and $w_n \in C(\mathbb{R};\mathcal{W})$, as claimed.

(iii) Recall that $[a,b]$ is finite and assume that $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$. By Definition 1.1, $\mathfrak{V}[a,b]$ is dense in $\mathfrak{W}^p[a,b]$, and we can find a sequence $\left[\begin{smallmatrix}x^k\\ w^k\end{smallmatrix}\right] \in \left[\begin{smallmatrix}C^1([a,b];\mathcal{X})\\ C([a,b];\mathcal{W})\end{smallmatrix}\right]$ of classical trajectories generated by V on $[a,b]$ converging to $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ in $\left[\begin{smallmatrix}BUC([a,b];\mathcal{X})\\ L^p([a,b];\mathcal{W})\end{smallmatrix}\right]$. Since $b-a<\infty$, all $x^k \in BUC^1([a,b];\mathcal{X})$ and $w^k \in BUC([a,b];\mathcal{W})$. We extend these to functions in $\left[\begin{smallmatrix}BUC^1(\mathbb{R};\mathcal{X})\\ BUC(\mathbb{R};\mathcal{W})\end{smallmatrix}\right]$, still denoting the extended functions by $\left[\begin{smallmatrix}x^k\\ w^k\end{smallmatrix}\right]$, in such a way that $\left[\begin{smallmatrix}x^k\\ w^k\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right]$ in $\left[\begin{smallmatrix}BUC(\mathbb{R};\mathcal{X})\\ L^p(\mathbb{R};\mathcal{W})\end{smallmatrix}\right]$ as $k \to \infty$.

Define $\left[\begin{smallmatrix}x_n^k\\ w_n^k\end{smallmatrix}\right] := \left[\begin{smallmatrix}x^k\\ w^k\end{smallmatrix}\right] * r_n$, so that $\dot{x}_n^k = \dot{x}^k * r_n$ in $L^1_{\mathrm{loc}}(\mathbb{R};\mathcal{X})$, according to [2, Thm 3.7.1(i)]. Moreover, $\left[\begin{smallmatrix}\dot{x}^k\\ x^k\\ w^k\end{smallmatrix}\right] \in BUC\left(\mathbb{R};\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]\right)$ implies that $\left[\begin{smallmatrix}\dot{x}^k\\ x^k\\ w^k\end{smallmatrix}\right] * r_n \in BUC\left(\mathbb{R};\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]\right)$, by [2, Thm 2.2.2(i)]. Therefore it holds *for all* $t \in \mathbb{R}$

that

$$\begin{bmatrix} \dot x_n^k(t) \\ x_n^k(t) \\ w_n^k(t) \end{bmatrix} = \left(\begin{bmatrix} \dot x^k \\ x^k \\ w^k \end{bmatrix} * r_n \right)(t) = \int_{-S_-/n}^{S_+/n} \begin{bmatrix} \dot x^k(t-s) \\ x^k(t-s) \\ w^k(t-s) \end{bmatrix} r_n(s)\,\mathrm{d}s. \tag{2.6}$$

Since $\rho_{[a,b]} \left[\begin{smallmatrix} x^k \\ w^k \end{smallmatrix}\right]$ is a classical trajectory generated by V on $[a,b]$, it satisfies $\left[\begin{smallmatrix} \dot x^k(t) \\ x^k(t) \\ w^k(t) \end{smallmatrix}\right] \in V$ for all $t \in (a,b)$. Therefore also $\left[\begin{smallmatrix} \dot x^k(t-s) \\ x^k(t-s) \\ w^k(t-s) \end{smallmatrix}\right] r_n(s) \in V$ for all $s \in [-S_-/n, S_+/n]$ and all $t \in (a+S_+/n, b-S_-/n)$. Recall that V is assumed to be closed in $\mathfrak{K}$ and that r takes scalar values. Therefore (2.6) belongs to V for all $t \in (a+S_+/n, b-S_-/n)$.

We have now established that $\left[\begin{smallmatrix} \dot x_n^k(t) \\ x_n^k(t) \\ w_n^k(t) \end{smallmatrix}\right] \in V$ for all $t \in (a+S_+/n, b-S_-/n)$ and the final step of the proof is to show that also $\left[\begin{smallmatrix} \dot x_n(t) \\ x_n(t) \\ w_n(t) \end{smallmatrix}\right] \in V$ for all t in this same interval by proving that

$$\begin{bmatrix} \dot x_n^k \\ x_n^k \\ w_n^k \end{bmatrix} \to \begin{bmatrix} \dot x_n \\ x_n \\ w_n \end{bmatrix} \quad \text{in} \quad BUC(\mathbb{R};V) \quad \text{as} \quad k \to \infty. \tag{2.7}$$

By construction $\left[\begin{smallmatrix} x^k \\ w^k \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \widetilde x \\ \widetilde w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^p(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ when $k \to \infty$, and it follows from [2, Theorem 2.2.2(i)] that $\left[\begin{smallmatrix} x_n^k \\ w_n^k \end{smallmatrix}\right] = \left[\begin{smallmatrix} x^k \\ w^k \end{smallmatrix}\right] * r_n \to \left[\begin{smallmatrix} \widetilde x \\ \widetilde w \end{smallmatrix}\right] * r_n = \left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^p(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ when $k \to \infty$. As in the proof of claim (i), we find that $x_n^k = x^k * r_n$ and $w_n^k = w^k * r_n$ are locally absolutely continuous and

$$\begin{bmatrix} \dot x_n^k(t) \\ \dot w_n^k(t) \end{bmatrix} = \left(\begin{bmatrix} x^k \\ w^k \end{bmatrix} * \mu_n \right)(t) \quad \text{a.e.,} \quad t \in \mathbb{R}. \tag{2.8}$$

By (2.5) and [2, Theorem 3.6.1(i)], $\left[\begin{smallmatrix} \dot x_n^k \\ \dot w_n^k \end{smallmatrix}\right] = \left[\begin{smallmatrix} x^k \\ w^k \end{smallmatrix}\right] * \mu_n \to \left[\begin{smallmatrix} \widetilde x \\ \widetilde w \end{smallmatrix}\right] * \mu_n = \left[\begin{smallmatrix} \dot x_n \\ \dot w_n \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^p(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ as $k \to \infty$. Since w_n^k and w_n are locally absolutely continuous and both $w_n^k \to w_n$ and $\dot w_n^k \to \dot w_n$ in $L^p(\mathbb{R};\mathcal{W})$ as $k \to \infty$, it follows from a Sobolev embedding theorem [1, Thm 5.4C on p. 97, with $m = n = 1$ and $j = 0$] that $w_n^k \to w_n$ in $BUC(\mathbb{R};\mathcal{W})$. This proves (2.7), and it completes the proof of Theorem 2.2 in the case where the interval $[a,b]$ is finite.

The cases where $a = -\infty$ and/or $b = \infty$ still remain to be treated. The proofs of these cases are in principle the same as above. The difference is that we no longer necessarily have $\left[\begin{smallmatrix} \widetilde x \\ \widetilde w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^p(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ but only $\left[\begin{smallmatrix} \widetilde x \\ \widetilde w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$. We therefore need to replace the results cited from [2] by analogous results that are valid under the weaker assumption that $\left[\begin{smallmatrix} \widetilde x \\ \widetilde w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$. These extended results can indeed be proved to be valid, and they depend crucially on the fact that the support of r is bounded. More precisely, they can be derived from the results cited in [2] above as follows.

By (2.2) the restriction of $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] * r_n$ to an arbitrary finite subinterval $[a', b']$ of $[a, b]$ is independent of the values of $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ outside of the interval $[a' - S_+, a' + S_-]$. Therefore it is possible to redefine $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ outside of the latter interval, for example by using the same type of formula as (2.1), so that the redefined functions satisfy $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^p(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$. Then we apply the argument given above in order to get the desired conclusion with the original interval $[a, b]$ replaced by $[a', b']$. As $[a', b']$ was an arbitrary closed finite subinterval of $[a, b]$, this gives the conclusions (i)–(iii) in the form stated in Theorem 2.2. We provide an example of this type of argument in the proof of Proposition 2.3 below, but we leave the remaing details of the proof of Theorem 2.2 to the reader. □

We can add the following conclusion to Theorem 2.2:

Proposition 2.3. *Let $r \in BV(\mathbb{R})$ have finite support and define $r_n(t) := nr(nt)$ for $t \in \mathbb{R}$ and $n \in \mathbb{Z}^+$, and let μ_n be the bounded measure defined in* (2.4)*. Let $-\infty \le a < b \le \infty$, let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C([a,b];\mathcal{X}) \\ L^p_{\mathrm{loc}}([a,b];\mathcal{W}) \end{smallmatrix}\right]$, define $\widetilde{x}$ and $\widetilde{w}$ by* (2.1)*, and set $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] := \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] * r_n$.*

We have $\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] * \mu_n \in \left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$. If r is locally absolutely continuous with distribution derivative $\dot{r} \in L^1(\mathbb{R})$, then $\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] * \dot{r}_n \in \left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$.

Proof. Let $[a', b']$ be an arbitrary finite subinterval of $\mathbb{R}$, and let the support of r be contained in the finite interval $[-S_-, S_+]$. Define $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ by the right-hand sides of (2.1) with $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ and $[a, b]$ replaced by $[a' - S_+, b' + S_-]$. Then $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^p(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$, which implies that $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right] * \mu_n \in \left[\begin{smallmatrix} BUC(\mathbb{R};\mathcal{X}) \\ L^p(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$ by [2, Thm 3.6.1(i,iii)].

The supports of r_n and μ_n are easily seen to be contained in $[-S_-/n, S_+/n]$, and therefore

$$\rho_{[a',b']} \begin{bmatrix} x_n \\ w_n \end{bmatrix} = \rho_{[a',b']} \left(\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} * r_n \right) = \rho_{[a',b']} \left(\begin{bmatrix} \widehat{x} \\ \widehat{w} \end{bmatrix} * r_n \right).$$

By (2.5) we then have

$$\rho_{(a',b')} \begin{bmatrix} \dot{x}_n \\ \dot{w}_n \end{bmatrix} = \rho_{(a',b')} \left(\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} * \mu_n \right) = \rho_{(a',b')} \left(\begin{bmatrix} \widehat{x} \\ \widehat{w} \end{bmatrix} * \mu_n \right). \tag{2.9}$$

We have proved that

$$\rho_{[a',b']} \begin{bmatrix} \dot{x}_n \\ \dot{w}_n \end{bmatrix} = \rho_{[a',b']} \left(\begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} * \mu_n \right) \in \begin{bmatrix} BUC([a',b'];\mathcal{X}) \\ L^p([a',b'];\mathcal{W}) \end{bmatrix}$$

for an arbitrary finite $[a', b'] \subset \mathbb{R}$, and this establishes $\left[\begin{smallmatrix} \dot{x}_n \\ \dot{w}_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] * \mu_n \in \left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$.

In order to prove the second claim, we assume that $\dot r \in L^1(\mathbb{R})$, which implies that $\dot r_n \in L^1(\mathbb{R})$. Then (2.9) holds with μ_n replaced by $\dot r_n$ according to [2, Cor. 3.7.2(ii)], and the same argument as above shows that $\left[\begin{smallmatrix} \dot x_n \\ \dot w_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde x \\ \widetilde w \end{smallmatrix}\right] * \dot r_n \in \left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W}) \end{smallmatrix}\right]$. □

In Theorem 2.2 it is common to choose $a = 0$ and

$$r(t) = \mathbf{1}_{[-1,0]} := \begin{cases} 1, & t \in [-1,0], \\ 0, & t \in \mathbb{R} \setminus [-1,0], \end{cases}$$

and in this case Theorem 2.2 simplifies to the following corollary:

Corollary 2.4. *The following claims are valid:*

(i) *Let* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p$ *be a generalised trajectory generated by* V *on* $\mathbb{R}^+$. *For each* $n \in \mathbb{Z}^+$ *and* $t \in \mathbb{R}^+$ *define*

$$\begin{bmatrix} x_n(t) \\ w_n(t) \end{bmatrix} := n \int_t^{t+1/n} \begin{bmatrix} x(s) \\ w(s) \end{bmatrix} \mathrm{d}s. \tag{2.10}$$

Then $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}$, *i.e., it is a classical trajectory generated by* V, *and* $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *in* $\left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ *as* $n \to \infty$.

(ii) *Let* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[0,T]$ *for some finite* $T > 0$, *and define* $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ *by* (2.10) *for* $n > 1/T$ *and* $t \in [0, T - 1/n]$. *Then* $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}[0, T-1/n]$ *and* $\rho_{[0,T-\varepsilon]} \left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ *tends to* $\rho_{[0,T-\varepsilon]} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ *in* $\left[\begin{smallmatrix} C([0,T-\varepsilon];\mathcal{X}) \\ L^p([0,T-\varepsilon];\mathcal{W}) \end{smallmatrix}\right]$ *for every* $\varepsilon \in (0,T)$ *as* $n \to \infty$.

Note that a finite right end-point T gives rise to technical complications if $S_- > 0$. We could have used $r = \mathbf{1}_{[0,1]}$ instead and moved the problems to the finite left end-point 0. Therefore $r = \mathbf{1}_{[0,1]}$ is a common choice of convolution kernel for left-infinite intervals.

The following corollary provides a test for determining if a given function pair is a generalised trajectory. The interval $\mathbb{R}^+$ can be replaced by $[a,\infty)$, $(-\infty, b]$ or $\mathbb{R}$ with trivial modifications of the statement and the proof.

Corollary 2.5. *Let* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ *and define* $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ *by* (2.10) *for all* $t \geq 0$ *and* $n \in \mathbb{Z}^+$. *Then* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p$ *if and only if* $\left[\begin{smallmatrix} \dot x_n(t) \\ x_n(t) \\ w_n(t) \end{smallmatrix}\right] \in V$ *for all* $t > 0$ *and* $n \in \mathbb{Z}^+$.

Proof. It follows from Theorem 2.2 that the sequence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ lies in $\left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$, that the sequence converges to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C(\mathbb{R}^+;\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$, and that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}$ for all $n \in \mathbb{Z}^+$ if $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p$.

Conversely, if $\left[\begin{smallmatrix} \dot x_n(t) \\ x_n(t) \\ w_n(t) \end{smallmatrix}\right] \in V$ for all $t > 0$, i.e., if the sequence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ lies in $\mathfrak{V}$, for all $n \in \mathbb{Z}^+$, then its limit $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ lies in $\mathfrak{W}^p$ by Definition 1.1. □

We end this section with a lemma establishing that a pair of functions, which is a generalised trajectory locally, in fact is a generalised trajectory globally.

Lemma 2.6. *The following claims are valid:*

(i) *A pair* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C([a,\infty);\mathcal{X}) \\ L^p_{\mathrm{loc}}([a,\infty);\mathcal{W}) \end{smallmatrix}\right]$, *where* $-\infty \le a < \infty$, *lies in* $\mathfrak{W}^p[a,\infty)$ *if and only if* $\rho_{[a,b]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$ *for all (finite)* $b \in (a,\infty)$.

(ii) *A pair* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C((-\infty,b];\mathcal{X}) \\ L^p_{\mathrm{loc}}((-\infty,b];\mathcal{W}) \end{smallmatrix}\right]$, *where* $-\infty < b \le \infty$, *lies in* $\mathfrak{W}^p(-\infty,b]$ *if and only if* $\rho_{[a,b]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$ *for all* $a \in (-\infty,b)$.

Proof. (i) It follows directly from Lemma 1.4(iii) for generalised trajectories that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,\infty)$ implies $\rho_{[a,b]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$ for all $b \in (a,\infty)$.

For the converse, assume that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C([a,\infty);\mathcal{X}) \\ L^p_{\mathrm{loc}}([a,\infty);\mathcal{W}) \end{smallmatrix}\right]$ satisfies $\rho_{[a,b]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$ for all $b \in (a,\infty)$. Define $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ by (2.10) for $n \in \mathbb{Z}^+$ and $t \in [a,\infty)$, so that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1([a,\infty);\mathcal{X}) \\ C([a,\infty);\mathcal{W}) \end{smallmatrix}\right]$ tends to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C([a,\infty);\mathcal{X}) \\ L^p_{\mathrm{loc}}([a,\infty);\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$ by Theorem 2.2(i,ii). Moreover, claim (iii) of that theorem also yields that $\left[\begin{smallmatrix} \dot{x}_n(t) \\ x_n(t) \\ w_n(t) \end{smallmatrix}\right] \in V$ for all $t \ge a$, since $\rho_{[a,t+1]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,t+1]$ by assumption. Definition 1.1 yields that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}[a,\infty)$ and that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,\infty)$.

(ii) This claim is proved in the same way as claim (i), using the convolution kernel $\mathbf{1}_{[0,1]}$ instead of $\mathbf{1}_{[-1,0]}$.

□

Before shifting the focus from trajectories to the generating subspace, we remark that Lemma 2.6 holds for classical trajectories if we replace $\mathfrak{W}^p$ by $\mathfrak{V}$. The proof is trivial.

3. Uniqueness of the generating subspace

We now prove that the generating subspace of a s/s system $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ is uniquely determined by $\mathfrak{W}^p$ through $V = V_{\max}$ given in (1.4).

Theorem 3.1. *Let* $\mathfrak{W}^p[a,b]$ *be the space of generalised trajectories generated by the closed subspace* $V \subset \mathfrak{K}$ *on* $[a,b]$, *where* $-\infty \le a < b \le \infty$. *Then the following claims are true:*

(i) *If* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$ *and for some* $t \in [a,b)$ *both*

$$z_t := \lim_{h\to 0^+} \frac{1}{h}\big(x(t+h) - x(t)\big) \quad \textit{and} \quad w_t := \lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} w(s)\,\mathrm{d}s \tag{3.1}$$

exist, then $\left[\begin{smallmatrix} z_t \\ x(t) \\ w_t \end{smallmatrix}\right] \in V$.

(ii) *We always have* $V_{\max} \subset V$, *where* $V_{\max}$ *is given in* (1.4). *If* V *has property (ii) of Definition 1.2, then we have* $V_{\max} = V$.

Proof. (i) Fix $t \in [a,b)$ and let $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ be the family of classical trajectory approximations of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ defined for $t \in [a, b-1/n]$ by (2.10), so that $\left[\begin{smallmatrix} \dot{x}_n(t) \\ x_n(t) \\ w_n(t) \end{smallmatrix}\right] \in V$ for

$n > 1/(b-t)$ by Theorem 2.2(iii). Then

$$\dot{x}_n(t) = n\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^{t+1/n} x(s)\,\mathrm{d}s - \int_a^t x(s)\,\mathrm{d}s\right) = n\big(x(t+1/n) - x(t)\big), \tag{3.2}$$

which implies that

$$V \ni \begin{bmatrix} \dot{x}_n(t) \\ x_n(t) \\ w_n(t) \end{bmatrix} = n \begin{bmatrix} x(t+1/n) - x(t) \\ \int_t^{t+1/n} x(s)\,\mathrm{d}s \\ \int_t^{t+1/n} w(s)\,\mathrm{d}s \end{bmatrix}. \tag{3.3}$$

If the limits in (3.1) exist, then the right-hand side of (3.3) tends to $\left[\begin{smallmatrix} z_t \\ x(t) \\ w_t \end{smallmatrix}\right]$ as $n \to \infty$, and since V is closed, it then follows that $\left[\begin{smallmatrix} z_t \\ x(t) \\ w_t \end{smallmatrix}\right] \in V$.

(ii) If $x \in C^1([a,b];\mathcal{X})$ then the limit z_t in (3.1) equals $\dot{x}(t)$ by the definition of (one-sided) derivative. If $w \in C([a,b];\mathcal{W})$ then the limit w_t exists and equals $w(t)$. Therefore the inclusion $V_{\max} \subset V$ holds. The converse inclusion is implied by (1.4) and Definition 1.2(ii), because for all $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$:

$$\exists \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V} \subset \mathfrak{W}^p \cap \begin{bmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{bmatrix} : \quad \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}$$

□

In [4, Sect. 6] we proved that every *well-posed* s/s system has a unique *maximal* generating subspace. In Theorem 3.1 of the present article we have established that every s/s system is generated by a unique s/s node. Thus the notion of a *maximal* generating subspace of a s/s system is no longer relevant.

We have the following one-to-one correspondence between a generating subspace V and the generalised trajectories is generates.

Corollary 3.2. *If V is closed with property (ii) in Definition 1.2, then it holds for every $[a,b]$, where $-\infty \le a < b \le \infty$, that:*

$$\mathfrak{V}[a,b] = \mathfrak{W}^p[a,b] \cap \begin{bmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{bmatrix}. \tag{3.4}$$

Proof. Choose an arbitrary $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p[a,b] \cap \left[\begin{smallmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{smallmatrix}\right]$. Then it follows immediately from Theorem 3.1 that $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V$ for all $t \in (a,b)$. We have now proved that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[a,b]$ and the converse inclusion follows immediately from Definition 1.1. □

A disadvantage with defining a s/s system using the generalised trajectories, as we did in [4], and therefore also Definition 1.2, is that the spaces $\mathfrak{W}^p$ depend on $p \in [1,\infty)$. Using $\mathfrak{W}^p$ to represent a s/s system leads to unnecessarily complicated formulations. A clearer way is to simply use the generating subspace V, since $\mathfrak{W}^p$

and V are in 1-1 correspondence for every given $p \in [1, \infty)$. This corresponds to talking about s/s *nodes* instead of s/s *systems*, cf. Definition 1.2.

Using the results obtained above we are able to simplify and clarify the theory of well-posed s/s systems presented in [4]. Some minor simplifications of [3] are also possible, but we leave these untreated. Theorem 3.1 leads to the following significant simplification of [4, Def. 3.3], and Definition 1.6 in the present article also simplifies similarly.

Definition 3.3. Let the s/s node $(V; \mathcal{X}, \mathcal{W})$ be L^p-well-posed with trajectories $\mathfrak{W}^p$.

The triple $\Sigma_{s/s} = (\mathfrak{W}^p; \mathcal{X}, \mathcal{W})$ is called the *L^p-well-posed state/signal system (well-posed s/s system)* on $(\mathcal{X}, \mathcal{W})$ *generated by* $(V; \mathcal{X}, \mathcal{W})$.

An i/o pair $(\mathcal{U}, \mathcal{Y})$ is admissible for the *system* Σ if it is admissible for $(V; \mathcal{X}, \mathcal{W})$.

Items (ii) and (iii) of [4, Cor. 3.12] collapse into one and the same statement that $(\mathcal{U}, \mathcal{Y})$ is admissible for the s/s node which generates Σ, given uniqueness of the s/s node, and this node is automatically well-posed. Moreover, this claim is by the new Definition 3.3 equivalent to admissibility of $(\mathcal{U}, \mathcal{Y})$ for Σ. Thus the simplified version of [4, Cor. 3.12] reads as follows:

Corollary 3.4. *Let $-\infty < a < b < \infty$, let $\Sigma_{s/s} = (\mathfrak{W}^p; \mathcal{X}, \mathcal{W})$ be an L^p-well-posed s/s system, and let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$.*

Then the i/o pair $(\mathcal{U}, \mathcal{Y})$ is admissible for the s/s system Σ if and only if the operator $\begin{bmatrix} 0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \end{bmatrix}$ maps $\mathfrak{W}_0^p[a, b]$ one-to-one onto $L^p([a, b]; \mathcal{U})$.

In the setup of item (ii) in [4, Thm 3.15], we now know that A automatically is the generator a C_0 semigroup, and thus there is no need to extend A; see [4, Def. 3.13] for definitions. A simplified but equivalent version of that theorem is thus the following:

Theorem 3.5. *Let $\mathcal{X}$ be a Banach space and let $V \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \end{smallmatrix}\right]$ be closed. Then V is the graph*

$$V = \begin{bmatrix} A \\ 1 \end{bmatrix} \operatorname{dom}(A) \tag{3.5}$$

of the generator A of a C_0 semigroup $\mathfrak{A}$ on $\mathcal{X}$ if and only if $(V; \mathcal{X}, \{0\})$ is an L^p-well-posed s/s node for some, or equivalently, for all $1 \le p < \infty$. In this case the L^p-well-posed s/s system is $(\mathfrak{W}^p; \mathcal{X}, \{0\})$, where

$$\mathfrak{W}^p = \{x \in C(\mathbb{R}^+; \mathcal{X}) \mid x(t) = \mathfrak{A}^t x_0,\ t \ge 0,\ x_0 \in \mathcal{X}\}.$$

Finally, claim (iv) in [4, Prop. 3.7] has an obvious simplification due to (3.4).

4. Denseness of externally generated classical trajectories

In [4] we often used the assumption that the space of externally generated classical trajectories is dense in the space of externally generated generalised trajectories on some interval $[a, b]$, i.e., that $\overline{\mathfrak{V}_0[a, b]} = \mathfrak{W}_0^p[a, b]$. However, we did not need to pay much attention to the actual meaning of the assumption, since it is always satisfied by *well-posed* s/s systems. We now provide more insight into this assumption.

It follows from Corollary 3.2 that for all intervals $[a,b]$, $-\infty \le a < b \le \infty$, the space of externally generated classical trajectories on $[a,b]$ is given by:

$$\mathfrak{V}_0[a,b] = \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[a,b] \cap \begin{bmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{bmatrix} \;\middle|\; \begin{bmatrix} x(a) \\ w(a) \end{bmatrix} = 0 \right\},$$

where we in the case $a = -\infty$ interpret $\left[\begin{smallmatrix} x(-\infty) \\ w(-\infty) \end{smallmatrix}\right] := \lim_{t\to-\infty} \left[\begin{smallmatrix} x(t) \\ w(t) \end{smallmatrix}\right]$. In this section we thus investigate what it means for $\mathfrak{V}_0[a,b]$ to be dense in the space

$$\mathfrak{W}_0^p[a,b] = \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p[a,b] \;\middle|\; x(a) = 0 \right\}.$$

We first introduce the notation π for the operator which extends its argument function by zero outside of a given subinterval of its domain:

$$(\pi_{[a,b]}x)(t) = \begin{cases} x(t), & t \in [a,b], \\ 0, & t \notin [a,b], \end{cases} \quad \text{where } [a,b] \subset \operatorname{dom}(x),$$

and abbreviate $\pi_+ := \pi_{[0,\infty)}$.

We can now give a necessary and sufficient condition for $\overline{\mathfrak{V}_0[a,b]} = \mathfrak{W}_0^p[a,b]$ in terms of extendability of generalised trajectories. Indeed, condition (4.1) below means that we can extend every externally generated generalised trajectory by zero to minus infinity. (The corresponding claim for classical trajectories generated by a s/s node always holds due to Lemma 1.4.)

Proposition 4.1. *Let $\mathfrak{V}_0[a,b]$ and $\mathfrak{W}_0^p[a,b]$ be the spaces of externally generated classical and generalised trajectories generated by a s/s node $(V;\mathcal{X},\mathcal{W})$ on $[a,b]$, where $-\infty < a < b \le \infty$.*

(i) *The space $\mathfrak{V}_0[a,b]$ is dense in $\mathfrak{W}_0^p[a,b]$ if and only if*

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}_0^p[a,b] \quad\Longrightarrow\quad \rho_{(-\infty,b]}\pi_{[a,b]} \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p(-\infty,b]. \tag{4.1}$$

(ii) *If $b = \infty$ then (4.1) is equivalent to the condition that $\mathfrak{W}_0^p := \mathfrak{W}_0^p[0,\infty)$ is right-shift invariant:*

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}_0^p \quad\Longrightarrow\quad \forall T \ge 0:\ \rho_+\tau^{-T}\pi_+ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}_0^p. \tag{4.2}$$

Proof. (i) Assume that $\overline{\mathfrak{V}_0[a,b]} = \mathfrak{W}_0^p[a,b]$, fix $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p[a,b]$ arbitrarily, and let $\left[\begin{smallmatrix} x^k \\ w^k \end{smallmatrix}\right]$ be a sequence in $\mathfrak{V}_0[a,b]$ which tends to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C([a,b];\mathcal{X}) \\ L^p_{\mathrm{loc}}([a,b];\mathcal{W}) \end{smallmatrix}\right]$. From Lemma 1.4 we have that $\rho_{(-\infty,b]}\pi_{[a,b]}\left[\begin{smallmatrix} x^k \\ w^k \end{smallmatrix}\right]$ all lie in $\mathfrak{V}(-\infty,b]$. Moreover, this sequence obviously tends to $\rho_{(-\infty,b]}\pi_{[a,b]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C((-\infty,b];\mathcal{X}) \\ L^p_{\mathrm{loc}}((-\infty,b];\mathcal{W}) \end{smallmatrix}\right]$ as $k \to \infty$, and the limit thus lies in $\mathfrak{W}^p(-\infty,b]$.

Now conversely assume that (4.1) holds and fix $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p[a,b]$ arbitrarily. Then $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] := \rho_{(-\infty,b]}\pi_{[a,b]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ is an extension of $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ to an externally generated generalised trajectory on $(-\infty,b]$ by assumption. Theorem 2.2 then yields that every element in the sequence of functions defined for $t \in (-\infty,b]$ by

$$\begin{bmatrix} x_n(t) \\ w_n(t) \end{bmatrix} := n\int_{t-1/n}^{t} \begin{bmatrix} \widetilde{x}(s) \\ \widetilde{w}(s) \end{bmatrix} \,\mathrm{d}s, \quad n \in \mathbb{Z}^+, \tag{4.3}$$

which corresponds to the convolution kernel $r = \mathbf{1}_{[0,1]}$, is a classical trajectory on $(-\infty,b]$, and that the sequence tends to $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ in $\mathfrak{W}^p(-\infty,b]$. Moreover, it is clear that $x(t)=0$ and $w(t)=0$ for all $t \le a$, and therefore $\rho_{[a,b]}\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ lies in $\mathfrak{V}_0[a,b]$. This sequence tends to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\mathfrak{W}^p[a,b]$ as $n \to \infty$, and we have proved that $\mathfrak{V}_0[a,b]$ is dense in $\mathfrak{W}_0^p[a,b]$.

(ii) Assume that $b=\infty$ and that (4.2) holds, and fix $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p[a,\infty)$ arbitrarily. We need to prove that $\pi_{[a,\infty)}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p(\mathbb{R})$. By Lemma 1.4(ii), we have $\tau^a\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p$ and therefore, by assumption, for all $T \ge 0$ that

$$\rho_+\tau^{-T}\pi_+\tau^a\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W}^p \quad \iff$$
$$\rho_{[-T,\infty)}\tau^a\pi_{[a,\infty)}\begin{bmatrix} x \\ w \end{bmatrix} \in \tau^T\mathfrak{W}^p = \mathfrak{W}^p[-T,\infty).$$

By Lemmas 1.4(ii) and 2.6(ii), we have that $\pi_{[a,\infty)}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \tau^{-a}\mathfrak{W}^p(\mathbb{R}) = \mathfrak{W}^p(\mathbb{R})$ and thus (4.1) holds.

We still need to prove that (4.1) implies (4.2) when $b=\infty$. Let therefore $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p$, so that $\pi_+\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p(\mathbb{R})$ by (4.1). By Lemma 1.4(ii) it is then clear that $\tau^{-T}\pi_+\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p(\mathbb{R})$ for all $T \ge 0$, and therefore also that $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] := \rho_+\tau^{-T}\pi_+\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^p$. We obviously have $\widetilde{x}(0)=0$ for all $T \ge 0$, and the proof is complete.

□

Proposition 4.1 should be compared to Lemma 3.6 in [4], where we proved that denseness of $\mathfrak{V}_0[0,T]$ in $\mathfrak{W}_0^p[0,T]$ implies that $\mathfrak{W}^p[0,T]$ is right-shift invariant.

Proposition 4.2. *If $\mathfrak{V}_0[a,b]$ generated by a s/s node is dense in $\mathfrak{W}_0^p[a,b]$ for all finite $b > a$ then $\mathfrak{V}_0[a,\infty)$ is dense in $\mathfrak{W}_0^p[a,\infty)$.*

Proof. We can cover the general case by proving only the case $a=0$ and then applying Lemma 1.4(ii). Assume thus that $\mathfrak{V}_0[0,b]$ is dense in $\mathfrak{W}_0^p[0,b]$ for all $b > 0$ and fix $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p[0,\infty)$ arbitrarily. Then we can for all $k \in \mathbb{Z}^+$ find a $\left[\begin{smallmatrix} \widetilde{x}^k \\ \widetilde{w}^k \end{smallmatrix}\right] \in \mathfrak{V}_0[0,k]$ such that

$$\left\| \begin{bmatrix} \widetilde{x}^k \\ \widetilde{w}^k \end{bmatrix} - \rho_{[0,k]}\begin{bmatrix} x \\ w \end{bmatrix} \right\|_{\left[\begin{smallmatrix} BUC([0,k];\mathcal{X}) \\ L^p([0,k];\mathcal{W}) \end{smallmatrix}\right]} < \frac{1}{k}.$$

By Definition 1.2(ii), every $\left[\begin{smallmatrix}\widetilde{x}^k\\ \widetilde{w}^k\end{smallmatrix}\right]$ has some extension to a trajectory $\left[\begin{smallmatrix}x^k\\ w^k\end{smallmatrix}\right]$ in $\mathfrak{V}_0[0,\infty)$, and by construction $\left[\begin{smallmatrix}x^k\\ w^k\end{smallmatrix}\right] \to \left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ in $\mathfrak{W}^p$ as $k \to \infty$. □

Note that the proof of Proposition 4.2 depends on property (ii) of a s/s node in Definition 1.2. Without additional assumptions we can not conclude from this proof that $\mathfrak{V}_0(-\infty,b]$ is dense in $\mathfrak{W}_0^p(-\infty,b]$ even if $\overline{\mathfrak{V}_0[a,b]} = \mathfrak{W}_0^p[a,b]$ for all finite $a<b$.

Remark 4.3. It is still an open problem if the converse of Proposition 4.2 is true. Another open problem is to give necessary and sufficient conditions for the existence of *some* $T>0$, such that $\overline{\mathfrak{V}_0[0,T]} = \mathfrak{W}_0^p[0,T]$, to imply that $\overline{\mathfrak{V}_0[0,T]} = \mathfrak{W}_0^p[0,T]$ for *all* $T>0$.

5. Derivatives and primitives of trajectories

We proved in Corollary 3.2 that every generalised trajectory of a s/s system that possesses sufficient smoothness is a classical trajectory. Thus it seems reasonable that the primitive of a generalised trajectory is a classical trajectory, and that the derivative of a classical trajectory is a generalised trajectory, at least in some cases. These questions are the topic of this section.

Let A be the generator of a C_0 semigroup $\mathfrak{A}$ on the Banach space $\mathcal{X}$. Then the intersection $\cap_{n\in\mathbb{Z}^+}\mathrm{dom}\,(A^n)$ is dense in $\mathcal{X}$, and for every x_0 in this intersection, the function $t \to \mathfrak{A}^t x_0$ lies in $C^\infty(\mathbb{R}^+;\mathcal{X})$; see [7, Thm 3.2.1(vi)]. We now give an analogous statement for the classical trajectories of s/s nodes.

Proposition 5.1. *Let $-\infty \le a < b \le \infty$, where $a=-\infty$ and/or $b=\infty$.*

The space $\mathfrak{V}^\infty[a,b] := \mathfrak{V}[a,b] \cap C^\infty\left([a,b];\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]\right)$ of infinitely many times differentiable classical trajectories is dense in the space $\mathfrak{W}^p[a,b]$ of generalised trajectories. If a is finite then

$$\left\{x(a) \mid \begin{bmatrix} x\\ w\end{bmatrix} \in \mathfrak{V}^\infty[a,\infty)\right\} \text{ is dense in } \left\{x(a) \mid \begin{bmatrix} x\\ w\end{bmatrix} \in \mathfrak{W}^p[a,\infty)\right\}. \quad (5.1)$$

If a is finite and $(\mathfrak{W}^p;\mathcal{X},\mathcal{W})$ is an L^p-well-posed s/s system, as described in Definition 1.6, then the second space in (5.1) equals $\mathcal{X}$.

Proof. If $b=\infty$ then we choose a r in $C^\infty(\mathbb{R})$ supported on $[-1,0]$. Then $\frac{\mathrm{d}}{\mathrm{d}t}^k r_n \in BV(\mathbb{R})$ for all $k\in\mathbb{Z}^+$ by [5, Ex. 6.23(b)] and it is easy to see that $\frac{\mathrm{d}}{\mathrm{d}t}^k r_n \in L^1(\mathbb{R})$. If $a=-\infty$ but b is finite then we choose a $r\in C^\infty(\mathbb{R})$ supported on $[0,1]$ and use the same proof as in the case $b=\infty$. In the rest of the proof we assume that $b=\infty$ and let r be supported on $[-1,0]$.

Fix $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \mathfrak{W}^p[a,\infty)$ arbitrarily and define $\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right]$ by (2.1). Then $\left[\begin{smallmatrix}x_n\\ w_n\end{smallmatrix}\right] := \left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right]*r_n$ tends to $\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right]$ in $\mathfrak{W}^p(\mathbb{R})$ and $\rho_{[a,\infty)}\left[\begin{smallmatrix}x_n\\ w_n\end{smallmatrix}\right] \in \mathfrak{V}[a,\infty)$ for all $n\in\mathbb{Z}^+$ by Theorem 2.2(i,iii). Moreover, by Proposition 2.3, for all $k\in\mathbb{Z}^+$ we have $\frac{\mathrm{d}}{\mathrm{d}t}^k\left[\begin{smallmatrix}x_n\\ w_n\end{smallmatrix}\right] = \left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{w}\end{smallmatrix}\right] *$

$\left(\frac{\mathrm{d}}{\mathrm{d}t}^k r_n\right)$ in $\left[\begin{smallmatrix} C(\mathbb{R};\mathcal{X}) \\ L^p_{\mathrm{loc}}(\mathbb{R};\mathcal{W})\end{smallmatrix}\right]$, and therefore $\rho_{[a,\infty)} \left[\begin{smallmatrix} x_n \\ w_n\end{smallmatrix}\right] \in C^\infty\big([a,\infty); \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W}\end{smallmatrix}\right]\big)$. The first claim is proved.

For the rest of the proof, assume that a is finite. Since $\left[\begin{smallmatrix} x_n \\ w_n\end{smallmatrix}\right] \to \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w}\end{smallmatrix}\right]$ in $\mathfrak{W}^p(\mathbb{R})$, we in particular have that $x_n \to x$ uniformly on the compact interval $[a, a+1]$, and consequently $x_n(a) \to \widetilde{x}(a) = x(a)$, which proves (5.1). If $(\mathfrak{W}^p; \mathcal{X}, \mathcal{W})$ is L^p-well posed, then the second space in (5.1) equals $\mathcal{X}$ by [4, Prop. 3.7(iii)]. □

If we consider a finite interval $[a, b]$ in Proposition 5.1, then we can approximate an arbitrary generalised trajectory on $[a, b]$ by a sequence of infinitely differentiable functions, which are classical trajectories on either $[a, b-\varepsilon]$ or $[a+\varepsilon, b]$, where $\varepsilon \in (0, b-a)$ can be taken arbitrarily small, depending on whether we choose r to be supported on $[-1, 0]$ or $[0, 1]$, respectively. (Compare this to Theorem 2.2(iii).) Unfortunately, this approach cannot be used to find an approximating classical trajectory on all of $[a, b]$ in the case where both a and b are finite, unless we have additional tools at our disposal.

We now complement Proposition 5.1 by proving that

$$\begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}^\infty[a,b] \quad \Longrightarrow \quad \begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} \in \mathfrak{V}^\infty[a,b] \tag{5.2}$$

when $a = -\infty$ and/or $b = \infty$. The following proposition establishes that the derivative of a classical trajectory on an infinite interval is a generalised trajectory whenever $\dot{w}$ is smooth enough to be part of a trajectory, i.e., whenever $\dot{w} \in L^p_{\mathrm{loc}}$.

Proposition 5.2. *Let $\left[\begin{smallmatrix} x \\ w\end{smallmatrix}\right] \in \mathfrak{V}[a,b]$, $-\infty \le a < b \le \infty$ where $a = -\infty$ and/or $b = \infty$, and assume that w is locally absolutely continuous with $w, \dot{w} \in L^p_{\mathrm{loc}}([a,b];\mathcal{W})$. Then $\left[\begin{smallmatrix} \dot{x} \\ \dot{w}\end{smallmatrix}\right] \in \mathfrak{W}^p[a,b]$.*

Proof. We again prove only the case $b = \infty$, since the case $a = \infty$ is a trivial modification of the first case. Fix $\left[\begin{smallmatrix} x \\ w\end{smallmatrix}\right] \in \mathfrak{V}[a,\infty)$ with $\dot{w} \in L^p_{\mathrm{loc}}([a,\infty);\mathcal{W})$ arbitrarily and define $\left[\begin{smallmatrix} x_n \\ w_n\end{smallmatrix}\right]$ by (2.10) for $t \ge a$. Then $\left[\begin{smallmatrix} x_n \\ w_n\end{smallmatrix}\right] \in \mathfrak{V}[a,\infty)$ and similarly to (3.2) we obtain

$$\begin{bmatrix} \ddot{x}_n(t) \\ \dot{x}_n(t) \\ \dot{w}_n(t) \end{bmatrix} = n \begin{bmatrix} \dot{x}_n(t+1/n) \\ x_n(t+1/n) \\ w_n(t+1/n) \end{bmatrix} - n \begin{bmatrix} \dot{x}_n(t) \\ x_n(t) \\ w_n(t) \end{bmatrix} \in V$$

for $t \ge a$, where obviously $\left[\begin{smallmatrix} \dot{x}_n \\ \dot{w}_n\end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1([a,\infty);\mathcal{X}) \\ C([a,\infty);\mathcal{W})\end{smallmatrix}\right]$, so that $\left[\begin{smallmatrix} \dot{x}_n \\ \dot{w}_n\end{smallmatrix}\right] \in \mathfrak{V}[a,\infty)$. Moreover, $\left[\begin{smallmatrix} \dot{x}_n(t) \\ \dot{w}_n(t)\end{smallmatrix}\right] = n \int_t^{t+1/n} \left[\begin{smallmatrix} \dot{x}(s) \\ \dot{w}(s)\end{smallmatrix}\right] \mathrm{d}s$, and consequently $\left[\begin{smallmatrix} \dot{x}_n \\ \dot{w}_n\end{smallmatrix}\right] \to \left[\begin{smallmatrix} \dot{x} \\ \dot{w}\end{smallmatrix}\right]$ in $\mathfrak{W}^p[a,\infty)$ as $n \to \infty$ by Theorem 2.2(i). □

Thus $\mathfrak{V}^\infty[a,b]$ is invariant under differentiation, i.e., (5.2) holds, when $a = -\infty$ and/or $b = \infty$. In this case we have the following analogue of Lemma 1.4(i):

$$\mathfrak{V}^\infty[a,b] = \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in C^\infty\left([a,b]; \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W}\end{smallmatrix}\right]\right) \;\middle|\; \begin{bmatrix} \dot{x} \\ x \\ w \end{bmatrix} \in C^\infty([a,b];V) \right\}.$$

It is also natural to ask if the primitive of a generalised trajectory is always a classical trajectory. The answer is no, as the following example based on Theorem 3.5 shows.

Example 5.3. Let A generate a C_0 semigroup $t \to \mathfrak{A}^t$, $t \geq 0$, on $\mathcal{X}$ and define V by (3.5). According to Theorem 3.5, $(V; \mathcal{X}, \{0\})$ is an L^p-well-posed s/s node for all $p \in [1, \infty)$, and for every $x_0 \in \mathcal{X}$, the function $x(t) := \mathfrak{A}^t x_0$, $t \geq 0$, is a generalised trajectory generated by V. However,

$$\widehat{x}(t) := \int_0^t x(s)\,\mathrm{d}s = \int_0^t \mathfrak{A}^s x_0\,\mathrm{d}s, \quad t \geq 0,$$

is not a classical trajectory generated by V if $x_0 \neq 0$. Indeed, by [7, Thm 3.2.1(ii)], $\widehat{x}(t) \in \mathrm{dom}\,(A)$ for all $t \geq 0$ and

$$A\widehat{x}(t) = \mathfrak{A}^t x_0 - \mathfrak{A}^0 x_0 = \left(\frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}\right)(t) - x_0 \neq \left(\frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}\right)(t).$$

The example raises the question if the primitive of an *externally generated* trajectory is a trajectory. The answer to this question is positive under the additional assumption (4.1).

Proposition 5.4. *Let V be a s/s node, let $-\infty < a < b \leq \infty$, and assume that $\mathfrak{V}_0[a, b]$ is dense in $\mathfrak{W}_0^p[a, b]$.*

If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}_0^p[a, b]$ then $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right] \in \mathfrak{V}_0[a, b]$, where

$$\begin{bmatrix} \widehat{x}(t) \\ \widehat{w}(t) \end{bmatrix} := \int_a^t \begin{bmatrix} x(s) \\ w(s) \end{bmatrix} \mathrm{d}s, \quad t \in [a, b]. \tag{5.3}$$

Proof. It is clear that $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1([a,b];\mathcal{X}) \\ C([a,b];\mathcal{W}) \end{smallmatrix}\right]$ and that $\widehat{x}(a) = 0$, $\widehat{w}(a) = 0$. By (3.4), we thus only need to prove that $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right] \in \mathfrak{W}^p[a, b]$. Define $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right]$ by (2.1) and note that $\rho_{(-\infty,b]} \left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \mathfrak{W}_0^p(-\infty, b]$ by assumption and Proposition 4.1(i).

Define $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ by (4.3) for $t \in (-\infty, b]$, so that $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}(-\infty, b]$ by Theorem 2.2(iii), and set $\left[\begin{smallmatrix} \widehat{x}_n(t) \\ \widehat{w}_n(t) \end{smallmatrix}\right] := \int_a^t \left[\begin{smallmatrix} x_n(s) \\ w_n(s) \end{smallmatrix}\right] \mathrm{d}s$ for $t \in [a, b]$. Noting that $x_n(a) = 0$, we obtain that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}_n\right)(t) = x_n(t) = x_n(t) - x_n(a) = \int_a^t \dot{x}_n(s)\,\mathrm{d}s,$$

and hence $\left[\begin{smallmatrix} \widehat{x}_n \\ \widehat{w}_n \end{smallmatrix}\right] \in \mathfrak{V}[a, b]$:

$$\begin{bmatrix} \left(\frac{\mathrm{d}}{\mathrm{d}t}\widehat{x}_n\right)(t) \\ \widehat{x}_n(t) \\ \widehat{w}_n(t) \end{bmatrix} = \int_a^t \begin{bmatrix} \dot{x}_n(s) \\ x_n(s) \\ w_n(s) \end{bmatrix} \mathrm{d}s \in V, \quad t \in (a, b).$$

Finally, we note that $\left[\begin{smallmatrix} \widehat{x}_n \\ \widehat{w}_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right]$ in $\mathfrak{W}^p[a, b]$, because

$$\left\| \begin{bmatrix} \widehat{x}_n(t) \\ \widehat{w}_n(t) \end{bmatrix} - \begin{bmatrix} \widehat{x}(t) \\ \widehat{w}(t) \end{bmatrix} \right\| \leq \int_a^t \left\| \begin{bmatrix} x_n(s) \\ w_n(s) \end{bmatrix} - \begin{bmatrix} x(s) \\ w(s) \end{bmatrix} \right\| \mathrm{d}s, \tag{5.4}$$

where $x_n \to x$ uniformly and $w_n \to w$ in L^p on compact subintervals of $[a,b]$. Indeed, it is easy to show that (5.4) implies that we for all finite $b' \in (a,b]$ have

$$\|\rho_{[a,b']}(\widehat{x}_n - \widehat{x})\|_{C([a,b'];\mathcal{X})} \le (b'-a)\|\rho_{[a,b']}(x_n - x)\|_{C([a,b'];\mathcal{X})} \quad \text{and}$$
$$\|\rho_{[a,b']}(\widehat{w}_n - \widehat{w})\|^p_{L^p([a,b'];\mathcal{X})} \le (b'-a)\|\rho_{[a,b']}(w_n - w)\|^p_{L^p([a,b'];\mathcal{X})}.$$

□

Note that we assumed a to be finite in Proposition 5.4, but there was no need to assume that $b = \infty$. The assumption that $\mathfrak{V}_0[a,b]$ is dense in $\mathfrak{W}^p_0[a,b]$ was used at the utilisation of (4.1) in the first paragraph of the proof. It follows from Proposition 5.4 that $\mathfrak{V}_0[a,b]$ and $\mathfrak{W}^p_0[a,b]$ are invariant under integration, but in general neither of $\mathfrak{V}[a,b]$ and $\mathfrak{W}^p[a,b]$ is invariant.

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