On passive and conservative state/signal systems in continuous time

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Abstract. This article is devoted to a study of continuous-time passive and conservative systems within the state/signal framework. The main idea of the state/signal approach is to not a priori distinguish between inputs and outputs, but rather to combine these two into a single external signal. The so-called node space is introduced as the product of two copies of the state space of the system and one copy of the space where the external signals of the system live. This node space is equipped with a sesquilinear product that makes it a Kreĭn space. A generating subspace is defined as a closed subspace of the node space which determines the trajectories of a state/signal system. One of the main results of this article is that a subspace of the node space generates a passive state/signal system if and only if it is a maximally nonnegative subspace of the node space and it satisfies a certain nondegeneracy condition. In this case the generating subspace can be interpreted as the graph of a scattering-passive input/state/output system node.

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1. Introduction

In this paper we study continuous-time passive and conservative linear systems within the so-called state/signal framework. This framework allows us to treat inputs and outputs on an equal basis. Indeed, a limitation of the input/state/output approach to systems theory is that inputs and outputs are considered to be ideal. When systems are interconnected, however, every input also acts as an output and vice versa, because the subsystems will always influence each other mutually. The state/signal formulation is useful

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for instance for modelling interconnections where a partial collapse of the state space takes place. This situation, which is not covered by the standard feedback theory, see [26], is illustrated in Example 4.9, where we model the interconnection of two capacitors in parallel.

State/signal systems in continuous time were introduced in [14] and in the present article we continue the development of their theory. The first steps in the direction of conservative continuous-time state/signal theory were taken already in Ball and Staffans [9], and Malinen and Staffans [18]. The theory of discrete-time state/signal systems has been developed by Arov and Staffans in [4, 5, 6, 7, 8]. For an overview how the state/signal theory unifies the theories of different types of passive discrete-time systems; see [27]. The present article mainly gives continuous-time analogues of some of the results in [5].

The state/signal framework is similar to the behavioural theory developed by Jan Willems and his coauthors and followers; see [23] for a good introduction. The main difference between these two formulations is that the system state plays an important role in the state/signal setting, whereas in the behavioural formulation this seems not to be the case. In addition, the state/signal theory is much more developed than the corresponding behavioural theory for infinite-dimensional systems. Here we use mainly energy-based methods, whereas the finite-dimensional behavioural theory is built mostly using algebraic tools.

Another approach to modelling conservative systems, which is closely related to the state/signal approach, uses the concepts of port-Hamiltonian systems and Dirac structures. Van der Schaft and Maschke are two of the main authors in this field, which originates from the modelling of conservative physical systems which are often nonlinear, see [15, 20, 21, 30]. Although some work has been done to extend the port-Hamiltonian system approach to distributed-parameter systems, most of the theory still concerns finite-dimensional systems. In the port-Hamiltonian approach the existence of solutions if often motivated by physical arguments, but in this article we are also interested in mathematical proofs for existence of system trajectories.

The theory of boundary triplets and their application to solving boundary control problems is now classical; see [13]. The concept of boundary triplets has been generalised to that of boundary relations by a group of authors in papers such as [10] and [12]. This work is also closely connected to the state/signal theory.

Passive infinite-dimensional input/state/output systems in continuous time have previously been studied a.o. in [1, 2, 3, 17, 18, 19, 24, 25, 29, 31].

We now proceed to describe the contents of the paper in more detail. To avoid unnecessary repetition we introduce the abbreviations i/s/o for input/state/output, i/o for input/output and s/s for state/signal.

A common model for a linear continuous-time time-invariant system is

$$\Sigma: \qquad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t > 0, \; x(0) = x_0. \tag{1.1}$$

Here A, B, C and D are linear operators, and $\dot{x} = \frac{\partial}{\partial t}x$. The function x is called the *state trajectory*, u is the *input signal*, y is the output signal, and together they form an *input/state/output (i/s/o) trajectory* (u, x, y) of Σ. The state trajectory x takes values in the *state space* $\mathcal{X}$, the input u lives in the *input space* $\mathcal{U}$, and the output y in the *output space* $\mathcal{Y}$. For the moment the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is assumed to map $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ continuously into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$, but we will soon replace $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ by an *unbounded operator node*, which we define in Definition 2.4, in order to allow a larger set of applications. A system reminiscent of (1.1) is commonly known as an i/s/o system.

We formalise the idea of equal treatment of inputs and outputs in (1.1) by considering the input space $\mathcal{U}$ and the output space $\mathcal{Y}$ to be closed subspaces of a *combined external signal space*: $\mathcal{W} := \mathcal{U} \dotplus \mathcal{Y}$. We can always achieve this by setting $\mathcal{W} := \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ and identifying $\mathcal{Y} = \left[\begin{smallmatrix} \mathcal{Y} \\ \{0\} \end{smallmatrix}\right]$ and $\mathcal{U} = \left[\begin{smallmatrix} \{0\} \\ \mathcal{U} \end{smallmatrix}\right]$. Then we rewrite the system (1.1) in *graph form* to get rid of the explicit input $u(t)$ and output $y(t)$:

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V, \quad t > 0, \; x(0) = x_0, \quad \text{where} \tag{1.2}$$

$$\begin{aligned} V &= \left\{ \begin{bmatrix} z \\ x \\ u+y \end{bmatrix} \,\middle|\, \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\} \\ &= \begin{bmatrix} A & B \\ 1_{\mathcal{X}} & 0 \\ C & D + 1_{\mathcal{U}} \end{bmatrix} \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}. \end{aligned} \tag{1.3}$$

We call a system of the type (1.2) a *differential s/s model* of the system Σ in (1.1).

We now return to the general infinite-dimensional setting. In the study of passive systems it is natural to require $\mathcal{W}$ to be a Kreĭn space, as we will see later. Some theory of Kreĭn spaces and brief definitions of the function spaces which we need throughout this article can be found in the appendix. The following definition should be compared to (1.2).

Definition 1.1. Let I be a subinterval of $\mathbb{R}$ with positive length, let $\mathcal{X}$ be a Hilbert space, and let $\mathcal{W}$ be a Kreĭn space. Let V be a subspace of $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$.

The space $\mathfrak{V}(I)$ of *classical trajectories* on I generated by V consists of all pairs $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{W}) \end{smallmatrix}\right]$, such that $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V$ for all interior points t in I. We abbreviate $\mathfrak{V} := \mathfrak{V}[0, \infty)$.

If $\dot{x}$, x and w are all continuous on I, and $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V$ for all internal points t of I, then the inclusion holds at every $t \in I$, with one-sided derivatives at any end-points of I. Thus we often replace the statement that the inclusion holds for all internal points t of I by the shorter statement that it holds for all $t \in I$.

The state space $\mathcal{X}$ in Definition 1.1 represents the internal memory of the system, whereas the external signal space is used to interconnect the s/s system to the outside world. The external signal space can be decomposed into a direct-sum decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ of an *input space* $\mathcal{U}$ and an *output space* $\mathcal{Y}$ in various ways and different decompositions yield different *i/s/o representations* (1.1). Indeed, the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ used to describe V in (1.3) corresponds to the given decomposition $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$. The different i/s/o representations can have different properties; they can for instance be passive with respect to different supply rates.

More precisely, a direct-sum decomposition of $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ forms an *admissible i/o pair* $(\mathcal{U}, \mathcal{Y})$ for the subspace V if there exists an *operator node* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, which we will define precisely in Definition 2.4, with input space $\mathcal{U}$, state space $\mathcal{X}$ and output space $\mathcal{Y}$, such that V can be written as the graph of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ in the following way:

$$V = \begin{bmatrix} A\&B \\ \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} \\ C\&D + \begin{bmatrix} 0 & 1_{\mathcal{U}} \end{bmatrix} \end{bmatrix} \operatorname{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right). \tag{1.4}$$

Denote the projection of $\mathcal{W}$ onto $\mathcal{U}$ along $\mathcal{Y}$ by $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}$ and the complementary projection by $\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}}$. Then admissibility of the i/o pair $(\mathcal{U}, \mathcal{Y})$ yields the following i/s/o representation: a pair $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ is a classical trajectory generated by V if and only if

$$\begin{bmatrix} \dot{x}(t) \\ \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(t) \end{bmatrix}, \quad t \geq 0,$$

where $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the operator node in (1.4). This is a very general representation of a linear time-invariant system with input signal $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w$, state trajectory x and output signal $\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w$. Operator node representations are studied in more detail in Section 2.

We call any subspace V of the triple $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$ a *generating subspace*, meaning that it generates some space $\mathfrak{V}$ of classical trajectories. Let us now impose some additional structure on a generating subspace in order to make it a *state/signal node.*

Definition 1.2. Let $\mathcal{X}$ be a Hilbert space and $\mathcal{W}$ a Kreĭn space, and let $V \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W} \end{smallmatrix}\right]$. We say that $(V; \mathcal{X}, \mathcal{W})$ is an *ordinary state/signal node (shortly s/s node)* if V has the following properties:

(i) The space V is closed in the norm

$$\left\| \begin{bmatrix} z \\ x \\ w \end{bmatrix} \right\| = \sqrt{\|z\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + \|w\|_{\mathcal{W}}^2},$$

where $\|\cdot\|_{\mathcal{W}}$ denotes an arbitrary admissible norm on $\mathcal{W}$; see Definition A.3.

(ii) The space V has the property $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Longrightarrow z = 0$.

(iii) There exists some $T > 0$ such that

$$\forall \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix} \in V \; \exists \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T] : \quad \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} z_0 \\ x_0 \\ w_0 \end{bmatrix}.$$

Comparing condition (ii) to Definition 1.1, we see that (ii) means that the derivative of the current state is uniquely determined by the current state and the current external signals at any given time. The condition in fact says that we have chosen an appropriate state space, as we will see in Proposition 4.7. Condition (iii) implies that every trajectory on an arbitrary interval $[0, T]$ can be extended in the forward-time direction to a trajectory on $\mathbb{R}^+$.

According to the following definition, a *s/s system* is essentially a collection of trajectories generated by a s/s node, and as such it is a very general object.

Definition 1.3. Let $(V; \mathcal{X}, \mathcal{W})$ be a s/s node and I a subinterval of $\mathbb{R}$ with positive length.

The space $\mathfrak{W}(I)$ of *generalised trajectories* generated by V on I is the closure of $\mathfrak{V}(I)$ in $\left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{loc}(I;\mathcal{W}) \end{smallmatrix}\right]$. By this we mean that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$ if and only if there exists a sequence of $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}(I)$ that tends to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{loc}(I;\mathcal{W}) \end{smallmatrix}\right]$ as $n \to \infty$. We abbreviate $\mathfrak{W}[0, \infty)$ by $\mathfrak{W}$.

The triple $\Sigma_{s/s} = (\mathfrak{W}; \mathcal{X}, \mathcal{W})$ is called the *state/signal system (s/s system)* induced by $(V; \mathcal{X}, \mathcal{W})$.

Any system can intuitively be called *passive* if it lacks internal energy sources. We now describe how this translates to s/s systems. The energy stored in state $x_0 \in \mathcal{X}$ is given by the norm of x_0 squared: $\|x_0\|^2_{\mathcal{X}} = (x_0, x_0)_{\mathcal{X}}$ and similarly we let the inner product on $\mathcal{W}$ describe how the trajectories generated by V exchange power with the surroundings via the external signal w, so that $[w(t), w(t)]_{\mathcal{W}}$ measures the amount of energy absorbed from the surroundings per time unit at time t. This energy exchange is inherently indefinite, because energy can flow in both directions. Therefore we need to allow $\mathcal{W}$ to have an indefinite inner product, i.e., we need to let $\mathcal{W}$ be a Kreĭn space.

We conclude that a passive s/s system should have the following property: For every generalised (or equivalently for every classical) trajectory, the energy stored in the state should at all times be at most the energy of the initial state plus the total energy absorbed from the surroundings, i.e.,

$$\forall \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{W} : \quad \|x(t)\|^2_{\mathcal{X}} \le \|x(0)\|^2_{\mathcal{X}} + \int_0^t [w(s), w(s)]_{\mathcal{W}} \, \mathrm{d}s, \quad t \ge 0. \quad (1.5)$$

It follows from the proof of Proposition 4.3 that (1.5) is equivalent to the statement

$$\forall \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V} : \quad (\dot{x}(t), x(t))_{\mathcal{X}} + (x(t), \dot{x}(t))_{\mathcal{X}} \le [w(t), w(t)]_{\mathcal{W}}, \quad t \ge 0. \quad (1.6)$$

Noting that

$$\big(\dot{x}(t), x(t)\big)_{\mathcal{X}} + \big(x(t), \dot{x}(t)\big)_{\mathcal{X}} = \left(\frac{\partial}{\partial t}\left(\|x\|_{\mathcal{X}}^2\right)\right)(t), \tag{1.7}$$

we can interpret (1.6) as a statement that the change of energy stored in the state at no time instance exceeds the power input from the outside world. We need to consider classical trajectories in (1.6), because the state part of a generalised trajectory is in general not differentiable. Motivated by this discussion we now introduce a so-called *power product* in order to be able to measure the amount of energy dissipated by a given trajectory at a given time.

Definition 1.4. Let $\mathcal{X}$ be a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{X}}$ and let $\mathcal{W}$ be a Kreĭn space with indefinite inner product $[\cdot,\cdot]_{\mathcal{W}}$. The *(continuous-time) node space* is $\mathfrak{K} := \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{X}\\ \mathcal{W}\end{smallmatrix}\right]$ equipped with the sesquilinear *power product*

$$\left[\left[\begin{matrix}z^1\\ x^1\\ w^1\end{matrix}\right], \left[\begin{matrix}z^2\\ x^2\\ w^2\end{matrix}\right]\right]_{\mathfrak{K}} := [w^1, w^2]_{\mathcal{W}} - (z^1, x^2)_{\mathcal{X}} - (x^1, z^2)_{\mathcal{X}}. \tag{1.8}$$

The power product in (1.8) can be interpreted in the following way. Let $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right] \in \mathfrak{V}$ be a classical trajectory, let $t \geq 0$ and denote

$$p(t) := \left[\left[\begin{matrix}\dot{x}(t)\\ x(t)\\ w(t)\end{matrix}\right], \left[\begin{matrix}\dot{x}(t)\\ x(t)\\ w(t)\end{matrix}\right]\right]_{\mathfrak{K}}.$$

If $p(t) > 0$, then the trajectory $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ *dissipates energy* at a rate of $p(t)$ per time unit at time t. If $p(t) < 0$, then $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ *accumulates energy* at a rate of $|p(t)|$ per time unit and if $p(t) = 0$ then $\left[\begin{smallmatrix}x\\ w\end{smallmatrix}\right]$ *preserves energy* at time t. As a remark, in Section 3 we introduce s/s systems systems whose trajectories evolve with time going in the backwards direction. A trajectory of such a time-reflected s/s system dissipates energy at time t if $p(t) < 0$, because a unit of time is negative in this case.

It turns out that it is natural to define the dual of the s/s node $(V;\mathcal{X},\mathcal{W})$ by $(V^{[\perp]};\mathcal{X},\mathcal{W})$, where the *orthogonal companion* $V^{[\perp]}$ of V in $\mathfrak{K}$ is the space

$$V^{[\perp]} := \{k \in \mathfrak{K} \mid \forall v \in V : \quad [v,k]_{\mathfrak{K}} = 0\} \tag{1.9}$$

of vectors that are $[\cdot,\cdot]_{\mathfrak{K}}$-orthogonal to V. See Section 3 for a more detailed exposition on the dual of a s/s node.

A s/s node $(V;\mathcal{X},\mathcal{W})$ is *passive* if V is a maximally nonnegative subspace of the Kreĭn space $\mathfrak{K}$. This means that $[v,v]_{\mathfrak{K}} \geq 0$ for all $v \in V$ and that V has no proper extension which preserves this property. The maximality requirement is related to the fact that also the dual should be passive in the time-reflected sense that $V^{[\perp]} \leq 0$. A s/s node $(V;\mathcal{X},\mathcal{W})$ is *conservative* if $V = V^{[\perp]}$, which means that all trajectories of the primal node as well as all those of its dual preserve the energy at all times. Section 4 contains general results on passive and conservative s/s systems.

Theorem 4.5 yields that every triple $(V;\mathcal{X},\mathcal{W})$, where V is a maximally nonnegative subspace of $\mathfrak{K}$, such that $\left[\begin{smallmatrix} z\\0\\0\end{smallmatrix}\right] \in V \Longrightarrow z = 0$, is a passive s/s node. The theorem moreover says that every fundamental decomposition, as defined in Definition A.1, of the external signal space $\mathcal{W}$ induces an admissible i/o pair for a passive s/s node. Operator node representations arising from fundamental decompositions of the external signal space $\mathcal{W}$ are called *scattering representations*, because every scattering representation of a passive or conservative s/s node is scattering passive or scattering conservative, respectively; see Definition 5.4. We study scattering representations and their connection to passivity of s/s nodes in Section 5. There we also clarify how passivity relates to the notion of L^2-well-posedness, which was introduced in [14]. If we drop the maximality assumption on the generating subspace V, i.e., we only assume that $V \geq 0$, then by Example 5.3 there is no guarantee of the existence of a scattering representation.

The particular citations to external results that we make in this article are chosen because they are formulated suitably for our needs. Many of these results have been formulated earlier in other contexts. The book [26] collects much of the background that we need and for simplicity we often cite results from this book. The reader may consult this source for further references to the original versions of the various results.

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2. Operator node representation of state/signal nodes

We begin this section by listing some useful properties of the classical trajectories generated by a s/s node. Thereafter we move on to introduce operator nodes and study how these can be used to represent s/s nodes.

In order to proceed in this way we need to introduce some operators for manipulating trajectories. We denote the operator which shifts its argument function to the left by an amount $c \in \mathbb{R}$ by τ^c, so that $(\tau^c f)(t) = f(t+c)$ for all t such that $t + c \in \mathrm{Dom}\,(f)$. The operator that restricts its argument function f to $I \subset \mathrm{Dom}\,(f)$ is denoted by ρ_I, so that $(\rho_I f)(t) = f(t)$ for $t \in I$. By $\bowtie_c$ we denote the concatenation operator at $c \in \mathbb{R}$, i.e.:

$$(f \bowtie_c g)(t) = \begin{cases} f(t), & t < c,\ t \in \mathrm{Dom}\,(f) \\ g(t), & t \geq c,\ t \in \mathrm{Dom}\,(g) \end{cases}.$$

Lemma 2.1. *Let $(V;\mathcal{X},\mathcal{W})$ be a s/s node with classical trajectories $\mathfrak{V}$. Then the following claims are valid:*

(i) *For all $-\infty < a < b < \infty$ and $c \in \mathbb{R}$:*

$$\mathfrak{V}[a,b] = \tau^c \mathfrak{V}[a+c, b+c] \quad \text{and} \quad \mathfrak{V}[a,\infty) = \tau^c \mathfrak{V}[a+c,\infty).$$

(ii) *For all positive-length subintervals I' of the interval I we have $\rho_{I'}\mathfrak{V}(I) \subset \mathfrak{V}(I')$ and, moreover,*

$$\forall b' \in (a,b] : \rho_{[a,b']}\mathfrak{V}[a,b] = \mathfrak{V}[a,b'] \quad \textit{and}$$
$$\forall b' > a : \rho_{[a,b']}\mathfrak{V}[a,\infty) = \mathfrak{V}[a,b'].$$

(iii) *Let $-\infty < a < c < b < \infty$ and assume that $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right] \in \mathfrak{V}[a,c]$ and $\left[\begin{smallmatrix} x_2 \\ w_2 \end{smallmatrix}\right] \in \mathfrak{V}[c,b]$. Then $\left[\begin{smallmatrix} x_1 \\ w_1 \end{smallmatrix}\right] \bowtie_c \left[\begin{smallmatrix} x_2 \\ w_2 \end{smallmatrix}\right] \in \mathfrak{V}[a,b]$ if and only if $x_1(c) = x_2(c)$ and $w_1(c) = w_2(c)$.*

(iv) *For all $T > 0$ we have*

$$V = \left\{ \begin{bmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ w \end{bmatrix} \in \mathfrak{V}[0,T] \right\}. \tag{2.1}$$

This claim is also valid for $T = \infty$ in the sense that it remains true if we replace $\mathfrak{V}[0,T]$ by $\mathfrak{V}$.

(v) *The spaces $\mathfrak{V}[0,T]$, $0 < T \le \infty$, are uniquely determined by V and vice versa.*

(vi) *Property (iii) of Definition 1.2 holds for some $T > 0$ if and only if it holds for all $T > 0$.*

(vii) *A pair $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$ lies in $\mathfrak{V}$ if and only if $\rho_{[0,T]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ for all $T > 0$.*

Proof. All of these claims were proved in [14, Section 2], except for claim (vii), which we now prove. If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$ then $\rho_{[0,T]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ for all $T > 0$ by claim (ii). Conversely assume only that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{W}) \end{smallmatrix}\right]$. If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \notin \mathfrak{V}$, then there by Definition 1.1 exists a $t_0 > 0$ such that $\left[\begin{smallmatrix} \dot{x}(t_0) \\ x(t_0) \\ w(t_0) \end{smallmatrix}\right] \notin V$. This implies that $\rho_{[0,t_0]}\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \notin \mathfrak{V}[0,t_0]$. □

Property (iv) says that for every vector $v_0 \in V$, we can find a trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ on an interval of arbitrary length, such that the initial value $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right]$ of the trajectory is the given vector v_0. A consequence of this result is given in claim (ii), which says that classical trajectories can always be restricted and extended in the forward-time direction to arbitrary intervals. Although most of the results in this paper are given for the interval $[0,T]$, they can be generalised immediately to all intervals $[a,b]$, $a < b$, or $[a,\infty)$, because of claim (ii) and the shift invariance property in claim (i).

An operator node is a relatively complicated mathematical object and we therefore need to recall some terminology before we can define it properly.

Definition 2.2. Let A be a closed operator on the Banach space $\mathcal{X}$.

The *resolvent set* $\mathrm{Res}\,(A)$ of A is the set of all $\lambda \in \mathbb{C}$ such that $\lambda - A$ maps $\mathrm{Dom}\,(A)$ one-to-one onto $\mathcal{X}$.

Fix $\alpha \in \mathrm{Res}\,(A)$, assume that $\mathcal{X}_1 := \mathrm{Dom}\,(A)$ is dense in $\mathcal{X}$, and equip $\mathcal{X}_1$ with the norm $\|x\|_1 := \|(\alpha - A)x\|_{\mathcal{X}}$. Denote by $\mathcal{X}_{-1}$ the completion of

$\mathcal{X}$ with respect to the norm $\|x\|_{-1} := \|(\alpha - A)^{-1}x\|_{\mathcal{X}}$. This norm is weaker than the norm $\|\cdot\|_{\mathcal{X}}$, because $\|x\|_{-1} \leq \|(\alpha - A)^{-1}\|\|x\|_{\mathcal{X}}$ for all $x \in \mathcal{X}$.

The spaces $\mathcal{X}_1$ and $\mathcal{X}_{-1}$ defined above satisfy $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_{-1}$ with dense and continuous embeddings. This construction is sometimes referred to as "rigging". Different choices of $\alpha \in \operatorname{Res}(A)$ give rise to the same triple $(\mathcal{X}_1, \mathcal{X}, \mathcal{X}_{-1})$ of spaces, because although the norms on $\mathcal{X}_1$ and $\mathcal{X}_{-1}$ depend on α, all the norms on $\mathcal{X}_1$ (all the norms on $\mathcal{X}_{-1}$) are equivalent. The norm on $\mathcal{X}_1$ is also equivalent to the graph norm of A. If $\mathcal{X}$ is a Hilbert space, then, so are $\mathcal{X}_1$ and $\mathcal{X}_{-1}$. The triple $(\mathcal{X}_1, \mathcal{X}, \mathcal{X}_{-1})$ is also called the *Gelfand triple* induced by A, see e.g. [26, Sec. 3.6] for more details.

Assume that $\beta \in \operatorname{Res}(A)$. Then $\beta - A$ maps $\mathcal{X}_1 = \operatorname{Dom}(A)$ isomorphically onto $\mathcal{X}$. The operator A can also be considered as a continuous operator which maps the dense subspace $\mathcal{X}_1$ of $\mathcal{X}$ into $\mathcal{X}_{-1}$ and we denote the unique continuous extension of A to an operator $\mathcal{X} \to \mathcal{X}_{-1}$ by $A|_{\mathcal{X}}$. Then the operator $\beta - A|_{\mathcal{X}}$ maps $\mathcal{X}$ isomorphically onto $\mathcal{X}_{-1}$ and $(\beta - A|_{\mathcal{X}})^{-1}$ is the continuous extension of $(\beta - A)^{-1}$ from $\mathcal{X}$ to $\mathcal{X}_{-1}$.

Definition 2.3. Let $\mathcal{X}$ be a Banach space. A family $t \to \mathfrak{A}^t$, $t \geq 0$, of bounded linear operators on $\mathcal{X}$ is a *semigroup* on $\mathcal{X}$ if $\mathfrak{A}^0 = 1$ and $\mathfrak{A}^{s+t} = \mathfrak{A}^s\mathfrak{A}^t$ for all $s, t \geq 0$.

The semigroup is *strongly continuous*, or shorter C_0, if $\lim_{t\to 0^+} \mathfrak{A}^t x_0 = x_0$ for all $x_0 \in \mathcal{X}$. A C_0 semigroup $\mathfrak{A}$ is a *contraction semigroup* if the norm of $\mathfrak{A}^t$ as an operator on $\mathcal{X}$ satisfies $\|\mathfrak{A}^t\| \leq 1$ for all $t \geq 0$.

The *generator* $A : \mathcal{X} \supset \operatorname{Dom}(A) \to \mathcal{X}$ of $\mathfrak{A}$ is the (in general unbounded) linear operator defined by

$$Ax_0 := \lim_{t\to 0^+} \frac{1}{t}(\mathfrak{A}^t x_0 - x_0) \tag{2.2}$$

with $\operatorname{Dom}(A)$ consisting of those $x_0 \in \mathcal{X}$ for which the limit (2.2) exists in $\mathcal{X}$.

The generator A of a C_0 semigroup on $\mathcal{X}$ is closed and $\operatorname{Dom}(A)$ is dense in $\mathcal{X}$; see [22, Thm 1.2.7]. Moreover, according to [22, Thm 1.2.6], a C_0 semigroup $\mathfrak{A}$ is uniquely determined by its generator A in the following way. For every $x_0 \in \operatorname{Dom}(A)$, the function $x : t \to \mathfrak{A}^t x_0$, $t \geq 0$, is the unique continuously differentiable solution of the initial value problem $\dot{x}(t) = Ax(t)$, $t \geq 0$, $x(0) = x_0$. The operators $\mathfrak{A}^t$, $t \geq 0$, are then extended by continuity to all of $\mathcal{X}$. It may therefore be said that A *generates* $\mathfrak{A}$. From [26, Thm 3.2.9(i)] we know that $\operatorname{Res}(A) \neq \emptyset$ for every C_0-semigroup generator. The following definition is essentially Definition 4.7.2 of [26].

Definition 2.4. By an *i/s/o-operator node (shortly operator node)* on the triple $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ of Banach spaces we mean a linear operator

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \operatorname{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

with the following properties:

(i) The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is closed.

(ii) The so-called *main operator* $A\colon \mathrm{Dom}\,(A) \to \mathcal{X}$ of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, defined by

$$Ax = A\&B \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ on } \mathrm{Dom}\,(A) = \left\{ x \in \mathcal{X} \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathrm{Dom}\,(S) \right\}, \tag{2.3}$$

has domain dense in $\mathcal{X}$ and nonempty resolvent set.

(iii) The operator $A\&B$ can be extended to an operator $\begin{bmatrix} A|_{\mathcal{X}} & B \end{bmatrix}$ that maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ continuously into $\mathcal{X}_{-1}$.

(iv) The domain of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ satisfies the condition

$$\mathrm{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \mid A|_{\mathcal{X}} x + Bu \in \mathcal{X} \right\}.$$

An operator node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is called an *i/s/o system node* if its main operator A generates a C_0 semigroup. The operator node is a *time-reflected i/s/o system node* if $-A$ generates a C_0 semigroup, and in this case we say that A *generates a* C_0 *semigroup in backward time.*

The triple (u, x, y) is said to be a *classical i/s/o trajectory* of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ if $u \in C(\mathbb{R}^+;\mathcal{U})$, $x \in C^1(\mathbb{R}^+;\mathcal{X})$, $y \in C(\mathbb{R}^+;\mathcal{Y})$, and $\left[\begin{smallmatrix} \dot{x}(t) \\ y(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ for all $t \geq 0$.

We return to time reflection and motivate the choice of the term "time-reflected i/s/o system node" in the next section. The following definition says that admissibility of a given i/o pair for a generating subspace means that the latter can be written as the graph of an operator node.

Definition 2.5. Let $V \subset \mathfrak{K}$ and $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$.

We say that $(\mathcal{U}, \mathcal{Y})$ is an *admissible i/o pair* of V if there exists an operator node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$, such that

$$V = \begin{bmatrix} A\&B \\ \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} \\ C\&D + \begin{bmatrix} 0 & 1_{\mathcal{U}} \end{bmatrix} \end{bmatrix} \mathrm{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right). \tag{2.4}$$

In this case we call $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ an *operator node representation* of V and write $V_{op} = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$.

If $(V; \mathcal{X}, \mathcal{W})$ is a s/s node and $V_{op} = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$, then we call $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ an *operator node representation* of both *the s/s node* $(V; \mathcal{X}, \mathcal{W})$ and of *the s/s system* Σ that the node generates.

An i/o pair $(\mathcal{U}, \mathcal{Y})$ is admissible for the *system* Σ if it is admissible for at least one of its generating s/s nodes.

An operator node representation is in a sense an input/state/output representation of a s/s node, but for clarity we call it an operator node representation, because the term i/s/o representation was given a different meaning in [14, Def. 4.5]. Note that $(V; \mathcal{X}, \mathcal{W})$ is not necessarily a s/s node even if $V \subset \mathfrak{K}$ has an admissible i/o pair, because condition (iii) of Definition 1.2 might be violated. We proceed to investigate this issue.

It follows from conditions (iii) and (iv) of Definition 2.4 that the top half $A\&B$ of an operator node is a closed operator from $\operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ to $\mathcal{X}$; see the proof of [26, Lemma 4.3.10]. Therefore the domain of an operator node is a Banach space with the graph norm of $A\&B$:

$$\left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{\operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)} = \sqrt{\left\| A\&B \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + \|u\|_{\mathcal{U}}^2}. \tag{2.5}$$

The closedness of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ implies that the operator node is continuous with respect to this norm. If $\mathcal{X}$ is a Hilbert space and $\mathcal{U}$ has a Hilbert-space topology, then the norm of $\operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ determines a Hilbert-space inner product by polarisation.

By taking $u = 0$ in (2.5) we obtain the following graph norm of A for $\operatorname{Dom}(A)$:

$$\|x\|_{\operatorname{Dom}(A)} = \sqrt{\|Ax\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2}.$$

This norm makes A a Banach space, and if $\mathcal{X}$ is a Hilbert space, then the norm defines a Hilbert-space inner product on $\operatorname{Dom}(A)$. The operator A is trivially continuous with respect to this norm.

Lemma 2.6. *Let A be a densely defined operator on the Hilbert space $\mathcal{X}$ such that* $\operatorname{Res}(A) \neq \emptyset$. *The homogeneous Cauchy problem*

$$\dot{x}(t) = Ax(t), \quad t > 0,\ x(0) = x_0, \tag{2.6}$$

has a unique solution $x \in C^1(\mathbb{R}^+; \mathcal{X})$ for every initial value $x_0 \in \operatorname{Dom}(A)$ if and only if A generates a C_0 semigroup on $\mathcal{X}$.

A proof can be found for instance in [22, Thm 4.1.3]. The lemma allows us to explain the difference between operator nodes and i/s/o system nodes, i.e. the existence of a semigroup, in terms of existence and uniqueness of classical trajectories. See Definition A.10 for a description of the function space $H^1_{loc}(I; \mathcal{X})$.

Proposition 2.7. *Let $I = [a, b]$ or $I = [a, \infty)$, where $a < b$, let $V \subset \mathfrak{K}$, and let $V_{op} = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node representation. Then the following claims are all true:*

(i) *If $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a system node, then the triple $(V; \mathcal{X}, \mathcal{W})$ is a s/s node with admissible i/o pair $(\mathcal{U}, \mathcal{Y})$.*

(ii) *Assume that the pair $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, where $x \in H^1_{loc}(I; \mathcal{X})$ and $u \in L^2_{loc}(I; \mathcal{U})$, satisfies the equation $\dot{x}(t) = A|_{\mathcal{X}} x(t) + Bu(t)$ in $\mathcal{X}_{-1}$ almost everywhere on I. Then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C\left(I; \operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)\right)$ if and only if $x \in C^1(I; \mathcal{X})$ and $u \in C(I; \mathcal{U})$.*

If these conditions all hold, then $\dot{x}(t) = A\&B \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ for all $t \in I$.

(iii) *A pair $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ lies in $\mathfrak{V}(I)$ if and only if $\left[\begin{smallmatrix} x \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{smallmatrix}\right] \in C\left(I; \operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)\right)$ and*

$$\begin{bmatrix} \dot{x}(t) \\ \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w(t) \end{bmatrix} \quad \textit{for all } t \in I. \tag{2.7}$$

(iv) *For all* $x_a \in \operatorname{Dom}(A)$ *there exists a unique classical trajectory* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$*, such that* $x(a) = x_a$ *and* $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$*, if and only if* A *generates a* C_0 *semigroup on* $\mathcal{X}$*. In this case* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is a system node.*

Proof. Claim (ii) was shown to hold as a part of the proof of [14, Lemma 5.7].

We now show how claim (iii) follows almost directly from Definition 1.1 and claim (ii). Assume that $\left[\begin{smallmatrix} x \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{smallmatrix}\right] \in C\left(I; \operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)\right)$ and that (2.7) holds. Then $\mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w \in C(I;\mathcal{Y})$ by [14, Lemma 5.6] and, moreover, (2.4) yields that $\begin{bmatrix} \dot x(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for all $t \in I$. Thus $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$. Conversely assume that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$, so that $\begin{bmatrix} \dot x(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for all $t \in I$. According to Definition 1.1, we have $x \in C^1(I;\mathcal{X})$ and $w \in C(I;\mathcal{W})$, and then $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \in C(I;\mathcal{U})$ because the projection is continuous. The inclusion $\begin{bmatrix} \dot x(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ again means that (2.7) holds for all $t \in I$ and thus $\left[\begin{smallmatrix} x \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w \end{smallmatrix}\right] \in C\left(I; \operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)\right)$ by claim (ii).

We use Definition 1.2 to prove claim (i). The graph V in (2.4) of any operator node is closed by Definition 2.4, and moreover, $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \implies z = A\&B \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = 0$. Let now $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ be arbitrary, so that $\left[\begin{smallmatrix} z_0 \\ \mathcal{P}_{\mathcal{Y}}^{\mathcal{U}} w_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] \left[\begin{smallmatrix} x_0 \\ \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_0 \end{smallmatrix}\right]$ by (2.4). We need to construct a classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$, such that $\begin{bmatrix} \dot x(0) \\ x(0) \\ w(0) \end{bmatrix} = \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$. According to [26, Lemma 4.7.8], we can define $u(t) := \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_0$ for $t \geq 0$ and let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ be the unique solution in $\left[\begin{smallmatrix} C^1(\mathbb{R}^+;\mathcal{X}) \\ C(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$ of the equation $\left[\begin{smallmatrix} \dot x(t) \\ y(t) \end{smallmatrix}\right] = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$, $t \geq 0$, $x(0) = x_0$, so that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ lies in $C\left(\mathbb{R}^+; \operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)\right)$ by claim (ii). Then $\left[\begin{smallmatrix} x \\ u+y \end{smallmatrix}\right] \in \mathfrak{V}$ by claim (iii) and $\begin{bmatrix} \dot x(0) \\ x(0) \\ u(0)+y(0) \end{bmatrix} = \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$ by construction.

We split the proof of claim (iv) into two parts: one for the case $I = \mathbb{R}^+$ and one for the case $I = [0, b]$, where $b > 0$. It is sufficient to consider these two cases, because we can assume that $a = 0$ without loss of generality, due to Lemma 2.1(i) and the shift invariance of the equation $\dot x(t) = Ax(t)$.

Part 1: Assume that $I = \mathbb{R}^+$. The operator A is densely defined in $\mathcal{X}$ and $\operatorname{Res}(A)$ is nonempty by Definition 2.4. We may thus use Lemma 2.6.

Fix $x_0 \in \operatorname{Dom}(A)$ arbitrarily. If $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$, then $x \in C^1(\mathbb{R}^+;\mathcal{X})$ and $\dot x(t) = Ax(t)$ for all $t \geq 0$ by claim (iii). Conversely, if $x \in C^1(\mathbb{R}^+;\mathcal{X})$ and $\dot x(t) = Ax(t)$ for all $t \geq 0$, then $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in C\left(\mathbb{R}^+; \operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)\right)$ by claim (ii). Defining $w(t) := C\&D \left[\begin{smallmatrix} x(t) \\ 0 \end{smallmatrix}\right]$ for all $t \geq 0$ we then get $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$, according to claim (iii), and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$ by construction.

The above argument shows that $x \in C^1(\mathbb{R}^+;\mathcal{X})$ satisfies $\dot x(t) = Ax(t)$, $t \geq 0$, if and only if there exists a w such that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$. This implies that for every $x_0 \in \operatorname{Dom}(A)$, there exists a unique $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$, such that $x(0) = x_0$

and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$, if and only if the equation $\dot{x}(t) = Ax(t)$, $t \geq 0$ and $x(0) = x_0$, has a unique continuously differentiable solution for all $x_0 \in \mathrm{Dom}\,(A)$. By Lemma 2.6, this holds if and only if A generates a C_0 semigroup on $\mathcal{X}$.

Regarding the last claim of (iv), if A generates a C_0 semigroup, then the operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a system node by Definition 2.4. This finishes the proof of claim (iv) for the case $I = [a, \infty)$.

Part 2: Now assume that $I = [0, b]$, where $b > 0$. We reduce this case to to the case $I = \mathbb{R}^+$ by proving that there exists a unique $\left[\begin{smallmatrix} x_b \\ w_b \end{smallmatrix}\right] \in \mathfrak{V}[0, b]$, such that $x_b(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_b = 0$ if and only if there exists a unique $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$, such that $x(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$.

First assume that there for all $x_0 \in \mathrm{Dom}\,(A)$ exists a unique $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$, such that $x(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$. Then fix $x_0 \in \mathrm{Dom}\,(A)$ arbitrarily and let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$ satisfy $x(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$. By Lemma 2.1(ii), $\left[\begin{smallmatrix} x_b \\ w_b \end{smallmatrix}\right] := \rho_{[0,b]} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ lies in $\mathfrak{V}[0, b]$, and trivially $x_b(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_b = 0$. In order to prove uniqueness, we let also $\left[\begin{smallmatrix} x_c \\ w_c \end{smallmatrix}\right] \in \mathfrak{V}[0, b]$ with $x_c(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_c = 0$. Then $x_c(b) \in \mathrm{Dom}\,(A)$ by claim (iii) and (2.3). By the assumption at the beginning of this paragraph, there exists a $\left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right] \in \mathfrak{V}$ such that $\widehat{x}(0) = x_c(b)$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widehat{w} = 0$. Claims (i) and (iii) of Lemma 2.1 yields that $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] := \left[\begin{smallmatrix} x_c \\ w_c \end{smallmatrix}\right] \bowtie_b \tau^{-b} \left[\begin{smallmatrix} \widehat{x} \\ \widehat{w} \end{smallmatrix}\right] \in \mathfrak{V}$ with $\widetilde{x}(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w} = 0$. We also assumed that the initial state $\xi(0)$ uniquely determines a trajectory $\left[\begin{smallmatrix} \xi \\ \omega \end{smallmatrix}\right] \in \mathfrak{V}$ with zero input: $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \omega = 0$. This implies that $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ and therefore we obtain $\left[\begin{smallmatrix} x_c \\ w_c \end{smallmatrix}\right] = \rho_{[0,b]} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_b \\ w_b \end{smallmatrix}\right]$. We have shown that there for every $x_0 \in \mathrm{Dom}\,(A)$ exists a unique $\left[\begin{smallmatrix} x_b \\ w_b \end{smallmatrix}\right] \in \mathfrak{V}[0, b]$, such that $x_b(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_b = 0$.

Now conversely assume that there for every $x_0 \in \mathrm{Dom}\,(A)$ exists a unique $\left[\begin{smallmatrix} x_b \\ w_b \end{smallmatrix}\right] \in \mathfrak{V}[0, b]$, such that $x_b(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_b = 0$. In order to prove that there for all $x_0 \in \mathrm{Dom}\,(A)$ exists a unique $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$, such that $x(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$, we first fix $x_0 \in \mathrm{Dom}\,(A)$ arbitrarily. By assumption we can find a sequence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}[0, b]$, such that $x_1(0) = x_0$ and $x_{n+1}(0) = x_n(b)$, $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w_n = 0$ for all $n \geq 1$, since $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}[0, b]$ implies that $x_n(b) \in \mathrm{Dom}\,(A)$. By claims (i), (iii) and (vii) of Lemma 2.1 we have that

$$\begin{bmatrix} x \\ w \end{bmatrix} := \begin{bmatrix} x_1 \\ w_1 \end{bmatrix} \bowtie_b \tau^{-b} \begin{bmatrix} x_2 \\ w_2 \end{bmatrix} \bowtie_{2b} \tau^{-2b} \ldots \in \mathfrak{V}$$

and obviously $x(0) = x_1(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$.

If also $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] \in \mathfrak{V}$ with $x(0) = x_0$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} w = 0$, then

$$\begin{bmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{bmatrix} := \rho_{[0,b]} \tau^{(n-1)b} \begin{bmatrix} \widetilde{x} \\ \widetilde{w} \end{bmatrix} \in \mathfrak{V}[0, b], \quad n \geq 1,$$

according to claims (i) and (ii) of Lemma 2.1. Moreover, $\widetilde{x}_1(0) = x_1(0)$ and $\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}} \widetilde{w}_n = 0$ for all $n \geq 1$. By assumption this implies that $\widetilde{x}_1 = x_1$. Using induction over n, we obtain that $\left[\begin{smallmatrix} \widetilde{x}_n \\ \widetilde{w}_n \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right]$ for all $n \geq 1$ and thus we arrive at $\left[\begin{smallmatrix} \widetilde{x} \\ \widetilde{w} \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$. The proof is now done. □

We end this section by describing in which sense this article also covers boundary control. The following definition is [17, Def. 1.1].

Definition 2.8. A triple (L, K, G) is a *boundary i/s/o node* on the triple $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ of Banach spaces if it has the following properties:

(i) The linear operators L, K and G have the same domain $\mathcal{Z}$.

(ii) The operator $\left[\begin{smallmatrix} L \\ K \\ G \end{smallmatrix}\right] : \mathcal{Z} \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]$ is closed.

(iii) The operator G is surjective and has dense kernel $\mathcal{N}(G)$.

(iv) The operator $A := L|_{\mathcal{N}(G)}$ has a nonempty resolvent set.

If all of these conditions hold and A generates a C_0 semigroup on $\mathcal{X}$, then the boundary i/s/o node is *internally well-posed.* If the conditions (i)–(iv) hold and $-A$ generates a C_0 semigroup, then the boundary i/s/o node is *internally well-posed in backward time.*

Recall that if A generates a C_0 semigroup on $\mathcal{X}$, then $\mathrm{Dom}\,(A) = \mathcal{N}(G)$ is dense in $\mathcal{X}$ and the resolvent set is of A is nonempty. In this case (L, K, G) satisfies conditions (iii) and (iv) of Definition 2.8 if G is surjective.

If (L, K, G) is a boundary i/s/o node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ then we, according to [17, Thm 2.3], always obtain an operator node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ by defining $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] := \left[\begin{smallmatrix} L \\ K \end{smallmatrix}\right] \left[\begin{smallmatrix} 1_{\mathcal{X}} \\ G \end{smallmatrix}\right]^{-1}$ on $\mathrm{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right) = \mathrm{Ran}\left(\left[\begin{smallmatrix} 1_{\mathcal{X}} \\ G \end{smallmatrix}\right]\right)$. In this case the operator node representation

$$V = \begin{bmatrix} A\&B \\ \begin{bmatrix} 1_{\mathcal{X}} & 0 \end{bmatrix} \\ C\&D + \begin{bmatrix} 0 & 1_{\mathcal{U}} \end{bmatrix} \end{bmatrix} \mathrm{Dom}\,(S) \tag{2.8}$$

can be written as

$$V = \begin{bmatrix} L \\ 1_{\mathcal{X}} \\ K + G \end{bmatrix} \mathrm{Dom}\,(L)\,.$$

This representation is formally independent of the i/o pair $(\mathcal{U}, \mathcal{Y})$, but note that conditions (iii) and (iv) in Definition 2.8 still depend on the choice of i/o pair.

Example 2.9. Equip $\mathbb{C}$ with the usual Hilbert-space inner product $\left(u^1, u^2\right)_{\mathbb{C}} = u^1\overline{u^2}$ and let $\mathcal{X} := L^2(\mathbb{R}^+; \mathbb{C})$. Set $\mathcal{Z} := H^1(\mathbb{R}^+; \mathbb{C})$. The elements of $\mathcal{Z}$ are continuous and we may therefore define the point-evaluation operator at 0 on $\mathcal{Z}$ by δ_0, so that $\delta_0 x = x(0)$ for all $x \in \mathcal{Z}$.

In this example we show that $\Xi := \begin{bmatrix} -\frac{\partial}{\partial z} \\ \delta_0 \end{bmatrix}$ with domain $\mathcal{Z}$ is a boundary i/s/o node both on $(\mathbb{C}, L^2(\mathbb{R}^+; \mathbb{C}), \{0\})$, i.e., with input space $\mathcal{U} = \mathbb{C}$ and output space $\mathcal{Y} = \{0\}$, and $(\{0\}, L^2(\mathbb{R}^+; \mathbb{C}), \mathbb{C})$, i.e. $\mathcal{U} = \{0\}$ and $\mathcal{Y} = \mathbb{C}$, by checking the conditions listed in Definition 2.8. In the first case $G = \delta_0$ and $K = 0$, and in the second case $G = 0$ and $K = \delta_0$. We prove that the first of these two boundary i/s/o nodes is internally well-posed (in forward time) and that the second one is internally well-posed in backward time.

Condition (i) of Definition 2.8 is met by Ξ according to the definition of Ξ. We proceed to verify that Ξ satisfies condition (iii) in the case $G = \delta_0$. Note therefore that the space of functions $x \in C^\infty(\mathbb{R}^+; \mathbb{C})$, such that $x(0) = 0$, is dense in $\mathcal{Z}$ and that every such function lies in $\mathcal{N}(G)$. Moreover, G is surjective, because for every $a \in \mathbb{C}$, the function $x_a := \frac{a}{1+z} \in H^1(\mathbb{R}^+;\mathbb{C})$ and $x_a(0) = a$. Condition (iii) is trivial in the case $G = 0$.

We now prove that Ξ satisfies condition (iv), beginning with the case where $G = 0$. According to [26, Ex. 2.3.2, 3.2.3 and 3.3.1], the operator $\frac{\partial}{\partial z} : \mathcal{Z} \to L^2(\mathbb{R}^+;\mathbb{C})$ has resolvent set $\mathbb{C}^+$ and it generates the incoming left-shift C_0 semigroup τ_+ on $L^2(\mathbb{R}^+;\mathbb{C})$:

$$(\tau_+^t x)(z) = x(z+t), \quad x \in L^2(\mathbb{R}^+;\mathbb{C}),\ t \geq 0,\ \text{for almost all } z \geq 0.$$

Consequently, the resolvent set of $-\frac{\partial}{\partial z}\big|_{\mathcal{Z}}$ is $\mathbb{C}^-$.

We next look at the case where $G = \delta_0$. Example 3.5.11(ii) of [26] yields that the operator $-\frac{\partial}{\partial z} : \mathcal{Z} \cap \mathcal{N}(\delta_0) \to L^2(\mathbb{R}^+;\mathbb{C})$ generates the adjoint semigroup τ_+^* of τ_+. This adjoint semigroup is the outgoing right-shift C_0 semigroup on $L^2(\mathbb{R}^+;\mathbb{C})$:

$$\big((\tau_+^*)^t x\big)(z) = \begin{cases} x(z-t), & t \geq 0,\ z \geq t, \\ 0, & t \geq 0,\ z < t, \end{cases} \quad \text{for almost all } z \geq 0.$$

The operator $-\frac{\partial}{\partial z}\big|_{\mathcal{N}(\delta_0)}$ is therefore the adjoint of $\frac{\partial}{\partial z}\big|_{\mathcal{Z}}$, cf. [26, Thm 3.5.6(v)]. Thus the resolvent set of this operator is also $\mathbb{C}^+$, because $\alpha \in \operatorname{Res}(A)$ if and only if $\overline{\alpha} \in \operatorname{Res}(A^*)$.

Condition (ii) is satisfied by Ξ both when $G = 0$ and when $G = \delta_0$. Indeed, by the above, the resolvent set of $-\frac{\partial}{\partial z}$ with domain $\mathcal{Z}$ is nonempty, and therefore this operator is closed. Moreover, δ_0 is continuous with respect to the graph norm of $-\frac{\partial}{\partial z}$, which is the standard Sobolev norm on $H^1(\mathbb{R}^+;\mathbb{C})$, and therefore Ξ is a closed operator.

Thus $\begin{bmatrix} -\frac{\partial}{\partial z} \\ \delta_0 \end{bmatrix}$ is a boundary i/s/o node with both i/o space pairs $(\mathbb{C}, \{0\})$ and $(\{0\}, \mathbb{C})$, i.e., with both $G = \delta_0, K = 0$ and $G = 0, K = \delta_0$, respectively. In the first case, the boundary i/s/o node is internally well-posed, because it has the semigroup τ_+^*. The boundary i/s/o node in the second case is internally well-posed in backward time since $\frac{\partial}{\partial z}\big|_{\mathcal{Z}}$ generates τ_+. If the second boundary i/s/o node were to be internally well-posed also in forward time, then the resolvent set of $-\frac{\partial}{\partial z}\big|_{\mathcal{Z}}$ would have to contain some right-half-plane; see [26, Thm 3.2.9(i)]. This is clearly not the case because by the above we know that this resolvent set equals $\mathbb{C}^-$. The same argument can be used to show that $\frac{\partial}{\partial z}\big|_{\mathcal{Z} \cap \mathcal{N}(\delta_0)}$ does not generate a C_0 semigroup on $\mathcal{X}$.

According to [17, Thm 2.3], both

$$S := -\frac{\partial}{\partial z} \begin{bmatrix} 1 \\ \delta_0 \end{bmatrix}^{-1} : \begin{bmatrix} L^2(\mathbb{R}^+;\mathbb{C}) \\ \mathbb{C} \end{bmatrix} \supset \operatorname{Dom}(S) \to L^2(\mathbb{R}^+;\mathbb{C})$$

with domain $\left[\begin{smallmatrix} 1 \\ \delta_0 \end{smallmatrix}\right] \mathcal{Z}$ and its so-called flow inverse

$$S^\times := \begin{bmatrix} -\frac{\partial}{\partial z} \\ \delta_0 \end{bmatrix} : L^2(\mathbb{R}^+;\mathbb{C}) \supset \operatorname{Dom}(S^\times) \to \begin{bmatrix} L^2(\mathbb{R}^+;\mathbb{C}) \\ \mathbb{C} \end{bmatrix},$$

$\operatorname{Dom}(S^\times) = \mathcal{Z}$, are therefore operator nodes.

Summarising the example, we have established that S and $-S^\times$ are i/s/o system nodes, that $-S$ and $S^\times$ are time-reflected i/s/o system nodes, and that all of these four operators are operator nodes.

Scattering- and impedance-conservative boundary i/s/o nodes are studied in [17] and [18]. In [16], Malinen studies five examples of boundary control systems after developing the tools necessary for this task. Some results applicable to interconnection of impedance-conservative boundary control systems are given in [15]. We return to flow inversion at the end of Section 4.

3. Time-reflected and dual state/signal nodes

The dynamics of the systems we have considered so far evolve with increasing time. The intuitive idea of a time-reflected s/s node, which we now introduce, is a s/s node whose trajectories evolve in backward time. Later in this section we study state/signal duals. Time-reflected and dual s/s nodes generalise the corresponding notions given for i/s/o systems in [28].

Definition 3.1. Let $V \subset \mathfrak{K}$ and $T < 0$. We call $(V; \mathcal{X}, \mathcal{W})$ a *time-reflected s/s node* if V has the following properties:

(i) V is closed,

(ii) $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \Longrightarrow z = 0$ and

(iii) for all $v_0 \in V$ there exists a $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[T,0]$ such that $\begin{bmatrix} \dot x(0) \\ x(0) \\ w(0) \end{bmatrix} = v_0$.

Comparing Definition 3.1 to Definition 1.2, we see that the difference between ordinary and time-reflected s/s nodes is that time-reflected s/s nodes are initialised at the right endpoint of the time interval, $t = 0$ in this case, and evolve in backward time. This determines the time direction of the s/s node. Note, however, that the generating subspaces V in Definition 1.1 have no time direction, because we only require that the generated trajectories $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ should satisfy $\begin{bmatrix} \dot x(t) \\ x(t) \\ w(t) \end{bmatrix} \in V$ for all internal points of the appropriate time interval.

In order to be able to formulate the next proposition, we need to introduce the *time-reflection operator* $\boldsymbol{\mathfrak{R}}$, which reflects its argument function about zero, so that $(\boldsymbol{\mathfrak{R}} f)(t) = f(-t)$, for $-t \in \mathrm{Dom}\,(f)$. We denote $\mathbb{R}^- = (-\infty, 0]$.

Proposition 3.2. *Let $T > 0$, let $V \subset \mathfrak{K}$ and let I be a positive-length subinterval of $\mathbb{R}$. Define the so-called* time-reflection $V^{\leftarrow}$ *of V by* $V^{\leftarrow} := \left[\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] V$ *and denote the space of classical trajectories generated by $V^{\leftarrow}$ on I by $\mathfrak{V}^{\leftarrow}(I)$.*

Then $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[0,T]$ if and only if $\boldsymbol{\mathfrak{R}} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}^{\leftarrow}[-T,0]$, and $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$ if and only if $\boldsymbol{\mathfrak{R}} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}^{\leftarrow}(\mathbb{R}^-)$. The triple $(V^{\leftarrow}; \mathcal{X}, \mathcal{W})$ is a time-reflected s/s node if and only if $(V; \mathcal{X}, \mathcal{W})$ is an ordinary s/s node and vice versa. We *have*

$$V = \begin{bmatrix} A\&B \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ C\&D + \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \operatorname{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \iff$$
$$V^{\leftarrow} = \begin{bmatrix} -A\&B \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ C\&D + \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \operatorname{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \tag{3.1}$$

and the sets of admissible i/o pairs for V and $V^{\leftarrow}$ coincide.

Proof. By Definition 1.1, $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[0,T]$ if and only if $x \in C^1([0,T];\mathcal{X})$, $w \in C([0,T];\mathcal{W})$ and $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V$ for all $t \in (0,T)$. This is obviously equivalent to $\boldsymbol{\mathfrak{R}} x \in C^1([-T,0];\mathcal{X})$, $\boldsymbol{\mathfrak{R}} w \in C([-T,0];\mathcal{W})$ and

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mathfrak{R}} x)(-t) \\ (\boldsymbol{\mathfrak{R}} x)(-t) \\ (\boldsymbol{\mathfrak{R}} w)(-t) \end{bmatrix} = \begin{bmatrix} -\dot{x}(t) \\ x(t) \\ w(t) \end{bmatrix} \in V^{\leftarrow}, \quad t \in [0,T],$$

i.e., to $\boldsymbol{\mathfrak{R}} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^{\leftarrow}[-T,0]$. This computation remains valid if we replace $[0,T]$ by $\mathbb{R}^+$ and $[-T,0]$ by $\mathbb{R}^-$.

Properties (i) and (ii) in Definition 3.1 are the same as in Definition 1.2 and they are invariant under premultiplication of V by $\left[\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$. Assume that these conditions are met by V and note that $\left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V$ if and only if $\left[\begin{smallmatrix} -z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right] \in V^{\leftarrow}$. Then V is a s/s node if and only if for all such elements, there exists a trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[0,T]$, such that $\left[\begin{smallmatrix} \dot{x}(0) \\ x(0) \\ w(0) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_0 \\ x_0 \\ w_0 \end{smallmatrix}\right]$. By the computation we just made, that same trajectory then satisfies $\boldsymbol{\mathfrak{R}} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}^{\leftarrow}[-T,0]$ and

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mathfrak{R}} x)(0) \\ (\boldsymbol{\mathfrak{R}} x)(0) \\ (\boldsymbol{\mathfrak{R}} w)(0) \end{bmatrix} = \begin{bmatrix} -\dot{x}(0) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} -z_0 \\ x_0 \\ w_0 \end{bmatrix}.$$

Therefore $(V;\mathcal{X},\mathcal{W})$ is a s/s node if and only if $(V^{\leftarrow};\mathcal{X},\mathcal{W})$ is a time-reflected s/s node.

The equivalence (3.1) is trivial. Looking at Definition 2.4, we see that conditions (i), (iii) and (iv) are independent of the sign of $A\&B$. Actually, condition (ii) is also independent of the sign of $A\&B$, because $\alpha \in \operatorname{Res}(A)$ if and only if $-\alpha \in \operatorname{Res}(-A)$. Thus V has operator node representation $V_{op} = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ if and only if $V^{\leftarrow}$ has operator node representation $V_{op}^{\leftarrow} = \left(\left[\begin{smallmatrix} -A\&B \\ C\&D \end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$. □

One can formulate properties of the trajectories generated by a time-reflected s/s node similar to those listed in Lemma 2.1 using Definition 3.1 and Proposition 3.2.

Preparing for the next definition, we recall that $V^{[\perp]}$ denotes the space of all vectors which are orthogonal to V in $\mathfrak{K}$, as given in (1.9).

Definition 3.3. Let $(V;\mathcal{X},\mathcal{W})$ be a (time-reflected or ordinary) s/s node. The triple $(V^{[\perp]};\mathcal{X},\mathcal{W})$ is the *s/s dual* of $(V;\mathcal{X},\mathcal{W})$.

For any subinterval I of $\mathbb{R}$ with positive length, we denote the space of classical trajectories generated by $V^{[\perp]}$ on I by $\mathfrak{V}^d(I)$. By $\mathfrak{W}^d(I)$ we denote the space of generalised trajectories generated by $V^{[\perp]}$, i.e., $\mathfrak{W}^d(I)$ is the closure of $\mathfrak{V}^d(I)$ in $\left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{loc}(I;\mathcal{W}) \end{smallmatrix}\right]$. We shortly write $\mathfrak{V}^d := \mathfrak{V}^d(\mathbb{R}^-)$ and $\mathfrak{W}^d := \mathfrak{W}^d(\mathbb{R}^-)$.

The following example shows that the dual of a s/s node is in general neither a s/s node nor a time-reflected s/s node.

Example 3.4. The space $V := \{0\} \subset \mathbb{C}^3$ is a s/s node but $V^{[\perp]}$ violates condition (ii) of Definition 1.2, because $\left[\begin{smallmatrix} \mathbb{C} \\ \{0\} \\ \{0\} \end{smallmatrix}\right] \subset V^{[\perp]}$.

Note that the s/s node in Example 3.4 lacks operator node representations. We show that the s/s dual of a s/s node which has an operator node representation is a time-reflected s/s node in Theorem 3.6.

Proposition 3.5. *The dual of a s/s node is always closed. The double dual of a s/s node is the s/s node itself. For all $V \subset \mathfrak{K}$, the property $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V^{[\perp]} \implies z = 0$ is equivalent to denseness of $\left[\begin{smallmatrix} 0 & 1 & 0 \end{smallmatrix}\right] V$ in $\mathcal{X}$.*

Proof. The first two claims follow from standard Kreĭn-space theory, since $V^{[\perp]}$ is always closed and $(V^{[\perp]})^{[\perp]} = \overline{V}$. For the last claim, note that $\left[\begin{smallmatrix} z' \\ 0 \\ 0 \end{smallmatrix}\right] \in V^{[\perp]}$ if and only if $\left[\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right], \left[\begin{smallmatrix} z' \\ 0 \\ 0 \end{smallmatrix}\right]\right]_{\mathfrak{K}} = -(x,z')_{\mathcal{X}} = 0$ for all $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$, which is equivalent to $z' \in \left(\left[\begin{smallmatrix} 0 & 1 & 0 \end{smallmatrix}\right] V\right)^{\perp}$. This orthogonal complement contains only the vector 0 if and only if $\left[\begin{smallmatrix} 0 & 1 & 0 \end{smallmatrix}\right] V$ is dense in $\mathcal{X}$. ☐

We identify the dual $\mathcal{X}'$ of $\mathcal{X}$ with $\mathcal{X}$ itself, as is common for Hilbert spaces, and moreover, we identify the dual of $\mathcal{W}$ with $\mathcal{W}$ itself as well. The correctness of the following argument follows from [7, Sec. 2.3], where the reader can also find more details. Note, however, that the dual of $\mathcal{W}$ is identified with $-\mathcal{W}$ in [6]. We explain this discrepancy after Definition 4.1.

By [7, Lemma 2.3], if $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ then also $\mathcal{W} = \mathcal{Y}^{[\perp]} \dotplus \mathcal{U}^{[\perp]}$. This allows us to identify the duals $\mathcal{U}'$ and $\mathcal{Y}'$ of $\mathcal{U}$ and $\mathcal{Y}$ as

$$\mathcal{U}' = \mathcal{Y}^{[\perp]} \quad \text{and} \quad \mathcal{Y}' = \mathcal{U}^{[\perp]}$$

using the duality pairings

$$\begin{aligned} \langle u, u' \rangle_{\langle \mathcal{U}, \mathcal{Y}^{[\perp]} \rangle} &= [u, u']_{\mathcal{W}}\,, \quad u \in \mathcal{U},\ u' \in \mathcal{Y}^{[\perp]} \quad \text{and} \\ \langle y, y' \rangle_{\langle \mathcal{Y}, \mathcal{U}^{[\perp]} \rangle} &= [y, y']_{\mathcal{W}}\,, \quad y \in \mathcal{Y},\ y' \in \mathcal{U}^{[\perp]}. \end{aligned} \tag{3.2}$$

We thus obtain the following duality pairings between $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ and their respective duals:

$$\begin{aligned}\left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} z' \\ u' \end{bmatrix} \right\rangle_{\left\langle \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}^{[\perp]}\end{smallmatrix}\right]\right\rangle} &= (x, z')_{\mathcal{X}} + \langle u, u'\rangle_{\langle \mathcal{U}, \mathcal{Y}^{[\perp]}\rangle} \quad \text{and} \\ \left\langle \begin{bmatrix} z \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\left\langle \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}^{[\perp]}\end{smallmatrix}\right]\right\rangle} &= (z, x')_{\mathcal{X}} + \langle y, y'\rangle_{\langle \mathcal{Y}, \mathcal{U}^{[\perp]}\rangle}.\end{aligned} \tag{3.3}$$

Adjoint operators computed with respect to these duality pairings are denoted by †. For instance, if $S : \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \supset \mathrm{Dom}\,(S) \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ is densely defined, then $S^\dagger : \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}^{[\perp]}\end{smallmatrix}\right] \supset \mathrm{Dom}\left(S^\dagger\right) \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}^{[\perp]}\end{smallmatrix}\right]$ is the maximally defined operator, such that for all $\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \mathrm{Dom}\,(S)$ and $\left[\begin{smallmatrix}x'\\ y'\end{smallmatrix}\right] \in \mathrm{Dom}\left(S^\dagger\right)$:

$$\left\langle S \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\left\langle \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}^{[\perp]}\end{smallmatrix}\right]\right\rangle} = \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, S^\dagger \begin{bmatrix} x' \\ y' \end{bmatrix} \right\rangle_{\left\langle \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}^{[\perp]}\end{smallmatrix}\right]\right\rangle}. \tag{3.4}$$

Theorem 3.6. *Let $V \subset \mathfrak{K}$ and $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$. Assume that there exists a densely defined operator $S = \left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right] : \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \supset \mathrm{Dom}\,(S) \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$, such that V has the graph representation*

$$V = \begin{bmatrix} A\&B \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ C\&D + \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \mathrm{Dom}\,(S). \tag{3.5}$$

Let $S^\dagger$ be the adjoint of S, as given in (3.4), *and define*

$$\begin{aligned} S^d := \begin{bmatrix} A\&B^d \\ C\&D^d \end{bmatrix} &:= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} S^\dagger \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{on} \\ \mathrm{Dom}\left(S^d\right) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathrm{Dom}\left(S^\dagger\right) \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{U}^{[\perp]} \end{bmatrix}. \end{aligned} \tag{3.6}$$

Then $V^{[\perp]}$ is given by

$$V^{[\perp]} = \begin{bmatrix} A\&B^d \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ C\&D^d + \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \mathrm{Dom}\left(S^d\right). \tag{3.7}$$

If S is an operator node, then so are S^d and $S^\dagger$. In this case, the main operator of S^d is $-A^*$, where A^* is the adjoint of A as an unbounded operator on the Hilbert space $\mathcal{X}$.

If S is an ordinary i/s/o system node, then so is $S^\dagger$ and in this case S^d is a time-reflected i/s/o system node; see Definition 2.4.

Proof. In the described setup, $\left[\begin{smallmatrix} z' \\ x' \\ u'+y' \end{smallmatrix}\right] \in V^{[\perp]}$ with $u' \in \mathcal{Y}^{[\perp]}$ and $y' \in \mathcal{U}^{[\perp]}$ if and only if for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{Dom}(S)$ we have:

$$\begin{aligned} 0 &= \left[\left[\begin{bmatrix} A\&B \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ C\&D + \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} z' \\ x' \\ w' \end{bmatrix}\right]\right]_{\mathfrak{K}} \\ &= \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} -z' \\ u' \end{bmatrix} \right\rangle_{\left\langle \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y}^{[\perp]} \end{smallmatrix}\right] \right\rangle} \\ &\quad - \left\langle \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} x' \\ -y' \end{bmatrix} \right\rangle_{\left\langle \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U}^{[\perp]} \end{smallmatrix}\right] \right\rangle}. \end{aligned} \tag{3.8}$$

Due to the assumed denseness of $\operatorname{Dom}(S)$ in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$, (3.8) holds for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{Dom}(S)$ if and only if $\left[\begin{smallmatrix} x' \\ -y' \end{smallmatrix}\right] \in \operatorname{Dom}(S^\dagger)$ and $\left[\begin{smallmatrix} -z' \\ u' \end{smallmatrix}\right] = S^\dagger \left[\begin{smallmatrix} x' \\ -y' \end{smallmatrix}\right]$. The latter condition is easily seen to be equivalent to (3.7).

According to [26, Lem. 6.2.14], the adjoint $S^\dagger$ of an operator node S is an operator node with main operator A^*, and $S^\dagger$ is an i/s/o system node if and only if S is an i/s/o system node. By Definition 2.4, it is immediate that $S^\dagger$ is an i/s/o system node if and only if $S^d = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right] S^\dagger \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is a time-reflected i/s/o system node. □

Remark 3.7. The operator $S^\dagger =: \left[\begin{smallmatrix} A\&B' \\ C\&D' \end{smallmatrix}\right]$ in (3.4) is usually referred to as the *causal dual* of S. Looking at (3.6) and (3.7), we see that the s/s dual $(V^{[\perp]}; \mathcal{X}, \mathcal{W})$ corresponds to the so-called *anti-causal dual* $\left[\begin{smallmatrix} -A\&B' \\ C\&D' \end{smallmatrix}\right]$ of S.

Note that the Gelfand triple $(\mathcal{X}_1^d, \mathcal{X}, \mathcal{X}_{-1}^d)$ induced by $-A^*$, the main operator of the dual S^d, in general differs from $(\mathcal{X}_1, \mathcal{X}, \mathcal{X}_{-1})$ when A is unbounded on $\mathcal{X}$. To be more precise, we identify the dual of $\mathcal{X}_1^d$ by $\mathcal{X}_{-1}$ and the dual of $\mathcal{X}_{-1}^d$ by $\mathcal{X}_1$, using $\mathcal{X}$ as pivot space. The Gelfand triple induced by A^*, the main operator of $S^\dagger$ is the same as that induced by $-A^*$.

We have the following important corollary to Theorem 3.6.

Corollary 3.8. *An i/o pair $(\mathcal{U}, \mathcal{Y})$ is admissible for the (ordinary or time-reflected) s/s node $(V; \mathcal{X}, \mathcal{W})$ if and only if the* "dual i/o pair" $(\mathcal{U}^{[\perp]}, \mathcal{Y}^{[\perp]})$ *is admissible for the dual s/s node $(V^{[\perp]}; \mathcal{X}, \mathcal{W})$.*

In Theorem 3.6 we characterised the dual s/s node in terms of the primal s/s node. We now characterise the classical trajectories of the dual in terms of the classical trajectories of the primal s/s node.

Proposition 3.9. *Let $(V; \mathcal{X}, \mathcal{W})$ be a s/s node, let $T < 0$ and let $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1([T,0];\mathcal{X}) \\ C([T,0];\mathcal{W}) \end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{V}^d[T,0]$, i.e., $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right]$ is a trajectory generated by $V^{[\perp]}$ on $[T,0]$, if and only if for all a and b, such that $T \le a < b \le 0$, and* *for all* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[a,b]$*, it holds that:*

$$\big(x(b), x^d(b)\big)_{\mathcal{X}} - \big(x(a), x^d(a)\big)_{\mathcal{X}} = \int_a^b \big[w(s), w^d(s)\big]_{\mathcal{W}}\, \mathrm{d}s. \tag{3.9}$$

Moreover, $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^-;\mathcal{X}) \\ C(\mathbb{R}^-;\mathcal{W}) \end{smallmatrix}\right]$ *lies in* $\mathfrak{V}^d(\mathbb{R}^-)$ *if and only if* (3.9) *holds for all* $a < b \le 0$ *and for all* $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[a,b]$.

Proof. First of all, (3.9) is equivalent to

$$\int_a^b \left[\!\left[\begin{bmatrix} \dot{x}(s) \\ x(s) \\ w(s) \end{bmatrix}, \begin{bmatrix} \dot{x}^d(s) \\ x^d(s) \\ w^d(s) \end{bmatrix} \right]\!\right]_{\mathfrak{K}} \mathrm{d}r = 0, \tag{3.10}$$

because

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\,(x(s), x^d(s))_{\mathcal{X}} &= -(\dot{x}(s), x^d(s))_{\mathcal{X}} - (x(s), \dot{x}^d(s))_{\mathcal{X}} \\ &= \left[\!\left[\begin{bmatrix} \dot{x}(s) \\ x(s) \\ 0 \end{bmatrix}, \begin{bmatrix} \dot{x}^d(s) \\ x^d(s) \\ 0 \end{bmatrix} \right]\!\right]_{\mathfrak{K}}. \end{aligned}$$

If $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{V}^d[T,0]$ then $\dot{x}^d$, x^d and w^d are all continuous on $(T,0)$ and $\left[\begin{smallmatrix} \dot{x}^d(s) \\ x^d(s) \\ w^d(s) \end{smallmatrix}\right] \in V^{[\perp]}$ for all $s \in [T,0]$, according to Definition 3.3. This implies (3.10) for all $T \le a < b \le 0$. If $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{V}^d(\mathbb{R}^-)$, then for every $a < 0$ we have $\rho_{[a,0]} \left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{V}^d[a,0]$ according to Proposition 3.2 and Lemma 2.1(ii). Therefore (3.10) holds for every $a < b \le 0$.

Conversely, fix $T < 0$ and assume that (3.10) holds for all $T \le a < b \le 0$. Fix $a \in [T,0)$ and $b \in (a,0]$ arbitrarily. By Definition 1.2, we can let $\left[\begin{smallmatrix} z_a \\ x_a \\ w_a \end{smallmatrix}\right] \in V$ be arbitrary and find a $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[a,b]$ such that $\left[\begin{smallmatrix} \dot{x}(a) \\ x(a) \\ w(a) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_a \\ x_a \\ w_a \end{smallmatrix}\right]$. Divide both sides of (3.10) by $b - a > 0$ and let $b \to a^+$. By the assumed continuity from the right of $\dot{x}$, x, w, $\dot{x}^d$, x^d and w^d at a we get that

$$0 = \frac{1}{b-a} \int_a^b \left[\!\left[\begin{bmatrix} \dot{x}(s) \\ x(s) \\ w(s) \end{bmatrix}, \begin{bmatrix} \dot{x}^d(s) \\ x^d(s) \\ w^d(s) \end{bmatrix} \right]\!\right]_{\mathfrak{K}} \mathrm{d}s \to \left[\!\left[\begin{bmatrix} \dot{x}(a) \\ x(a) \\ w(a) \end{bmatrix}, \begin{bmatrix} \dot{x}^d(a) \\ x^d(a) \\ w^d(a) \end{bmatrix} \right]\!\right]_{\mathfrak{K}}. \tag{3.11}$$

Thus the limit in (3.11) is zero for all $\left[\begin{smallmatrix} \dot{x}(a) \\ x(a) \\ w(a) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_a \\ x_a \\ w_a \end{smallmatrix}\right] \in V$, which implies that $\left[\begin{smallmatrix} \dot{x}^d(a) \\ x^d(a) \\ w^d(a) \end{smallmatrix}\right] \in V^{[\perp]}$ for all $a \in [T,0)$. By the closedness of $V^{[\perp]}$ and continuity from the left of x^d, $\dot x^d$ and w^d at 0 we obtain that also

$$\begin{bmatrix} \dot x^d(0) \\ x^d(0) \\ w^d(0) \end{bmatrix} = \lim_{t\to 0^-} \begin{bmatrix} \dot x^d(t) \\ x^d(t) \\ w^d(t) \end{bmatrix} \in V^{[\perp]}.$$

We have now proved that $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{W}^d[T,0]$.

We still need to show that if $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(\mathbb{R}^-;\mathcal{X}) \\ C(\mathbb{R}^-;\mathcal{W}) \end{smallmatrix}\right]$ and (3.9) holds for all $a<b\le 0$, then $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{W}^d(\mathbb{R}^-)$. Combining Lemma 2.1(vii) and Proposition 3.2, we see that $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{W}^d(\mathbb{R}^-)$ if and only if $\rho_{[T',0]}\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{W}^d[T',0]$ for all $T'<0$. Now fix $T'<0$ arbitrarily and note that $\rho_{[T',0]}\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1([T',0];\mathcal{X}) \\ C([T',0];\mathcal{W}) \end{smallmatrix}\right]$ and that (3.9) by assumption holds for all $T'\le a<b\le 0$ and for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}[a,b]$. The finite-interval case of this proposition yields that $\rho_{[T',0]}\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{W}^d[T',0]$, and we have proved that $\left[\begin{smallmatrix} x^d \\ w^d \end{smallmatrix}\right] \in \mathfrak{W}^d$. □

We are now finally ready to introduce passive s/s systems properly.

4. Passive and conservative state/signal nodes

In this section we add the concept of passivity to the s/s framework and study what additional structure passive and conservative s/s nodes have. We begin with the following definition, which was motivated in the introduction of the article.

Definition 4.1. An *ordinary s/s node* $(V;\mathcal{X},\mathcal{W})$ is *dissipative* (in the forward-time direction) if $V\ge 0$. A *time-reflected s/s node* $(V^{\leftarrow};\mathcal{X},\mathcal{W})$ is *dissipative* (in the backward-time direction) if $V^{\leftarrow}\le 0$.

An ordinary or time-reflected s/s node $(V;\mathcal{X},\mathcal{W})$ is *energy preserving* if V is neutral: $[v,v]_{\mathfrak{K}}=0$ for all $v\in V$.

An ordinary or time-reflected s/s node is *passive* or *conservative* if both $(V;\mathcal{X},\mathcal{W})$ and its dual $(V^{[\perp]};\mathcal{X},\mathcal{W})$ are dissipative or energy preserving, respectively.

An ordinary or time-reflected *s/s system* is said to be dissipative, passive, energy preserving or conservative if one of its generating s/s nodes is of the corresponding type.

In Definition 4.1 we deviate slightly from the terminology used by Arov and Staffans in [5]. In the setting of [5] all systems evolve in forwards time, and this makes it natural for Arov and Staffans to identify the dual $\mathcal{W}'$ of $\mathcal{W}$ by $-\mathcal{W}$. Theorem 3.6 implies that the dual of a s/s node often is a time-reflected s/s node in our setting.

Arov and Staffans call a dissipative s/s node “forward passive” and by a “backward passive” node they mean a s/s node whose dual s/s node is forward passive. While the terminology of Arov and Staffans is appropriate in their setting, it becomes confusing when some systems evolve in backward time. We also give the term "dissipative" a different meaning than Willems in [32, 33]. More precisely, the class of s/s systems which are "dissipative" in Willems' terminology are precisely those which we call passive.

Recall that $V \subset \mathfrak{K}$ is called *maximally nonnegative* if $[v,v]_{\mathfrak{K}} \geq 0$ for all $v \in V$ and V has no proper extension that preserves this property. The subspace V is *neutral* if and only if $V \subset V^{[\perp]}$ and it is *Lagrangian* if $V = V^{[\perp]}$; see Lemma A.7. We have the following immediate corollary to Definition 4.1.

Corollary 4.2. *An ordinary s/s node is passive or conservative if and only if V is maximally nonnegative or Lagrangian, respectively. Every conservative s/s node is passive.*

Proof. According to Proposition A.8, a closed subspace $V \geq 0$ is maximally nonnegative if and only if $V^{[\perp]} \leq 0$. Trivially both V and $V^{[\perp]}$ are neutral, i.e., $V \subset V^{[\perp]}$ and $V^{[\perp]} \subset V$ if and only if $V = V^{[\perp]}$. In particular, $V = V^{[\perp]}$ by Proposition A.8 implies that $V \geq 0$ and $V^{[\perp]} \leq 0$. □

Let $\mathcal{X} \neq \{0\}$ and $\mathcal{W} = \{0\}$. Note that $V := \begin{bmatrix} \mathcal{X} \\ \{0\} \\ \{0\} \end{bmatrix}$ is a Lagrangian subspace of $\mathfrak{K}$, which does not satisfy $\begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix} \in V \implies z = 0$. Therefore the triple $(V;\mathcal{X},\mathcal{W})$ needs not be a s/s node even if V is maximally nonnegative or Lagrangian. This is in contrast to the discrete-time case described in [5, Prop. 5.12].

We now characterise dissipative and energy-preserving s/s nodes in terms of their trajectories.

Proposition 4.3. *Let $(V;\mathcal{X},\mathcal{W})$ be an ordinary s/s node and let $I = [a,b]$, with $b > a$, or $I = [a,\infty)$.*

The s/s node $(V;\mathcal{X},\mathcal{W})$ is dissipative if and only if the inequality

$$\forall t \in I: \quad \|x(t)\|_{\mathcal{X}}^2 - \|x(a)\|_{\mathcal{X}}^2 \leq \int_a^t [w(s),w(s)]_{\mathcal{W}}\,\mathrm{d}s \tag{4.1}$$

holds for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$, or equivalently, for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$.

The s/s node is energy preserving if and only if (4.1) *holds with equality instead of inequality for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$, or equivalently, for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$.*

Proof. The proof is divided into two parts for readability.
Part 1: We begin by proving that $(V;\mathcal{X},\mathcal{W})$ is dissipative if and only if all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$ satisfy (4.1). Assume therefore first that $(V;\mathcal{X},\mathcal{W})$ is dissipative, i.e., that $V \geq 0$. Select $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$ and $t \in I$ arbitrarily. Then we by (1.7) for all $s \in [a,t]$ have

$$0 \leq \left[\begin{bmatrix} \dot{x}(s) \\ x(s) \\ w(s) \end{bmatrix}, \begin{bmatrix} \dot{x}(s) \\ x(s) \\ w(s) \end{bmatrix}\right]_{\mathfrak{K}} = [w(s),w(s)]_{\mathcal{W}} - \frac{\partial}{\partial s}\big(\|x(s)\|_{\mathcal{X}}^2\big) \tag{4.2}$$

and integrating this from a to t we get that (4.1) holds for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$.

Conversely assume that (4.1) holds for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$. Let $\left[\begin{smallmatrix} z_a \\ x_a \\ w_a \end{smallmatrix}\right] \in V$ be arbitrary and choose $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$ such that $\left[\begin{smallmatrix} \dot{x}(a) \\ x(a) \\ w(a) \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_a \\ x_a \\ w_a \end{smallmatrix}\right]$, cf. Lemma 2.1(iv). By (4.1) we for all $h > 0$, such that $a + h \in I$, have

$$\frac{1}{h}\int_a^{a+h} [w(s), w(s)]_{\mathcal{W}} \,\mathrm{d}s - \frac{1}{h}\left(\|x(a+h)\|_{\mathcal{X}}^2 - \|x(a)\|_{\mathcal{X}}^2\right) \geq 0. \tag{4.3}$$

Letting $h \to 0^+$, we get (4.2) with $s = a$ and thus

$$\left[\left[\begin{bmatrix} z_a \\ x_a \\ w_a \end{bmatrix}, \begin{bmatrix} z_a \\ x_a \\ w_a \end{bmatrix}\right]\right]_{\mathfrak{K}} \geq 0 \quad \text{for all} \quad \begin{bmatrix} z_a \\ x_a \\ w_a \end{bmatrix} \in V. \tag{4.4}$$

Part 2: If (4.1) holds for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$ then (4.1) trivially holds for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$, because every classical trajectory is also generalised according to Definition 1.3.

Now conversely assume that (4.1) holds for all $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}(I)$, so that:

$$\forall t \in I: \quad \|x_n(t)\|_{\mathcal{X}}^2 - \|x_n(a)\|_{\mathcal{X}}^2 - \int_a^t [w_n(s), w_n(s)]_{\mathcal{W}} \,\mathrm{d}s \leq 0. \tag{4.5}$$

Let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$ be arbitrary and let the sequence $\left[\begin{smallmatrix} x_n \\ w_n \end{smallmatrix}\right] \in \mathfrak{V}(I)$, $n \geq 1$, tend to $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{loc}(I;\mathcal{W}) \end{smallmatrix}\right]$. Then we for all $t \in I$ have $\lim_{n\to\infty} x_n(t) = x(t)$ and

$$\lim_{n\to\infty} \int_a^t [w_n(s), w_n(s)]_{\mathcal{W}} \,\mathrm{d}s = \int_a^t [w(s), w(s)]_{\mathcal{W}} \,\mathrm{d}s.$$

We now obtain (4.1) for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$ by letting $n \to \infty$ in (4.5).

The claim that $V \subset V^{[\perp]}$ if and only if (4.1) holds with equality for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$, or equivalently for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$, is proved by replacing the inequality signs in (4.1), (4.2), (4.3), (4.4) and (4.5) by equality signs. □

Note that the conditions in Proposition 4.3 hold for some subinterval $I \subset \mathbb{R}$ of the type $[a, b]$, $b > a$, or $[a, \infty)$ if and only if they hold for all such subintervals, because the claim that the s/s node is dissipative (or energy preserving) does not depend on the choice of I. In order to characterise also passive and conservative s/s nodes, we need the following counterpart of Proposition 4.3 for time-reflected s/s nodes.

Corollary 4.4. *Let $I = [a, b]$, where $a < b$, or $I = (-\infty, b]$. A time-reflected s/s node $(V; \mathcal{X}, \mathcal{W})$ is dissipative if and only if we for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$, or equivalently, for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$ have:*

$$\forall t \in I: \quad \|x(b)\|_{\mathcal{X}}^2 - \|x(t)\|_{\mathcal{X}}^2 \geq \int_t^b [w(s), w(s)]_{\mathcal{W}} \,\mathrm{d}s. \tag{4.6}$$

The s/s node is energy preserving if and only if (4.6) *holds with equality instead of inequality for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}(I)$, or equivalently, for all $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{W}(I)$.*

Proof. The argument is analogous to the proof of Proposition 4.3. First assume that $V \le 0$ and integrate (4.2) with a reversed inequality sign from t to b in order to obtain (4.6).

Conversely, assume that (4.6) holds and let $h > 0$ be such that $b-h > a$. Letting $t = b - h$ in (4.6) and dividing the inequality by h, we obtain $V \le 0$ by letting $h \to 0^+$. □

The following theorem is of fundamental importance for the theory of passive s/s systems, because it establishes a.o. the very useful fact that every fundamental i/o pair is admissible for a passive s/s node.

Theorem 4.5. *Assume that V is a maximally nonnegative subspace of $\mathfrak{K}$, that $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Longrightarrow z = 0$ and that $\mathcal{W} = (\mathcal{W}_+, \mathcal{W}_-)$ is a fundamental i/o pair.*

The following claims are valid:

(i) *The triple $(V; \mathcal{X}, \mathcal{W})$ is a passive s/s node for which $(\mathcal{W}_+, \mathcal{W}_-)$ is admissible.*
(ii) *In the operator node representation $V_{op} = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right)$, the operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{smallmatrix}\right] \supset \mathrm{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W}_- \end{smallmatrix}\right]$ is an i/s/o system node with a contraction semigroup $\mathfrak{A}$ on $\mathcal{X}$. The generator A of $\mathfrak{A}$ satisfies $\mathbb{C}^+ \subset \mathrm{Res}\,(A)$.*

Proof. We use [14, Prop. 6.7] to prove the claims we made. Define $\mathfrak{K}_\pm$ by (A.2) and $\mathcal{P}_{\mathfrak{K}_\pm}^{\mathfrak{K}_\mp}$ by (A.3) with $\alpha = 1$, so that $(\mathfrak{K}_+, \mathfrak{K}_-)$ is a fundamental decomposition of $\mathfrak{K}$ and $\mathcal{P}_{\mathfrak{K}_\pm}^{\mathfrak{K}_\mp}$ the corresponding fundamental projections, according to Proposition A.2. That proposition also yields that

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} \end{bmatrix} V = \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix}, \tag{4.7}$$

because $\mathcal{P}_{\mathfrak{K}_+}^{\mathfrak{K}_-} V = \mathfrak{K}_+$ by the assumed maximal nonnegativity of V and Proposition A.8. The maximal nonnegativity also implies that V is closed; see Remark A.6. Thus conditions (a) and (d) of [14, Prop. 6.7] are fulfilled by V.

Letting $|\mathcal{W}_-|$ be as given in Definition A.1, we see that the nonnegativity property $V \ge 0$ means that:

$$\begin{bmatrix} z \\ x \\ w \end{bmatrix} \in V \quad \Longrightarrow \quad \|\mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} w\|_{\mathcal{W}_+}^2 - \|\mathcal{P}_{\mathcal{W}_-}^{\mathcal{W}_+} w\|_{|\mathcal{W}_-|}^2 - 2\mathrm{Re}\,(z, x)_{\mathcal{X}} \ge 0. \tag{4.8}$$

This implies that the space $V_z := \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} \end{smallmatrix}\right] V$ is closed, as we now show. Let therefore $\left[\begin{smallmatrix} z_n \\ x_n \\ u_n \end{smallmatrix}\right] \in V_z$, $n \in \mathbb{Z}^+$, tend to some $\left[\begin{smallmatrix} z \\ x \\ u \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{W}_+ \end{smallmatrix}\right]$. The inclusion $\left[\begin{smallmatrix} z_n \\ x_n \\ u_n \end{smallmatrix}\right] \in V_z$ means that there exists a sequence $y_n \in \mathcal{W}_-$, such that $\left[\begin{smallmatrix} z_n \\ x_n \\ u_n + y_n \end{smallmatrix}\right] \in V$. Then $\left[\begin{smallmatrix} z_n \\ x_n \\ u_n + y_n \end{smallmatrix}\right] - \left[\begin{smallmatrix} z_m \\ x_m \\ u_m + y_m \end{smallmatrix}\right] \in V$ for all $m, n \in \mathbb{Z}^+$ and (4.8) yields that

$$\|y_n - y_m\|_{|\mathcal{W}_-|}^2 \le \|u_n - u_m\|_{\mathcal{W}_+}^2 - 2\mathrm{Re}\,(z_n - z_m, x_n - x_m)_{\mathcal{X}}$$

for all $n, m \in \mathbb{Z}^+$. The right-hand side tends to zero as $m, n \to \infty$, because z_n, x_n and u_n are all convergent, and thus Cauchy, sequences. Then also y_n is a Cauchy sequence which tends to some y in the complete space $\mathcal{W}_-$. By the closedness of V, we have $\left[\begin{smallmatrix} z \\ x \\ u+y \end{smallmatrix}\right] \in V$, and from the closedness of $\mathcal{W}_+$ and $\mathcal{W}_-$ we obtain that $u \in \mathcal{W}_+$ and $y \in \mathcal{W}_-$, i.e., that $\left[\begin{smallmatrix} z \\ x \\ u \end{smallmatrix}\right] \in V_z$. We have proved that V_z is closed.

Another consequence of (4.8) is that V is given by

$$V = \begin{bmatrix} A\&B \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \\ C\&D + \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \mathrm{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right) \tag{4.9}$$

for some operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, which maps

$$\mathrm{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} \end{bmatrix} V \subset \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix} \text{ into } \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_- \end{bmatrix}.$$

Indeed, if $\left[\begin{smallmatrix} z \\ 0 \\ y \end{smallmatrix}\right] \in V$ and $y \in \mathcal{W}_-$ then

$$0 \le -\|y\|^2_{|\mathcal{W}_-|} - 2\mathrm{Re}\,(z, 0) = -\|y\|^2_{|\mathcal{W}_-|},$$

i.e., $y = 0$, and thus $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V$, which by assumption implies that $z = 0$. Defining the main operator A and the observation operator C of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ by

$$\begin{bmatrix} A \\ C \end{bmatrix} x := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad x \in \left\{ x_0 \mid \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \in \mathrm{Dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right) \right\}, \tag{4.10}$$

we obtain that $\left[\begin{smallmatrix} z \\ x \\ y \end{smallmatrix}\right] \in V$ with $y \in \mathcal{W}_-$ if and only if $x \in \mathrm{Dom}\,(A)$ and $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right] x$.

It still remains to prove that A generates a contraction semigroup on $\mathcal{X}$ and that $\mathbb{C}^+ \subset \mathrm{Res}\,(A)$. We use the Lumer-Phillips theorem [22, Thm 1.4.6] for this purpose. We thus need to show that $\mathrm{Dom}\,(A)$ is dense in $\mathcal{X}$, that $1 - A$ is surjective and that A is dissipative, i.e., that $\mathrm{Re}\,(Ax, x) \le 0$ for all $x \in \mathrm{Dom}\,(A)$. According to [22, Thm 1.4.6], denseness of $\mathrm{Dom}\,(A)$ is implied by the two latter properties, because $\mathcal{X}$ is a Hilbert space.

It follows from (4.8), (4.9) and (4.10) that A is dissipative, because $x \in \mathrm{Dom}\,(A)$ implies that $\left[\begin{smallmatrix} Ax \\ x \\ Cx \end{smallmatrix}\right] \in V$ and then

$$\|0\|^2_{\mathcal{W}_+} - \|Cx\|^2_{|\mathcal{W}_-|} - 2\mathrm{Re}\,(Ax, x)_{\mathcal{X}} \ge 0 \quad \Longrightarrow \quad 2\mathrm{Re}\,(Ax, x)_{\mathcal{X}} \le 0.$$

Moreover, $1 - A$ is surjective, because by (4.7) and (4.9) there for all $\xi \in \mathcal{X}$ exists a $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V$, such that

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} \end{bmatrix} \begin{bmatrix} z \\ x \\ w \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} - A\&B \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} w \end{bmatrix} = \begin{bmatrix} \xi \\ 0 \end{bmatrix},$$

which means that $x \in \mathrm{Dom}\,(A)$ and $(1 - A)x = \xi$.

We have now proved that A generates a contraction semigroup on $\mathcal{X}$ and by [22, Cor. 1.3.6] we have $\mathbb{C}^+ \subset \operatorname{Res}(A)$. According to Proposition 2.7(i), $(V;\mathcal{X},\mathcal{W})$ is a s/s node with admissible i/o pair $(\mathcal{W}_+,\mathcal{W}_-)$. This s/s node is passive by Corollary 4.2, because V was assumed to be maximally nonnegative. □

The next section deals with operator node representations that correspond to fundamental i/o pairs. In the rest of this section we present a few results which do not refer to specific i/o pairs.

Remark 4.6. According to Definition 1.2, Corollary 4.2 and Theorem 4.5, the triple $(V;\mathcal{X},\mathcal{W})$ is a passive s/s node if and only if V is a maximally nonnegative subspace of $\mathfrak{K}$ and $\left[\begin{smallmatrix} z\\0\\0\end{smallmatrix}\right] \in V \Longrightarrow z=0$.

We now show that the condition $\left[\begin{smallmatrix} z\\0\\0\end{smallmatrix}\right] \in V \Longrightarrow z=0$ is not crucial in the context of passive s/s systems, because we can turn every maximally nonnegative subspace $V\subset\mathfrak{K}$ into a s/s node by removing a certain "degenerate" part of V and shrinking the state space.

Proposition 4.7. *Let V be a maximally nonnegative subspace of $\mathfrak{K}$ and define $\mathcal{X}_0 := \left\{z\in\mathcal{X} \mid \left[\begin{smallmatrix} z\\0\\0\end{smallmatrix}\right]\in V\right\}$, $V_0 := \left[\begin{smallmatrix} \mathcal{X}_0\\0\\0\end{smallmatrix}\right]$ and $\mathcal{X}_1 := \mathcal{X}\ominus\mathcal{X}_0$. Let $\mathfrak{K}_1 := \left[\begin{smallmatrix} \mathcal{X}_1\\\mathcal{X}_1\\\mathcal{W}\end{smallmatrix}\right]$ inherit the indefinite inner product from $\mathfrak{K}$ and set $V_1 := V\cap\mathfrak{K}_1$.*

The following claims are true:

(i) *We have $V = V_0 \dotplus V_1$, where $V_0[\perp]V_1$ and V_0 is neutral: $V_0\subset V_0^{[\perp]}$.*
(ii) *The orthogonal companion $V_1^{[\perp]_1}$ of V_1 in $\mathfrak{K}_1$ is given by $V_1^{[\perp]_1} = V^{[\perp]}\cap\mathfrak{K}_1$ and, moreover, $V^{[\perp]} = V_0 \dotplus V_1^{[\perp]_1}$.*
(iii) *The triple $(V_1;\mathcal{X}_1,\mathcal{W})$ is a passive s/s node, which is conservative if and only if V is Lagrangian: $V=V^{[\perp]}$.*
(iv) *The spaces V and V_1 generate the same trajectories: $\mathfrak{V}=\mathfrak{V}_1$ and $\mathfrak{W}=\mathfrak{W}_1$. The only trajectory generated by V_0 is the zero trajectory: $\mathfrak{V}_0 = \mathfrak{W}_0 = \{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\}$.*

Proof. First fix a fundamental decomposition $\mathcal{W}=(\mathcal{W}_+,\mathcal{W}_-)$ and let $J := \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} - \mathcal{P}_{\mathcal{W}_-}^{\mathcal{W}_+}$ be the corresponding fundamental symmetry, cf. Definitions A.1 and A.3. Then it is readily verified that

$$\left(\begin{bmatrix} z^1\\x^1\\w^1\end{bmatrix},\begin{bmatrix} z^2\\x^2\\w^2\end{bmatrix}\right)_{\mathfrak{K}} := (z^1,z^2)_{\mathcal{X}} + (x^1,x^2)_{\mathcal{X}} + [w^1,Jw^2]_{\mathcal{W}} \tag{4.11}$$

is the admissible inner product on $\mathfrak{K}$ corresponding to the fundamental decomposition (A.2) of $\mathfrak{K}$ with $\alpha=1$. Also note that $V_0 = V\cap\left[\begin{smallmatrix}\mathcal{X}\\\{0\}\\\{0\}\end{smallmatrix}\right]$.

(i) The maximal nonnegativity of the space V implies that it is closed, see Remark A.6, and it is then immediate that also V_0 is closed. Define $\widetilde{V}_1 := V\ominus V_0$, where the orthogonality is taken with respect to the Hilbert-space inner product (4.11). Then $V = V_0\oplus\widetilde{V}_1$, by assumption we have $V \geq 0$, and it clearly holds that $V_0 \subset V_0^{[\perp]}$. Lemma A.7 yields that $V_0[\perp]\widetilde{V}_1$ and $\widetilde{V}_1 \geq 0$. We are done proving claim (i) once we have established that $\widetilde{V}_1 = V_1$.

Let $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \widetilde{V}_1$ be arbitrary, and note that $\widetilde{V}_1 \subset V$, $\widetilde{V}_1[\perp]V_0$ and $\widetilde{V}_1 \perp V_0$ imply that

$$(x, z_0)_{\mathcal{X}} = -\left[\begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z_0 \\ 0 \\ 0 \end{bmatrix}\right]_{\mathfrak{K}} = 0 = \left(\begin{bmatrix} z \\ x \\ w \end{bmatrix}, \begin{bmatrix} z_0 \\ 0 \\ 0 \end{bmatrix}\right)_{\mathfrak{K}} = (z, z_0)_{\mathcal{X}}$$

for all $z_0 \in \mathcal{X}_0$. This yields that $z, x \in \mathcal{X}_0^{\perp}$, i.e., that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \cap \mathfrak{K}_1 = V_1$. Conversely, if $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \cap \mathfrak{K}_1$, then in particular $z \in \mathcal{X}_1$, which implies that $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in V \ominus V_0 = \widetilde{V}_1$. We have proved that $\widetilde{V}_1 = V_1$.

(ii) Applying a slight modification of the procedure in the proof of claim (i) to $V^{[\perp]}$, which is maximally nonpositive by Proposition A.8, we obtain that $V^{[\perp]} = \widehat{V}_0 \dotplus \widehat{V}_1$, where $\widehat{V}_0 = V^{[\perp]} \cap \left[\begin{smallmatrix} \mathcal{X} \\ \{0\} \\ \{0\} \end{smallmatrix}\right]$ is neutral and $\widehat{V}_1 = V^{[\perp]} \cap \mathfrak{K}_1$ is nonpositive.

By Lemma A.7, we have $V_0 \subset V^{[\perp]}$ and by definition $V_0 \subset \left[\begin{smallmatrix} \mathcal{X} \\ \{0\} \\ \{0\} \end{smallmatrix}\right]$, and therefore $V_0 \subset \widehat{V}_0$. The same argument applied to $\widehat{V}_0$ yields that

$$\widehat{V}_0 \subset \left((V^{[\perp]})^{[\perp]}\right) \cap \begin{bmatrix} \mathcal{X} \\ \{0\} \\ \{0\} \end{bmatrix} = V_0$$

and we thus have $\widehat{V}_0 = V_0$. Furthermore, $\widehat{V}_1 \subset V_1^{[\perp]_1}$, i.e. $\widehat{V}_1[\perp]V_1$, because $V^{[\perp]} = V_0 + \widehat{V}_1$ is orthogonal to $V = V_0 + V_1$. On the other hand, assuming that $v_1' \in V_1^{[\perp]_1}$, we obtain

$$[v_0 + v_1, v_1']_{\mathfrak{K}} = [v_0, v_1']_{\mathfrak{K}} + [v_1, v_1']_{\mathfrak{K}_1} = 0, \quad v_i \in V_i,$$

which implies that $v_1' \in V^{[\perp]} \cap \mathfrak{K}_1 = \widehat{V}_1$. Therefore $\widehat{V}_1 = V_1^{[\perp]_1}$.

(iii) First note that if $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V_1$, then $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V_0 \cap V_1 = \{0\}$ and thus $z = 0$. We showed that $V_1 \geq 0$ in the proof of claim (i), the space $V_1 = V \cap \mathfrak{K}_1$ is closed since both V and $\mathfrak{K}_1$ are closed, and the proof of claim (ii) then yields that $V_1^{[\perp]_1} \leq 0$. According to Proposition A.8, we have that V_1 is a maximally nonnegative subspace of $\mathfrak{K}_1$, and Theorem 4.5 yields that $(V_1; \mathcal{X}_1, \mathcal{W})$ is a passive s/s node.

If, moreover, $V = V^{[\perp]}$, then V is also maximally nonpositive by Proposition A.8, and the above argument can be modified to yield that V_1 is a maximally nonpositive subspace of $\mathfrak{K}_1$. Then $V_1 = V_1^{[\perp]_1}$ by Proposition A.8, and thus $(V_1; \mathcal{X}_1, \mathcal{W})$ is a s/s node and V_1 is a Lagrangian subspace of $\mathfrak{K}_1$, i.e., $(V_1; \mathcal{X}_1, \mathcal{W})$ is a conservative s/s node.

Conversely, if $(V_1;\mathcal{X}_1,\mathcal{W})$ is a conservative s/s node, then $V_1 = V_1^{[\perp]_1}$ according to Corollary 4.2, and by claim (ii) we have $V^{[\perp]} = V_0 \dotplus V_1^{[\perp]_1} = V_0 \dotplus V_1 = V$.

(iv) Every classical trajectory generated by V_1 is trivially a classical trajectory generated by V, because $V_1 \subset V$. For the converse inclusion we let $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}$ be an arbitrary classical trajectory generated by V. Noting that $V_0[\perp]V_1$ is equivalent to $V \subset \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{X}_1 \\ \mathcal{W} \end{smallmatrix}\right]$, we obtain from Definition 1.1 that $x(t) \in \mathcal{X}_1$ for all $t > 0$. Therefore

$$\dot{x}(t) = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} \in \mathcal{X}_1, \quad t > 0,$$

and thus $\left[\begin{smallmatrix} \dot{x}(t) \\ x(t) \\ w(t) \end{smallmatrix}\right] \in V_1 = V \cap \left[\begin{smallmatrix} \mathcal{X}_1 \\ \mathcal{X}_1 \\ \mathcal{W} \end{smallmatrix}\right]$ for all $t > 0$.

Every classical trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$ generated by $V_0 \subset \left[\begin{smallmatrix} \mathcal{X} \\ \{0\} \\ \{0\} \end{smallmatrix}\right]$ trivially satisfies $x(t) = 0$ and $w(t) = 0$ for all $t \geq 0$. According to Definition 1.3, $\mathfrak{V}_1 = \mathfrak{V}$ and $\mathfrak{V}_0 = \{0\}$ imply that $\mathfrak{W}_1 = \mathfrak{W}$ and $\mathfrak{W}_0 = \{0\}$.

□

The following corollary follows directly from Proposition 4.7(ii).

Corollary 4.8. *Assume that V is maximally nonnegative. Then $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Longrightarrow z = 0$ if and only if $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V^{[\perp]} \Longrightarrow z = 0$.*

We now demonstrate how Proposition 4.7 can be applied in practice by connecting two capacitors in parallel.

Example 4.9. An ideal capacitor with capacitance C_i can be modelled by the equation

$$\begin{bmatrix} \dot{x}_i(t) \\ y_i(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{C_i} \\ 1/\sqrt{C_i} & 0 \end{bmatrix} \begin{bmatrix} x_i(t) \\ u_i(t) \end{bmatrix}, \tag{4.12}$$

where x_i is the charge in the capacitor divided by $\sqrt{C_i}$, u_i is the current entering the capacitor and y_i is the voltage over the capacitor. This system has generating subspace

$$V_i = \begin{bmatrix} 0 & 1/\sqrt{C_i} \\ 1 & 0 \\ 1/\sqrt{C_i} & 0 \\ 0 & 1 \end{bmatrix} \mathbb{C}^2, \tag{4.13}$$

where $\mathcal{U} = \left[\begin{smallmatrix} \{0\} \\ \mathbb{C} \end{smallmatrix}\right]$ and $\mathcal{Y} = \left[\begin{smallmatrix} \mathbb{C} \\ \{0\} \end{smallmatrix}\right]$, cf. (1.2).

The appropriate external signal space in this case is $\mathcal{W} = \mathbb{C}^2$, and we equip $\mathcal{W}$ with the power product $\left[\left[\begin{smallmatrix} y^1 \\ u^1 \end{smallmatrix}\right], \left[\begin{smallmatrix} y^2 \\ u^2 \end{smallmatrix}\right]\right]_{\mathcal{W}} = y^1\overline{u^2} + u^1\overline{y^2}$, because electrical power equals voltage times current. Therefore the node space is $\mathfrak{K}_i = \mathbb{C}^4$ with the power product

$$\left[\left[\begin{bmatrix} z^1 \\ x^1 \\ y^1 \\ u^1 \end{bmatrix}, \begin{bmatrix} z^2 \\ x^2 \\ y^2 \\ u^2 \end{bmatrix}\right]\right]_{\mathfrak{K}} = y^1\overline{u^2} + u^1\overline{y^2} - z^1\overline{x^2} - x^1\overline{z^2}.$$

Corollary A.9 yields that V_i is Lagrangian, because V_i is easily seen to be neutral and $\dim V_i = 2$, which is precisely half of the dimension of $\mathbb{C}^4$. This reflects the well-known fact that an ideal capacitor conserves energy.

In Figure 1 we have drawn two capacitors, which are initially not interconnected. We consider these two capacitors as a single system, the so-called *product* of the two individual capacitors. This product system has generating subspace

$$V = \begin{bmatrix} 0 & 0 & 1/\sqrt{C_1} & 0 \\ 0 & 0 & 0 & 1/\sqrt{C_2} \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1/\sqrt{C_1} & 0 & 0 & 0 \\ 0 & 1/\sqrt{C_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbb{C}^4,$$

which is a Lagrangian subspace of $\mathbb{C}^8$ with the appropriate power product obtained as the sum of the power products on $\mathfrak{K}_1$ and $\mathfrak{K}_2$. We use the horizontal lines to separate the two copies of the state space $\mathbb{C}^2$ from the external signal space $\mathbb{C}^4$.

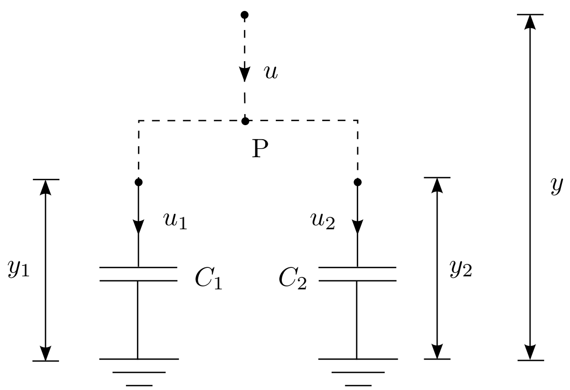

FIGURE 1. Two initially disconnected capacitors C_1 and C_2, which we interconnect by adding the dashed wire.

We now connect the two capacitors in parallel. Making the dashed connections in Figure 1 and applying Kirchhoff's laws at the junction P, we get the additional constraint that $y_1 = y_2$, i.e., $x_1/\sqrt{C_1} = x_2/\sqrt{C_2}$. Moreover, the total current flowing into the parallel coupling is $u_1 + u_2$. Let us therefore define $y := (y_1 + y_2)/2$ and $u := u_1 + u_2$, so that y is the voltage over the parallel coupling and u is the current flowing into it. Then the variables $\left[\begin{smallmatrix} z \\ x \\ y \\ u \end{smallmatrix}\right]$ of the interconnected system live in the subspace

$$V' = \left[\begin{array}{cccc} 0 & 0 & 1/\sqrt{C_1} & 0 \\ 0 & 0 & 0 & 1/\sqrt{C_2} \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1/2\sqrt{C_1} & 1/2\sqrt{C_2} & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \times \mathcal{N}\left(\left[\begin{array}{cccc} 1/\sqrt{C_1} & -1/\sqrt{C_2} & 0 & 0 \end{array}\right]\right). \tag{4.14}$$

The space V' is a Lagrangian subspace of $\mathfrak{K}' := \mathbb{C}^6$ equipped with the power product

$$\left[\!\left[\begin{bmatrix} z^1 \\ x^1 \\ y^1 \\ u^1 \end{bmatrix}, \begin{bmatrix} z^2 \\ x^2 \\ y^2 \\ u^2 \end{bmatrix} \right]\!\right]_{\mathfrak{K}'} = y^1\overline{u^2} + u^1\overline{y^2} - (z^1, x^2)_{\mathbb{C}^2} - (x^1, z^2)_{\mathbb{C}^2}.$$

However, V' is not the generating subspace of a s/s system, because it does not satisfy condition (ii) of Definition 1.2. Indeed, $\left[\begin{smallmatrix} z \\ 0 \\ 0 \\ 0 \end{smallmatrix}\right] \in V'$ if and only if $z \in \left[\begin{smallmatrix} -\sqrt{C_2} \\ \sqrt{C_1} \end{smallmatrix}\right] \mathbb{C} =: \mathcal{X}_0$, and this space is nontrivial. One easily verifies that

$$\mathcal{X}_1 := \mathcal{X} \ominus \mathcal{X}_0 = \begin{bmatrix} \sqrt{C_1} \\ \sqrt{C_2} \end{bmatrix} \mathbb{C} = \mathcal{N}\left(\left[\begin{array}{cc} 1/\sqrt{C_1} & -1/\sqrt{C_2} \end{array}\right]\right)$$

and setting $\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in \mathcal{X}_1$ in (4.14) yields that

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} u_1/\sqrt{C_1} \\ u_2/\sqrt{C_2} \end{bmatrix} = \begin{bmatrix} \sqrt{C_1} \\ \sqrt{C_2} \end{bmatrix} a, \quad a = (u_1 + u_2)/(C_1 + C_2).$$

Defining $V_1' := V' \cap \left[\begin{smallmatrix} \mathcal{X}_1 \\ \mathcal{X}_1 \\ \mathbb{C}^2 \end{smallmatrix}\right]$, we thus obtain that

$$V_1' = \left[\begin{array}{ccc} 0 & 0 & \sqrt{C_1}/(C_1 + C_2) \\ 0 & 0 & \sqrt{C_2}/(C_1 + C_2) \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1/2\sqrt{C_1} & 1/2\sqrt{C_2} & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \begin{bmatrix} \sqrt{C_1} \\ \sqrt{C_2} \end{bmatrix} \mathbb{C} \\ \mathbb{C} \end{array}\right].$$

By Proposition 4.7(iii), the triple $(V_1'; \mathcal{X}_1, \mathbb{C}^2)$ is a conservative s/s node. Obviously this s/s node has operator node representation

$$V_{1,op}' = \left(S_1'; \begin{bmatrix} \sqrt{C_1} \\ \sqrt{C_2} \end{bmatrix} \mathbb{C}, \begin{bmatrix} \{0\} \\ \mathbb{C} \end{bmatrix}, \begin{bmatrix} \mathbb{C} \\ \{0\} \end{bmatrix}\right), \tag{4.15}$$

where the system node S_1' is given by the restriction to $\left[\begin{smallmatrix} \mathcal{X}_1 \\ \mathcal{U} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \left[\begin{smallmatrix} \sqrt{C_1} \\ \sqrt{C_2} \end{smallmatrix}\right]\mathbb{C} \\ \left[\begin{smallmatrix} \{0\} \\ \mathbb{C} \end{smallmatrix}\right] \end{smallmatrix}\right]$

$$\text{of} \quad \begin{bmatrix} A_1' & B_1' \\ C_1' & D_1' \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & \sqrt{C_1}/(C_1+C_2) \\ 0 & 0 & 0 & \sqrt{C_2}/(C_1+C_2) \\ \hline 1/2\sqrt{C_1} & 1/2\sqrt{C_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This example seemingly turns a simple task into very complicated one, but it is interesting that the same approach can be applied to quite general infinite-dimensional systems, using the tools developed in Sections 4 and 5.

Some first steps in the direction of interconnection of conservative systems, which are relevant for s/s systems, were taken in [15], but we will study this topic in more detail and generality elsewhere. The operator node representation (4.15) is a special case of a so-called *impedance representation* of a s/s node, due to the fact that $\mathcal{U} = \left[\begin{smallmatrix} \{0\} \\ \mathbb{C} \end{smallmatrix}\right] = \mathcal{U}^{[\perp]}$ and $\mathcal{Y} = \left[\begin{smallmatrix} \mathbb{C} \\ \{0\} \end{smallmatrix}\right] = \mathcal{Y}^{[\perp]}$, so that $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ is a *Lagrangian decomposition.* We will also study more general impedance representations in a forthcoming article.

The next step is to characterise conservative s/s nodes, but in order to do this we first need to write down the following lemma.

Lemma 4.10. *If* $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is an operator node and* $\alpha \in \operatorname{Res}(A)$*, then the operator* $\begin{bmatrix} \begin{bmatrix} \alpha & 0 \end{bmatrix} - A\&B \\ \begin{bmatrix} 0 & 1_{\mathcal{U}} \end{bmatrix} \end{bmatrix}$ *maps* $\operatorname{Dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ *one-to-one onto* $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$.

Proof. Under the given assumptions, $\left[\begin{smallmatrix} [\alpha\ 0] - A\&B \\ [0\ 1_{\mathcal{U}}] \end{smallmatrix}\right]$ is injective, because

$$\begin{bmatrix} \begin{bmatrix} \alpha & 0 \end{bmatrix} - A\&B \\ \begin{bmatrix} 0 & 1_{\mathcal{U}} \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \quad \Longrightarrow \quad x \in \operatorname{Dom}(A) \quad \text{and} \quad (\alpha - A)x = 0,$$

and then $x = 0$ and $u = 0$.

Moreover the operator is surjective, as we will now show. Let therefore $z \in \mathcal{X}$ and $u \in \mathcal{U}$ be arbitrary and let x be such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{Dom}(S)$. Since $\alpha \in \operatorname{Res}(A)$, we can find an $x' \in \operatorname{Dom}(A)$ such that

$$(\alpha - A)x' = z - \left(\alpha x - A\&B \begin{bmatrix} x \\ u \end{bmatrix}\right) \in \mathcal{X}.$$

Then $\left[\begin{smallmatrix} x+x' \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] + \left[\begin{smallmatrix} x' \\ 0 \end{smallmatrix}\right] \in \operatorname{Dom}(S)$ and

$$\left(\begin{bmatrix} \alpha & 0 \end{bmatrix} - A\&B\right) \begin{bmatrix} x + x' \\ u \end{bmatrix} = z.$$

The proof is done. □

Recall from Corollary 3.8 that if $(\mathcal{U},\mathcal{Y})$ is an admissible i/o pair for the s/s node $(V;\mathcal{X},\mathcal{W})$ then $(\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ is an admissible i/o pair for the s/s dual $(V^{[\perp]};\mathcal{X},\mathcal{W})$. In this case we denote the main operator of the operator node representation $V_{op}^{[\perp]} = (S^d,\mathcal{X},\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ by A^d, cf. Definitions 2.4 and 2.5. If $(\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ is admissible also for the primal s/s node $(V;\mathcal{X},\mathcal{W})$, then we denote the corresponding operator node by $S^\times$ and its main operator by $A^\times$.

Theorem 4.11. *Assume that $V\subset\mathfrak{K}$ has the property that $\left[\begin{smallmatrix} z\\0\\0\end{smallmatrix}\right]\in V \Longrightarrow z=0$. Then the following claims are equivalent:*

(i) *The triple $(V;\mathcal{X},\mathcal{W})$ is a conservative s/s node.*
(ii) *The space V is a Lagrangian subspace of $\mathfrak{K}$: $V=V^{[\perp]}$.*
(iii) *Both $(V;\mathcal{X},\mathcal{W})$ and $(V^{[\perp]};\mathcal{X},\mathcal{W})$ are ordinary s/s nodes and, for* some *subinterval $[a,b]\subset\mathbb{R}$ of finite positive length, the spaces of classical trajectories generated by V and $V^{[\perp]}$ on $[a,b]$ coincide: $\mathfrak{V}[a,b]=\mathfrak{V}^d[a,b]$.*
(iv) *Both $(V;\mathcal{X},\mathcal{W})$ and $(V^{[\perp]};\mathcal{X},\mathcal{W})$ are ordinary s/s nodes and, for* every *positive-length subinterval $I\subset\mathbb{R}$, the spaces of classical trajectories generated by V and $V^{[\perp]}$ on I coincide: $\mathfrak{V}(I)=\mathfrak{V}^d(I)$.*
(v) *The triples $(V;\mathcal{X},\mathcal{W})$ and $(V^{[\perp]};\mathcal{X},\mathcal{W})$ are both passive ordinary s/s nodes, so that V and $V^{[\perp]}$ are maximally nonnegative.*
(vi) *Both $(V;\mathcal{X},\mathcal{W})$ and $(V^{[\perp]};\mathcal{X},\mathcal{W})$ are passive time-reflected s/s nodes: V and $V^{[\perp]}$ are maximally nonpositive.*
(vii) *The following two conditions hold for some decomposition $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$:*
 (a) *both $(\mathcal{U},\mathcal{Y})$ and $(\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ are admissible i/o pairs for $(V;\mathcal{X},\mathcal{W})$, and*
 (b) *the operator node representations $V_{op}=(S^\times;\mathcal{X},\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ and $V_{op}^{[\perp]}=(S^d;\mathcal{X},\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$ coincide: $S^\times=S^d$.*
(viii) *The following three conditions all hold:*
 (c) *the space V is neutral: $V\subset V^{[\perp]}$,*
 (d) *there exists a decomposition $\mathcal{W}=\mathcal{U}\dotplus\mathcal{Y}$, such that* (a) *holds and*
 (e) *the main operators of the operator node representations of V and $V^{[\perp]}$ corresponding to the i/o pair $(\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]})$, as given in* (b), *have non-disjoint resolvent sets:*

$$\operatorname{Res}\left(A^\times\right)\cap\operatorname{Res}\left(A^d\right)\neq\emptyset.$$

Assume that (d) *holds and let A be the main operator of $V_{op}=(S;\mathcal{X},\mathcal{U},\mathcal{Y})$. If either both $A^\times$ and $-A$, or both $-A^\times$ and A, generate C_0 semigroups on $\mathcal{X}$, then* (e) *also holds.*

Condition (a) of Theorem 4.11 needs some clarification. Let $(V;\mathcal{X},\mathcal{W})$ be a s/s node. By Definition 2.5, admissibility of the i/o pair $(\mathcal{U},\mathcal{Y})$ means that V satisfies (2.4) for some *operator node* $\left[\begin{smallmatrix} A\&B\\ C\&D\end{smallmatrix}\right]$ with input space $\mathcal{U}$ and output space $\mathcal{Y}$. Thus there is no implication that A generates a C_0 semigroup on $\mathcal{X}$. As a consequence, there is no guarantee that A or $A^\times$ generate semigroups even if (a) holds.

Proof Theorem 4.11. We begin by proving the last claim. Assume therefore that A generates a C_0 semigroup with growth bound ω; see [26, Def. 2.5.6]. By Theorem 3.6, $-A^d = A^*$ and, according to [26, Thm 3.5.6], A^* generates the C_0 semigroup $t \to (\mathfrak{A}^t)^*$, which also has growth bound ω. If $-A$ generates a C_0 semigroup $\mathfrak{A}'$, then $(-A)^* = A^d$ obviously generates the dual semigroup $(\mathfrak{A}')^*$.

Assume that $A^\times$ and $-A$ generate semigroups with growth bounds $\omega^\times$ and ω, respectively. By the argument we just made, A^d generates a semigroup with growth bound ω as well. Theorem 3.2.9(i) of [26] then yields that

$$\left\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \max\left\{\omega^\times, \omega\right\}\right\} \subset \operatorname{Res}\left(A^\times\right) \cap \operatorname{Res}\left(A^d\right),$$

which obviously implies (e).

Now drop the earlier assumptions on A and $A^\times$, and instead assume that both $-A^\times$ and A generate C_0 semigroups with growth bounds $\omega^\times$ and ω, respectively. Then (e) again holds, because

$$\left\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < -\max\left\{\omega^\times, \omega\right\}\right\} \subset \operatorname{Res}\left(A^\times\right) \cap \operatorname{Res}\left(A^d\right).$$

The following implications prove the equivalence of the claims (i)–(viii) listed in the theorem:

(i) $\iff$ (ii): If $V = V^{[\perp]}$, then V is maximally nonnegative by Proposition A.8. In this case $(V;\mathcal{X},\mathcal{W})$ is a (passive) s/s node by Theorem 4.5. The rest was shown in Corollary 4.2.

(ii) $\Longrightarrow$ (iv) and *(iv) $\Longrightarrow$ (iii):* These implications are both trivial once we know that $(V;\mathcal{X},\mathcal{W})$ is a s/s node.

(iii) $\Longrightarrow$ (ii): Let $\left[\begin{smallmatrix} z_a \\ x_a \\ w_a \end{smallmatrix}\right] \in V$. By Definition 1.2, V is closed and there exists a trajectory $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}[a,b]$ such that $\begin{bmatrix} \dot x(a) \\ x(a) \\ w(a) \end{bmatrix} = \left[\begin{smallmatrix} z_a \\ x_a \\ w_a \end{smallmatrix}\right]$. This by assumption implies that $\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathfrak{V}^d[a,b]$, i.e., that $\left[\begin{smallmatrix} z_a \\ x_a \\ w_a \end{smallmatrix}\right] \in V^{[\perp]}$. Thus $V \subset V^{[\perp]}$. For the converse inclusion, we apply the same argument to $V^{[\perp]}$ and obtain that $V^{[\perp]} \subset (V^{[\perp]})^{[\perp]} = V$.

(ii) $\iff$ (v): Claim (ii) implies claim (i), which says that $(V;\mathcal{X},\mathcal{W})$ is a conservative s/s node. In this case $(V^{[\perp]};\mathcal{X},\mathcal{W}) = (V;\mathcal{X},\mathcal{W})$ is a passive s/s node by Corollary 4.2.

Conversely, Corollary 4.2 yields that $(V;\mathcal{X},\mathcal{W})$ and $(V^{[\perp]};\mathcal{X},\mathcal{W})$ are passive s/s nodes if and only if V and $V^{[\perp]}$ are both maximally nonnegative. According to Proposition A.8 this is equivalent to V being Lagrangian.

(v) $\iff$ (vi): If V is maximally semidefinite, then V is closed by Remark A.6. According to Proposition A.8, V is then maximally nonnegative if and only if $V^{[\perp]}$ is maximally nonpositive.

(vii) $\Longrightarrow$ (ii): The assumptions (a) and (b) immediately imply that

$$V_{op}^{[\perp]} = (S^d;\mathcal{X},\mathcal{U}^{[\perp]},\mathcal{Y}^{[\perp]}) = V_{op},$$

i.e., that $V = V^{[\perp]}$, cf. Definition 2.5.

(ii) $\Longrightarrow$ (viii): Assume that $V = V^{[\perp]}$. Then trivially (c) holds and the triple $(V;\mathcal{X},\mathcal{W})$ is a passive s/s node by item (v). Thus every fundamental

decomposition $\mathcal{W} = (\mathcal{W}_+, \mathcal{W}_-)$ is admissible for $(V; \mathcal{X}, \mathcal{W})$ and $\mathbb{C}^+ \subset \mathrm{Res}\,(A)$ for the main operator of the corresponding operator node representation, according to Theorem 4.5.

Theorem 3.6 yields that $(\mathcal{W}_+^{[\perp]}, \mathcal{W}_-^{[\perp]})$ is admissible for $V^{[\perp]}$ and that the corresponding operator node representation S^d has main operator $A^d = -A^*$. One easily shows that $(\mathcal{W}_+^{[\perp]}, \mathcal{W}_-^{[\perp]}) = (\mathcal{W}_-, \mathcal{W}_+)$ and, by Corollary 3.8, this i/o pair is also admissible for V, because $V = V^{[\perp]}$. Thus (d) holds and $S^d = S^\times$ is immediate from $V = V^{[\perp]}$ and the notation in (b), see Definition 2.5 again. In particular, $A^d = A^\times$ and $\mathbb{C}^- \subset \mathrm{Res}\,(-A^*) = \mathrm{Res}\,\big(A^d\big) = \mathrm{Res}\,(A^\times)$, because $\alpha \in \mathrm{Res}\,(A)$ if and only if $\overline{\alpha} \in \mathrm{Res}\,(A^*)$, which is equivalent to $-\overline{\alpha} \in \mathrm{Res}\,(-A^*)$.

(viii) $\Longrightarrow$ *(vii):* Assume that $V \subset V^{[\perp]}$. Then $S^\times \subset S^d$, again by the notation in (b), and we are done if we can prove that $\mathrm{Dom}\,(S^\times) = \mathrm{Dom}\,\big(S^d\big)$.

Let now $\alpha \in \mathrm{Res}\,(A^\times) \cap \mathrm{Res}\,\big(A^d\big)$. By Lemma 4.10, $\left[\begin{smallmatrix} [\alpha\ 0] - A\&B^\times \\ [0\ 1] \end{smallmatrix}\right]$ maps $\mathrm{Dom}\,(S^\times)$ one-to one onto $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U}^{[\perp]} \end{smallmatrix}\right]$. The inclusion $S^\times \subset S^d$ implies that also $\left[\begin{smallmatrix} [\alpha\ 0] - A\&B^d \\ [0\ 1] \end{smallmatrix}\right]$ maps $\mathrm{Dom}\,(S^\times)$ one-to one onto $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U}^{[\perp]} \end{smallmatrix}\right]$. Since the latter operator maps $\mathrm{Dom}\,\big(S^d\big)$ one-to one onto $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U}^{[\perp]} \end{smallmatrix}\right]$, again by Lemma 4.10, we conclude that $\mathrm{Dom}\,(S^\times) = \mathrm{Dom}\,\big(S^d\big)$. □

We made the following observation in the proof of Theorem 4.11. Let $(V; \mathcal{X}, \mathcal{W})$ be a conservative s/s node. Then every *fundamental i/o pair* $(\mathcal{U}, \mathcal{Y}) = (\mathcal{W}_+, \mathcal{W}_-)$ satisfies condition (a) of Theorem 4.11 and

$$\mathbb{C}^- \subset \mathrm{Res}\,\big(A^\times\big) = \mathrm{Res}\,\big(A^d\big)\,,$$

where $A^\times$ and A^d are as given in (b) with $\mathcal{U}^{[\perp]} = \mathcal{W}_-$ and $\mathcal{Y}^{[\perp]} = \mathcal{W}_+$.

Theorem 4.11(vi) has the following consequence, which should be compared to [19, Rem. 4.3]. Let $\mathcal{W} = (\mathcal{U}, \mathcal{Y})$ be an *orthogonal i/o pair*, i.e., let $\mathcal{U}[\perp]\mathcal{Y}$. Then $\mathcal{U}^{[\perp]} = \mathcal{Y}$ and if condition (a) holds for such an i/o pair, then V has the operator node representations $V_{op} = (S; \mathcal{X}, \mathcal{U}, \mathcal{Y})$ and $V_{op} = (S^\times; \mathcal{X}, \mathcal{Y}, \mathcal{U})$. In this case $S^\times$ has the interpretation of being the *flow inverse* of S, because (u, x, y) is a classical i/s/o trajectory generated by S if and only if (y, x, u) is a classical i/s/o trajectory generated by $S^\times$, cf. Definition 2.4. The condition $S^\times = S^d$ thus means that the flow inverse equals the anti-causal dual of S introduced in Remark 3.7.

Flow inversion is described in more detail in [28] and Chapter 6 of [26]. I/s/o representations corresponding to general orthogonal i/o pairs are called *transmission representations*, and we will treat these in a forthcoming article. We conclude this section with an example of a conservative s/s node of boundary control type.

Example 4.12. We continue Example 2.9, using the notation we introduced there. Defining $V := \begin{bmatrix} -\frac{\partial}{\partial z} \\ 1 \\ \delta_0 \end{bmatrix} \mathcal{Z}$, we obviously get a subspace of $\mathfrak{K}$ with the property $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \implies z = 0$, and we may thus use Theorem 4.11 to show that $(V; L^2(\mathbb{R}^+;\mathbb{C}),\mathbb{C})$ is a conservative s/s node.

Combining the following short computation, where $x' = \frac{\partial}{\partial z}x$, with Lemma A.7(i), we obtain that V satisfies Theorem 4.11(c):

$$\begin{aligned}\left[\begin{bmatrix} -x' \\ x \\ x(0) \end{bmatrix}, \begin{bmatrix} -x' \\ x \\ x(0) \end{bmatrix}\right]_{\mathfrak{K}} &= |x(0)|^2 + \int_0^\infty \left(x'(\zeta)\,\overline{x(\zeta)} + x(\zeta)\,\overline{x'(\zeta)}\right) \mathrm{d}\zeta \\ &= |x(0)|^2 + \int_0^\infty \left(\frac{\partial}{\partial z}|x|^2\right)(\zeta)\,\mathrm{d}\zeta \\ &= |x(0)|^2 + \lim_{z\to+\infty} |x(z)|^2 - |x(0)|^2 = 0.\end{aligned}$$

From $\mathcal{W} = \mathbb{C}$ we obtain $\mathbb{C}^{[\perp]} = \{0\}$. By Example 2.9, V has operator node representations

$$\begin{aligned} V_{op} &= (S; L^2(\mathbb{R}^+;\mathbb{C}), \mathbb{C}, \{0\}) \quad \text{and} \\ V_{op} &= (S^\times; L^2(\mathbb{R}^+;\mathbb{C}), \{0\}, \mathbb{C}). \end{aligned}$$

The i/o pair $(\mathbb{C}, \{0\})$ thus proves that V satisfies Theorem 4.11(d) as well.

Note that $A^\times = -\frac{\partial}{\partial z}\Big|_{\mathcal{Z}}$ does not generate a C_0 semigroup on $\mathcal{X} = L^2(\mathbb{R}^+;\mathbb{C})$, but $-A^\times$ does. By the last claim of Theorem 4.11, condition (e) of that theorem also holds, and we conclude that $(V; L^2(\mathbb{R}^+;\mathbb{C}),\mathbb{C})$ is a conservative s/s node.

We now proceed to study operator node representations corresponding to fundamental i/o pairs in more detail.

5. Scattering representations and passivity

In this section we introduce the so-called scattering representations and use these to establish some additional properties of passive s/s systems. We also give a few characterisations of passive s/s systems in terms of scattering representations.

Definition 5.1. Let $V \subset \mathfrak{K}$ and assume that the fundamental decomposition $\mathcal{W} = (\mathcal{W}_+, \mathcal{W}_-)$ is an admissible i/o pair for V.

Then we call the the corresponding operator node representation a *scattering representation* of V and write $V_{sca} = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right)$.

If $(V; \mathcal{X}, \mathcal{W})$ is a s/s node, then we call V_{sca} a scattering representation of both $(V; \mathcal{X}, \mathcal{W})$ and the *s/s system* that the s/s node generates.

According to Definition A.1, every fundamental decomposition of a Kreĭn space is orthogonal. Thus a scattering representation is a special case of a transmission representation, as it was introduced in the previous section. The next theorem gives one characterisation of which dissipative s/s nodes are actually passive. Dissipativity is of course a necessary condition for passivity.

Theorem 5.2. *Let V be a nonnegative subspace of $\mathfrak{K}$ and $(\mathcal{W}_+, \mathcal{W}_-)$ a fundamental decomposition of $\mathcal{W}$. Then the following claims are equivalent:*

(i) *The triple $(V; \mathcal{X}, \mathcal{W})$ is a passive s/s node.*
(ii) *The subspace V has scattering representation $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right)$. The corresponding main operator A defined in* (2.3) *generates a contraction semigroup on $\mathcal{X}$ and $\mathbb{C}^+ \subset \operatorname{Res}(A)$.*
(iii) *The i/o pair $(\mathcal{W}_+, \mathcal{W}_-)$ is admissible for V and the resolvent set of the associated main operator given in* (2.3) *satisfies* $\operatorname{Res}(A) \cap \mathbb{C}^+ \neq \emptyset$.
(iv) *The space V is closed, $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \Longrightarrow z = 0$ and for some $\alpha \in \mathbb{C}^+$ we have*

$$\begin{bmatrix} -1 & \alpha & 0 \\ 0 & 0 & \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} \end{bmatrix} V = \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix}. \tag{5.1}$$

(v) *The space V is closed, $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \Longrightarrow z = 0$ and* (5.1) *holds for all $\alpha \in \mathbb{C}^+$.*

Proof. We again divide the proof into a series of implications.
(i) $\Longrightarrow$ (v): By Definition 1.2, every s/s node $(V; \mathcal{X}, \mathcal{W})$ has the properties that V is closed and $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \Longrightarrow z = 0$. By assumption (i) and Corollary 4.2, V is maximally nonnegative and, according to Propositions A.2 and A.8, we then have (5.1) for an arbitrary $\alpha \in \mathbb{C}^+$.
(v) $\Longrightarrow$ (iv) and *(ii) $\Longrightarrow$ (iii):* These implications are trivial.
(iv) $\Longrightarrow$ (i),(ii): By assumption the implication $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Longrightarrow z = 0$ holds. We now show that V is maximally nonnegative, whereafter Theorem 4.5 yields that claims (i) and (ii) are true.

Let $\alpha \in \mathbb{C}^+$ satisfy (5.1) and let $\mathfrak{K} = (\mathfrak{K}_+, \mathfrak{K}_-)$ be the fundamental decomposition and $\mathcal{P}_{\mathfrak{K}_+}^{\mathfrak{K}_-}$ the projection given in Proposition A.2, so that $\mathcal{P}_{\mathfrak{K}_+}^{\mathfrak{K}_-} V = \mathfrak{K}_+$. We assumed that $V \geq 0$ and thus V is maximally nonnegative according to Proposition A.8.
(iii) $\Longrightarrow$ (iv): By assumption there exists an operator node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, such that (4.9) holds, and in particular the implication $\left[\begin{smallmatrix} z \\ 0 \\ 0 \end{smallmatrix}\right] \in V \Longrightarrow z = A\&B \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = 0$ is valid. By Lemma 4.10, (5.1) holds for every $\alpha \in \operatorname{Res}(A) \cap \mathbb{C}^+$. ☐

Conditions (ii)–(v) of Theorem 5.2 hold for some fundamental decomposition if and only if they hold for all fundamental decompositions, because condition (i) is independent of the fundamental decomposition.

As the following example taken from [5, Ex. 5.5] shows, there exist energy-preserving s/s nodes, for which no fundamental i/o pair is admissible. If there on the contrary existed such an i/o pair, then the system would be passive by Theorem 5.2, because the condition $\operatorname{Res}(A) \cap \mathbb{C}^+ \neq \emptyset$ becomes trivial when $\mathcal{X} = \{0\}$.

Example 5.3. Let $\mathcal{X} = \{0\}$ and $\mathcal{W} = \mathbb{C}^3$ with the power product

$$\left[\begin{bmatrix} y_1^1 \\ y_2^1 \\ u^1 \end{bmatrix}, \begin{bmatrix} y_1^2 \\ y_2^2 \\ y^2 \end{bmatrix}\right]_{\mathcal{W}} = -y_1^1\overline{y_1^2} + y_2^1\overline{y_2^2} + u^1\overline{u^2}.$$

Then $V := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\mathbb{C}$ is neutral and $(V; \{0\}, \mathcal{W})$ is an energy-preserving s/s node.

Moreover, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in V^{[\perp]}$ and $\left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]_{\mathcal{W}} = 1 \geq 0$, so that $V^{[\perp]} \not\leq 0$. This implies that V is not maximally nonnegative, i.e. that the s/s node is not passive.

In Theorem 5.2 we characterised passivity under the assumption of dissipativity. We now proceed to define a scattering-passive i/s/o system node in order to be able to prove that every scattering representation of a passive s/s node is of this type. The following definition uses classical i/s/o trajectories of an operator node, as these were introduced in Definition 2.4.

Definition 5.4. Let $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an i/s/o system node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$.

The i/s/o system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is L^2-*well-posed* if there exists a $T > 0$ and a constant K_T such that all classical i/s/o trajectories (u, x, y) of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ satisfy

$$\forall t \in [0, T]: \quad \|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(s)\|_{\mathcal{Y}}^2 \, \mathrm{d}s \leq K_T \left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2 \, \mathrm{d}s \right). \tag{5.2}$$

The i/s/o system node is *scattering passive* if it is L^2-well-posed with $K_T = 1$, i.e., if all classical trajectories satisfy

$$\forall t \in [0, T]: \quad \|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(s)\|_{\mathcal{Y}}^2 \, \mathrm{d}s \leq \|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(s)\|_{\mathcal{U}}^2 \, \mathrm{d}s.$$

The i/s/o system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is *scattering energy preserving* if (5.2) holds with equality instead of inequality and $K_T = 1$. The i/s/o node is *scattering conservative* if both $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*$ are scattering energy preserving.

Comparing Remark 4.6 to Definitions 2.4 and 5.4, we see that a passive s/s node is indeed much easier to describe than a passive i/s/o system node. This is only one example of how it can be more natural to study systems theory in the s/s framework than in the i/s/o counterpart.

Remark 5.5. Theorem 4.7.13 in [26] says that a system node $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ is L^2-well-posed if and only if there exists a $T > 0, K_T > 0$ such that all i/s/o trajectories (u, x, y) of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ on $[0, T]$ satisfy the following for all $t \in [0, T]$:

$$\|x(t)\|_{\mathcal{X}} + \sqrt{\int_0^t \|y(s)\|_{\mathcal{Y}}^2 \, \mathrm{d}s} \leq K_T \left(\|x(0)\|_{\mathcal{X}} + \sqrt{\int_0^t \|u(s)\|_{\mathcal{U}}^2 \, \mathrm{d}s} \right). \tag{5.3}$$

Comparing this to (5.2), we see that the terms in (5.2) are the terms in (5.3) squared. This difference is non-essential, however, because (5.2) corresponds to using the norm $\|[\begin{smallmatrix} a \\ b \end{smallmatrix}]\| = \sqrt{|a|^2 + |b|^2}$ on $\mathbb{R}^2$ and (5.3) corresponds to using the norm $\|[\begin{smallmatrix} a \\ b \end{smallmatrix}]\| = |a| + |b|$, and all norms in $\mathbb{R}^2$ are equivalent. If we change the powers, to which the terms in (5.2) are raised, then we may be forced to increase K_T, but the claim that there exists some constant K_T is either true in both cases or false in both cases.

We use (5.2) as the definition for L^2-well-posedness, because it fits passive systems better than (5.3).

We now study passivity of s/s nodes which have a scattering representation.

Proposition 5.6. *Let the subspace $V \subset \mathfrak{K}$ have the scattering representation $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{W}_+, \mathcal{W}_-\right)$. Then the following conditions are equivalent:*

(i) *The triple $(V; \mathcal{X}, \mathcal{W})$ is a passive s/s node.*
(ii) *The subspace $V \subset \mathfrak{K}$ is nonnegative and* $\operatorname{Res}(A) \cap \mathbb{C}^+ \neq \{0\}$.
(iii) *We have $V \geq 0$ and $\mathbb{C}^+ \subset \operatorname{Res}(A)$.*
(iv) *The operator $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a scattering-passive i/s/o system node.*

The triple $(V; \mathcal{X}, \mathcal{W})$ is a conservative s/s node if and only if $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a scattering-conservative i/s/o system node. In this case conditions (i)–(iv) *above hold.*

Proof. We first note the following almost direct consequence of (4.9). It holds that $\left[\begin{smallmatrix} x \\ u+y \end{smallmatrix}\right] \in \mathfrak{V}$ with $u(t) \in \mathcal{W}_+$ and $y(t) \in \mathcal{W}_-$ for all $t \geq 0$ if and only if (u, x, y) is a classical i/s/o trajectory of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$. If $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a i/s/o system node, then it is scattering passive if and only if $V \geq 0$, because (5.2) with $K_T = 1$ is equivalent to (4.1) with $I = [0, T]$ when $\mathcal{U} = \mathcal{W}_+$ and $\mathcal{Y} = \mathcal{W}_-$.
(iv) $\Longrightarrow$ (iii): By Definition 2.3, $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ has a C_0 semigroup $\mathfrak{A}$. Theorem 3.2.9(i) of [26] yields that any $\alpha \in \mathbb{R}^+$ greater than the growth bound of $\mathfrak{A}$ lies in $\operatorname{Res}(A) \cap \mathbb{C}^+$, which is thus nonempty, cf. the beginning of the proof of Theorem 4.11. By the discussion at the beginning of this proof, $V \geq 0$ and Theorem 5.2(ii) then yields that $\mathbb{C}^+ \subset \operatorname{Res}(A)$.
(iii) $\Longrightarrow$ (ii): This implication is trivial.
(ii) $\Longrightarrow$ (i): This also follows from Theorem 5.2(iii).
(i) $\Longrightarrow$ (iv): Theorem 5.2 yields that the main operator A of $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ generates a contraction semigroup, i.e., that $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an i/s/o system node. According to Corollary 4.2, V is a (maximally) nonnegative subspace, so that (4.1) holds for $I = [0, T]$, $T > 0$, by Proposition 4.3. Therefore (5.2) also holds with $K_T = 1$ and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is scattering passive.

The last claim follows from [9, Prop. 4.9] and Theorem 4.11. □

We now prove that all passive s/s nodes are L^2-well-posed. In order to do this, we first need to recall the definition of an L^2-well-posed s/s node from [14].

Definition 5.7. The s/s node $(V;\mathcal{X},\mathcal{W})$ is L^2*-well-posed* if there exists a $T>0$ and a direct sum decomposition $\mathcal{W}=\mathcal{U}\dot{+}\mathcal{Y}$, such that $\mathfrak{V}[0,T]$ satisfies the following conditions:

(i) The space $\left\{x(0)\;\middle|\;\left[\begin{smallmatrix}x\\w\end{smallmatrix}\right]\in\mathfrak{V}[0,T]\right\}$ is dense in $\mathcal{X}$.

(ii) The operator $\left[\begin{array}{cc}0 & \mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}\end{array}\right]$ maps the space

$$\mathfrak{V}_0[0,T]:=\left\{\left[\begin{array}{c}x\\w\end{array}\right]\in\mathfrak{V}[0,T]\;\middle|\;\left[\begin{array}{c}x(0)\\w(0)\end{array}\right]=0\right\}\tag{5.4}$$

densely into $L^2([0,T];\mathcal{U})$.

(iii) There exists a $K_T>0$, such that all $\left[\begin{smallmatrix}x\\w\end{smallmatrix}\right]\in\mathfrak{V}[0,T]$ satisfy for all $t\in[0,T]$:

$$\|x(t)\|_{\mathcal{X}}^2+\int_0^t\|w(s)\|_{\mathcal{W}}^2\,\mathrm{d}s\le K_T\left(\|x(0)\|_{\mathcal{X}}^2+\int_0^t\|\mathcal{P}_{\mathcal{U}}^{\mathcal{Y}}w(s)\|_{\mathcal{U}}^2\,\mathrm{d}s\right).$$

In this case we call $(\mathcal{U},\mathcal{Y})$ an L^2*-admissible i/o pair* of the *s/s node* $(V;\mathcal{X},\mathcal{W})$.

The notion of an L^2-admissible i/o pair is related to the notion of admissibility given in Definition 2.5, but neither type of admissibility implies the other type. It follows from the following proposition that all of the theory developed in [14] is applicable to passive s/s systems.

Proposition 5.8. *Let $T>0$ and let $(V;\mathcal{X},\mathcal{W})$ be a passive s/s node with generalised trajectories $\mathfrak{W}[0,T]$. Then the following claims are true:*

(i) *Every fundamental i/o pair is L^2-admissible for $(V;\mathcal{X},\mathcal{W})$. In particular, the space V generates an L^2-well-posed s/s system Σ.*

(ii) *The space V is maximal in the sense that all subspaces that generate the same space $\mathfrak{W}[0,T]$ of generalised trajectories are contained in V.*

(iii) *For every $T>0$, the space of classical trajectories generated by V on $[0,T]$ satisfies*

$$\mathfrak{V}[0,T]=\mathfrak{W}[0,T]\cap\left[\begin{array}{c}C^1([0,T];\mathcal{X})\\C([0,T];\mathcal{W})\end{array}\right].\tag{5.5}$$

(iv) *The generating subspace V is uniquely determined by $\mathfrak{W}[0,T]$ through*

$$V=\left\{\left[\begin{array}{c}\dot{x}(0)\\x(0)\\w(0)\end{array}\right]\;\middle|\;\left[\begin{array}{c}x\\w\end{array}\right]\in\mathfrak{W}[0,T]\cap\left[\begin{array}{c}C^1([0,T];\mathcal{X})\\C([0,T];\mathcal{W})\end{array}\right]\right\}.$$

Any of the spaces V, $\mathfrak{V}[0,T]$, $\mathfrak{V}$, $\mathfrak{W}[0,T]$ and $\mathfrak{W}$ determine the other four of these spaces uniquely.

Proof. We begin by proving claims (i) and (ii) and we therefore fix a fundamental i/o pair $\mathcal{W}=(\mathcal{W}_+,\mathcal{W}_-)$. By Theorem 5.2, $\left(\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right];\mathcal{X},\mathcal{W}_+,\mathcal{W}_-\right)$ is a scattering representation of V, and by Proposition 5.6, $\left[\begin{smallmatrix}A\&B\\C\&D\end{smallmatrix}\right]$ is a scattering-passive system node. Thus (5.2) holds with $K_T=1$ and therefore every scattering representation of a passive s/s node node is L^2-well-posed. Theorem 6.4 of [14] yields that $(V;\mathcal{X},\mathcal{W})$ is an L^2-well-posed s/s node, which is maximal in the sense that all other generating subspaces of the same s/s system are included in V.

Claim (iii) was proved in the proof of [14, Thm 6.4]. Combining (2.1) with (5.5), we see that claim (iv) holds.

We fix $T > 0$ in order to prove the last claim. The generating subspace V determines $\mathfrak{V}$ uniquely by Definition 1.1, and $\mathfrak{V}$ in turn determines $\mathfrak{W}$ uniquely by Definition 1.3. Moreover, $\mathfrak{W}[0,T] = \rho_{[0,T]}\mathfrak{W}$ by [14, Prop. 3.9], and $\mathfrak{W}[0,T]$ determines $\mathfrak{V}[0,T]$ as in claim (iii). Finally, $\mathfrak{V}[0,T]$ determines V through (2.1). $\square$

One may ask if the converse of Proposition 5.8(ii) also is true, i.e., if assuming that $V \geq 0$ contains all generating subspaces of the same s/s system is enough to imply that $(V;\mathcal{X},\mathcal{W})$ is passive. Example 5.3 shows that the answer is no, and the explanation is the following. According to [14, Thm 6.4], the maximality of V as a generating subspace follows from the existence of an arbitrary L^2-well-posed i/o pair. However, for $(V;\mathcal{X},\mathcal{W})$ to be passive we need the i/o pair to be *fundamental*, which is a stronger condition, cf. Theorem 5.2.

Appendix A. Some basics of Kreĭn spaces

In this appendix we collect some standard terminology and results from the theory of Kreĭn spaces. More background can be found e.g. in [5] and [11].

Definition A.1. The vector space $(\mathcal{W};[\cdot,\cdot]_{\mathcal{W}})$, where $[\cdot,\cdot]_{\mathcal{W}}$ is an indefinite sesquilinear product, is an *anti-Hilbert space* if $-\mathcal{W} := (\mathcal{W};-[\cdot,\cdot]_{\mathcal{W}})$ is a Hilbert space. In this case we denote the Hilbert space $-\mathcal{W}$ by $|\mathcal{W}|$.

The space $(\mathcal{W};[\cdot,\cdot]_{\mathcal{W}})$ is a *Kreĭn space* if it admits a direct-sum decomposition $\mathcal{W} = \mathcal{W}_+ \dotplus \mathcal{W}_-$, such that:

(i) the spaces $\mathcal{W}_+$ and $\mathcal{W}_-$ are $[\cdot,\cdot]_{\mathcal{W}}$-orthogonal, i.e., $[w_+,w_-]_{\mathcal{W}} = 0$ for all $w_+ \in \mathcal{W}_+$ and $w_- \in \mathcal{W}_-$, and

(ii) the space $\mathcal{W}_+$ is a Hilbert space and $\mathcal{W}_-$ is an anti-Hilbert space.

In this case we call the decomposition $\mathcal{W} = \mathcal{W}_+ \dotplus \mathcal{W}_-$ a *fundamental decomposition* of $\mathcal{W}$, and we always denote it by $\mathcal{W} = (\mathcal{W}_+,\mathcal{W}_-)$, so that the second space in the pair is the anti-Hilbert space.

Let $\mathcal{U}$ and $\mathcal{Y}$ be subspaces of the Kreĭn space $\mathcal{W}$. By writing $\mathcal{U}[\perp]\mathcal{Y}$ we mean that $\mathcal{U}$ and $\mathcal{Y}$ are orthogonal to each other with respect to $[\cdot,\cdot]_{\mathcal{W}}$. The *orthogonal companion* of $\mathcal{U}$ is the space

$$\mathcal{U}^{[\perp]} := \{w \in \mathcal{W} \mid \forall u \in \mathcal{U}: \quad [u,w]_{\mathcal{W}} = 0\}. \tag{A.1}$$

We now prove that the node space $\mathfrak{K}$ in Definition 1.4 is a Kreĭn space.

Proposition A.2. *Let $\alpha \in \mathbb{C}^+$ and let $\mathcal{W}$ be a Kreĭn space with fundamental decomposition $\mathcal{W} = (\mathcal{W}_+,\mathcal{W}_-)$. Then the node space $\mathfrak{K}$ in Definition 1.4 is a Kreĭn space with fundamental decomposition $\mathfrak{K} = (\mathfrak{K}_+,\mathfrak{K}_-)$, where*

$$\mathfrak{K}_+ = \begin{bmatrix} \begin{bmatrix} -\overline{\alpha} \\ 1 \end{bmatrix}\mathcal{X} \\ \mathcal{W}_+ \end{bmatrix} \quad\text{and}\quad \mathfrak{K}_- = \begin{bmatrix} \begin{bmatrix} \alpha \\ 1 \end{bmatrix}\mathcal{X} \\ \mathcal{W}_- \end{bmatrix}. \tag{A.2}$$

The projections of $\mathfrak{K}$ onto $\mathfrak{K}_\pm$ along $\mathfrak{K}_\mp$ are given by

$$\begin{aligned}\mathcal{P}_{\mathfrak{K}_+}^{\mathfrak{K}_-} &:= \begin{bmatrix} \frac{1}{2\mathrm{Re}\,\alpha}\begin{bmatrix} -\overline{\alpha} \\ 1 \end{bmatrix}\begin{bmatrix} -1 & \alpha \end{bmatrix} & 0 \\ 0 & \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} \end{bmatrix} \quad \text{and} \\ \mathcal{P}_{\mathfrak{K}_-}^{\mathfrak{K}_+} &:= \begin{bmatrix} \frac{1}{2\mathrm{Re}\,\alpha}\begin{bmatrix} \alpha \\ 1 \end{bmatrix}\begin{bmatrix} 1 & \overline{\alpha} \end{bmatrix} & 0 \\ 0 & \mathcal{P}_{\mathcal{W}_-}^{\mathcal{W}_+} \end{bmatrix}.\end{aligned} \tag{A.3}$$

For every $V \subset \mathfrak{K}$, the condition $\mathcal{P}_{\mathfrak{K}_+}^{\mathfrak{K}_-} V = \mathfrak{K}_+$ is equivalent to

$$\begin{bmatrix} -1 & \alpha & 0 \\ 0 & 0 & \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} \end{bmatrix} V = \begin{bmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{bmatrix}.$$

Proof. The subspace $\mathfrak{K}_+$ is a Hilbert space, because $\mathcal{X}$ and $\mathcal{W}_+$ are both Hilbert spaces by assumption and

$$\begin{aligned}\left[\!\left[\begin{bmatrix} \begin{bmatrix} -\overline{\alpha} \\ 1 \end{bmatrix} x \\ w_+ \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} -\overline{\alpha} \\ 1 \end{bmatrix} x \\ w_+ \end{bmatrix} \right]\!\right]_{\mathfrak{K}} &= (w_+, w_+)_{\mathcal{W}_+} + (\overline{\alpha}x, x)_{\mathcal{X}} + (x, \overline{\alpha}x)_{\mathcal{X}} \\ &= \|w_+\|_{\mathcal{W}_+}^2 + (2\mathrm{Re}\,\alpha)\|x\|_{\mathcal{X}}^2.\end{aligned}$$

Replacing $-\overline{\alpha}$ by α and w_+ by w_-, we get that $\mathfrak{K}_-$ is an anti-Hilbert space. In particular $\mathfrak{K}_+$ and $\mathfrak{K}_-$ are both closed. Moreover, $\mathfrak{K}_+[\perp]\mathfrak{K}_-$, because for all $x, z \in \mathcal{X}$ and $w_\pm \in \mathcal{W}_\pm$ we have:

$$\left[\!\left[\begin{bmatrix} \begin{bmatrix} -\overline{\alpha} \\ 1 \end{bmatrix} x \\ w_+ \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} \alpha \\ 1 \end{bmatrix} z \\ w_- \end{bmatrix} \right]\!\right]_{\mathfrak{K}} = (\overline{\alpha}x, z)_{\mathcal{X}} - (x, \alpha z)_{\mathcal{X}} + [w_+, w_-]_{\mathcal{W}} = 0.$$

Checking that $\mathcal{P}_{\mathfrak{K}_+}^{\mathfrak{K}_-} + \mathcal{P}_{\mathfrak{K}_-}^{\mathfrak{K}_+} = 1_{\mathcal{W}}$ is trivial and it is also straightforward to verify that $\mathcal{P}_{\mathfrak{K}_\pm}^{\mathfrak{K}_\mp}$ are projections onto $\mathfrak{K}_\pm$, i.e., that $(\mathcal{P}_{\mathfrak{K}_\pm}^{\mathfrak{K}_\mp})^2 = \mathcal{P}_{\mathfrak{K}_\pm}^{\mathfrak{K}_\mp}$ and $\mathrm{Ran}\left(\mathcal{P}_{\mathfrak{K}_\pm}^{\mathfrak{K}_\mp}\right) = \mathfrak{K}_\pm$. This implies that $\mathfrak{K} = \mathfrak{K}_+ + \mathfrak{K}_-$, because any $k \in \mathfrak{K}$ can be written $k = k_+ + k_-$, where $k_\pm = \mathcal{P}_{\mathfrak{K}_\pm}^{\mathfrak{K}_\mp} k \in \mathfrak{K}_\pm$. The sum $\mathfrak{K} = \mathfrak{K}_+ + \mathfrak{K}_-$ is direct, because $\left[\begin{smallmatrix} z \\ x \\ w \end{smallmatrix}\right] \in \mathfrak{K}_+ \cap \mathfrak{K}_-$ implies that $z = \alpha x = -\overline{\alpha}x$ and $w \in \mathcal{W}_+ \cap \mathcal{W}_- = \{0\}$, and then, in particular, $(2\mathrm{Re}\,\alpha)x = 0$. As $\mathrm{Re}\,\alpha > 0$, we get $z = \alpha x = 0$.

The last claim is proved simply by noting that

$$\mathcal{P}_{\mathfrak{K}_+}^{\mathfrak{K}_-} = \begin{bmatrix} \frac{1}{2\mathrm{Re}\,\alpha}\begin{bmatrix} -\overline{\alpha} \\ 1 \end{bmatrix} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} -1 & \alpha \end{bmatrix} & 0 \\ 0 & \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_+} \end{bmatrix}$$

and that the first factor maps $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{W}_+ \end{smallmatrix}\right]$ one-to-one onto $\mathfrak{K}_+$. □

Let $\mathcal{W} = (\mathcal{W}_+, \mathcal{W}_-)$ be a fundamental decomposition of the Kreĭn space $\mathcal{W}$. Then it follows from Definition A.1 that all $w_+^1 + w_-^1, w_+^2 + w_-^2 \in \mathcal{W}$, where $w_\pm^1, w_\pm^2 \in \mathcal{W}_\pm$, satisfy

$$\begin{aligned}[][w_+^1 + w_-^1, w_+^2 + w_-^2]_{\mathcal{W}} &= [w_+^1, w_+^2]_{\mathcal{W}_+} + [w_-^1, w_-^2]_{\mathcal{W}_-} \\ &= (w_+^1, w_+^2)_{\mathcal{W}_+} - (w_-^1, w_-^2)_{|\mathcal{W}_-|}.\end{aligned} \tag{A.4}$$

Therefore we can turn $\mathcal{W}$ into a Hilbert space by changing the sign on the restriction of $[\cdot,\cdot]_{\mathcal{W}}$ to $\mathcal{W}_-$, as described in the following definition.

Definition A.3. We call the Hilbert-space inner products on $\mathcal{W}$ that arise from fundamental decompositions $\mathcal{W} = (\mathcal{W}_+, \mathcal{W}_-)$ through

$$(w_+^1 + w_-^1, w_+^2 + w_-^2)_{\mathcal{W}} = (w_+^1, w_+^2)_{\mathcal{W}_+} + (w_-^1, w_-^2)_{|\mathcal{W}_-|} \tag{A.5}$$

admissible inner products. The inner product (A.5) can be compactly written

$$(w^1, w^2)_{\mathcal{W}} = [w^1, Jw^2]_{\mathcal{W}},$$

where the so-called *fundamental symmetry* J is given by $J = \mathcal{P}_{\mathcal{W}_+}^{\mathcal{W}_-} - \mathcal{P}_{\mathcal{W}_-}^{\mathcal{W}_+}$.

A norm induced by an admissible inner product is called an *admissible norm.*

Only Hilbert and anti-Hilbert spaces have unique fundamental decompositions, but all admissible norms are equivalent. Every admissible inner product turns a closed subspace of a Kreĭn space into a Hilbert space, and thus in particular, every closed subspace of a Kreĭn space is a reflexive Banach space.

In contrast to Hilbert spaces, not every closed subspace $\mathcal{U}$ of a Kreĭn space $\mathcal{W}$ is itself a Kreĭn space. Indeed, in the state/signal theory we often encounter Lagrangian subspaces, which are closed non-Kreĭn subspaces of Kreĭn spaces. More precisely, a closed subspace $\mathcal{U}$ is a Kreĭn space if and only if it is ortho-complemented: $\mathcal{U} \dotplus \mathcal{U}^{[\perp]} = \mathcal{W}$; see [11, Thm V.3.4].

The orthogonal companion (A.1) of any subspace $\mathcal{U}$ of $\mathcal{W}$ is a closed subspace of $\mathcal{W}$ with respect to the admissible norms. Denoting the closure of a subspace $\mathcal{U} \subset \mathcal{W}$ with respect to any admissible norm by $\overline{\mathcal{U}}$, we have that $(\mathcal{U}^{[\perp]})^{[\perp]} = \overline{\mathcal{U}}$.

The following definition makes use of the continuous dual $\mathcal{U}'$ of a Banach space $\mathcal{U}$. Recall that this continuous dual is the space of all continuous linear functionals on $\mathcal{U}$.

Definition A.4. Let $\mathcal{W} = \mathcal{U} \dotplus \mathcal{Y}$ be a direct-sum decomposition of a Kreĭn space. According to [7, Lemma 2.3], we can identify the continuous duals of $\mathcal{U}$ and $\mathcal{Y}$ with $\mathcal{Y}^{[\perp]}$ and $\mathcal{U}^{[\perp]}$, respectively, using the following restrictions of $[\cdot,\cdot]_{\mathcal{W}}$ as duality pairings:

$$\begin{aligned}\langle u, u'\rangle_{\langle \mathcal{U},\mathcal{U}'\rangle} &= [u, u']_{\mathcal{W}}, \quad u \in \mathcal{U},\ u' \in \mathcal{Y}^{[\perp]} \quad \text{and} \\ \langle y, y'\rangle_{\langle \mathcal{Y},\mathcal{Y}'\rangle} &= [y, y']_{\mathcal{W}}, \quad y \in \mathcal{Y},\ y' \in \mathcal{U}^{[\perp]}.\end{aligned}$$

Let T map a dense subspace of $\mathcal{U}$ linearly into $\mathcal{Y}$. By $T^\dagger$ we denote the (possibly unbounded) *adjoint* of T computed with respect to these duality pairings, so that $T^\dagger : \mathcal{Y}' \to \mathcal{U}'$ is the maximally defined operator that satisfies

$$\forall u \in \operatorname{Dom}(T), y' \in \operatorname{Dom}\left(T^\dagger\right): \quad \langle Tu, y' \rangle_{\langle \mathcal{Y},\mathcal{Y}' \rangle} = \langle u, T^\dagger y' \rangle_{\langle \mathcal{U},\mathcal{U}' \rangle}. \tag{A.6}$$

Here $\operatorname{Dom}\left(T^\dagger\right)$ is the subspace consisting of those $y' \in \mathcal{Y}'$, for which there exists some $u' \in \mathcal{U}'$, such that $\langle Tu, y' \rangle_{\langle \mathcal{Y},\mathcal{Y}' \rangle} = \langle u, u' \rangle_{\langle \mathcal{U},\mathcal{U}' \rangle}$ for all $u \in \operatorname{Dom}(T)$.

The condition (A.6) can also be written

$$\forall u \in \operatorname{Dom}(T), y' \in \operatorname{Dom}\left(T^\dagger\right): \quad [Tu, y']_{\mathcal{W}} = [u, T^\dagger y']_{\mathcal{W}}, \tag{A.7}$$

but note that T is not densely defined on $\mathcal{W}$ in general, and therefore (A.7) does not uniquely determine $T^\dagger$ as an operator on $\mathcal{W}$. However, if $\mathcal{U} = \mathcal{Y} = \mathcal{W}$ and this is a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{W}}$, then the construction in Definition A.4 leads to an identification $\mathcal{W}' = \mathcal{W}$, using the standard Hilbert-space duality pairing $\langle w, w' \rangle_{\langle \mathcal{W},\mathcal{W}' \rangle} = (w, w')_{\mathcal{W}}$. In this case we denote the adjoint $T^\dagger$ of T by T^* in order to emphasise that the adjoint is computed with respect to a Hilbert-space inner product.

Definition A.5. The subspace $V \subset \mathcal{W}$ is *nonnegative (nonpositive)* if $[v,v]_{\mathcal{W}} \geq 0$ ($[v,v]_{\mathcal{W}} \leq 0$) for all $v \in V$. In both of these cases V is said to be *semidefinite* and V is *maximally semidefinite* if V has no proper extension to a semidefinite subspace of $\mathcal{W}$.

A *vector* $v \in \mathcal{W}$ is *neutral* if $[v,v]_{\mathcal{W}} = 0$. The *space* V is *neutral* if all $v \in V$ are neutral and V is *Lagrangian* if $V = V^{[\perp]}$.

Remark A.6. The closure of a semidefinite subspace is semidefinite and, therefore, every maximally semidefinite subspace is closed.

Obviously a subspace is neutral if and only if it is both nonnegative and nonpositive.

Lemma A.7. *Let $\mathcal{W}$ be a Kreĭn space, let $V_0, V_1 \subset \mathcal{W}$, and define $V := V_0 + V_1$. Then the following claims are true:*

(i) *The space V_0 is neutral, i.e., $[v,v]_{\mathcal{W}} = 0$ for all $v \in V$, if and only if $V_0 \subset V_0^{[\perp]}$, i.e., $[v,v']_{\mathcal{W}} = 0$ for all $v, v' \in V_0$.*
(ii) *Let V_0 be neutral. Then V is nonnegative or nonpositive if and only if V_1 is nonnegative or nonpositive, respectively, and $V_0[\perp]V_1$.*
(iii) *If V_0 is neutral then $V_0 \subset V^{[\perp]}$.*

Proof. We prove claim (ii) first. Assume therefore that $[v,v]_{\mathcal{W}} = 0$ for all $v \in V_0$. Then we for all $v_0 + v_1 \in V$, $v_i \in V_i$, have

$$[v_1 + v_0, v_1 + v_0]_{\mathcal{W}} = [v_1, v_1] + 2\mathrm{Re}\,[v_1, v_0] + [v_0, v_0] = [v_1, v_1] + 2\mathrm{Re}\,[v_1, v_0]. \tag{A.8}$$

Thus, $V_1 \geq 0$ and $V_0[\perp]V_1$ immediately imply that $V \geq 0$.

Now conversely assume that $V \geq 0$ and that $[v,v]_{\mathcal{W}} = 0$ for all $v \in V_0$. Then trivially $V_1 \subset V$ is also nonnegative. Moreover, if there exists $v_i \in V_i$ such that $[v_1, v_0]_{\mathcal{W}} =: \alpha \neq 0$, then for all $s \in \mathbb{R}^-$ we by (A.8) have $v_1 + s\alpha v_0 \in V$ and:

$$[v_1 + s\alpha v_0, v_1 + s\alpha v_0]_{\mathcal{W}} = [v_1, v_1] + 2s|\alpha|^2 \in \mathbb{R}.$$

This expression tends to $-\infty$ as $s \to -\infty$, which contradicts the assumption that $V \geq 0$ and therefore necessarily $V_0[\perp]V_1$.

We now give the proof of claim (i). If $[v,v']_{\mathcal{W}} = 0$ for all $v, v' \in V_0$ then trivially $[v,v]_{\mathcal{W}} = 0$ for all $v \in V_0$. Conversely, if $[v,v]_{\mathcal{W}} = 0$ for all $v \in V_0$, then $V_0 = V_0 + V_0$ is neutral and by item (ii) we have $V_0[\perp]V_0$, which is equivalent to $V_0 \subset V_0^{[\perp]}$.

Regarding claim (iii), note that if $V_0[\perp]V_1$ and $V_0 \subset V_0^{[\perp]}$, then by claim (i) we for all $v_0 + v_1 \in V$ and $v_0' \in V_0$ have that:

$$[v_0 + v_1, v_0'] = [v_0, v_0'] + [v_1, v_0'] = 0.$$

Thus $v_0' \in V^{[\perp]}$, i.e., $V_0 \subset V^{[\perp]}$. □

The following characterisation of semidefinite subspaces of a Kreĭn space is useful. For proof, see Theorems 11.7, 4.2 and 4.4, and Lemma 4.5 of [11].

Proposition A.8. *Let $\mathfrak{K} = (\mathfrak{K}_+, \mathfrak{K}_-)$ be a fundamental decomposition. Let $V \subset \mathfrak{K}$ and define $V_\pm := \mathcal{P}_{\mathfrak{K}_\pm}^{\mathfrak{K}_\mp} V$.*

The space V is nonnegative if and only if there exists a Hilbert-space contraction $\mathcal{A}_+ : \mathfrak{K}_+ \to |\mathfrak{K}_-|$, such that $V = (1 + \mathcal{A}_+)V_+$. The space V is maximally nonnegative if and only if $V_+ = \mathfrak{K}_+$.

The subspace V is nonpositive if and only if there exists a contraction $\mathcal{A}_- : |\mathfrak{K}_-| \to \mathfrak{K}_+$, such that $V = (1 + \mathcal{A}_-)V_-$. The space V is maximally nonpositive if and only if $V_- = \mathfrak{K}_-$.

The subspace V is neutral if and only if it is nonnegative with an isometric $\mathcal{A}_+$, which in turn is true if and only if V is nonpositive with an isometric $\mathcal{A}_-$. The subspace V is Lagrangian if and only if it is both maximally nonnegative and maximally nonpositive, in which case $\mathcal{A}_+$ and $\mathcal{A}_-$ are both unitary.

Let V be closed and nonnegative. Then $V^{[\perp]}$ is nonpositive if and only if V is maximally nonnegative. We can say even more, namely that

$$V = (1 + \mathcal{A}_+)\mathfrak{K}_+ \quad \Longrightarrow \quad V^{[\perp]} = (1 + \mathcal{A}_+^*)\mathfrak{K}_-,$$

where $\mathcal{A}_+^*$ is computed with respect to the inner product on $|\mathfrak{K}_-|$, i.e., for all $w_- \in \mathfrak{K}_-$ and $w_+ \in \mathfrak{K}_+$:

$$(\mathcal{A}_+ w_+, w_-)_{|\mathfrak{K}_-|} = -\left[\mathcal{A}_+ w_+, w_-\right]_{\mathfrak{K}} = \left(w_+, \mathcal{A}_+^* w_-\right)_{\mathfrak{K}_+}.$$

It is elementary to characterise Lagrangian subspaces V of finite-dimensional Kreĭn spaces. Indeed, the following corollary to Proposition A.8 shows that it suffices to check that V is neutral and has sufficiently large dimension.

Corollary A.9. *Assume that $\mathfrak{K}$ is a Kreĭn space with finite dimension n. If $V \subset V^{[\perp]} \subset \mathfrak{K}$ and $\dim V \geq n/2$ then $V = V^{[\perp]}$ and $\dim V = n/2$.*

Proof. Assume that $V \subset V^{[\perp]}$ and let $\mathfrak{K} = (\mathfrak{K}_+, \mathfrak{K}_-)$ be a fundamental decomposition. Then $V = (1 + \mathcal{A}_+)V_+$ for some isometric $\mathcal{A}_+ : \mathfrak{K}_+ \to |\mathfrak{K}_-|$. Moreover, $V_+ \subset \mathfrak{K}_+$ and therefore necessarily $n/2 \le \dim V \le \dim V_+ \le \dim \mathfrak{K}_+$. Dually, $V = (1 + \mathcal{A}_-)V_-$ for some isometric $\mathcal{A}_- : |\mathfrak{K}_-| \to \mathfrak{K}_+$ and $V_- \subset \mathfrak{K}_-$ with $n/2 \le \dim V \le \dim V_- \le \dim \mathfrak{K}_-$.

From $\mathfrak{K} = \mathfrak{K}_+ \dot{+} \mathfrak{K}_-$ we now get that $\dim \mathfrak{K}_+ + \dim \mathfrak{K}_- \le n$, which implies that $\dim \mathfrak{K}_\pm = n/2$. Then $V_\pm \subset \mathfrak{K}_\pm$ with $\dim V_\pm = \dim \mathfrak{K}_\pm = n/2$, i.e., $V_\pm = \mathfrak{K}_\pm$. Thus $\dim V \le n/2$ and Proposition A.8 yields that $V = V^{[\perp]}$. □

We now end this paper by listing a few function spaces, which are frequently used throughout the article.

Definition A.10. Let $\mathcal{U}$ be a closed subspace of a Kreĭn space and let $I = [a, b]$, where $b > a$, or $I = [a, \infty)$.

(i) The space of continuous $\mathcal{U}$-valued functions which are defined on I is denoted by $C(I;\mathcal{U})$. The space $C([a, b];\mathcal{U})$ is a Banach space with the supremum norm, whereas $C([a,\infty);\mathcal{U})$ is a Fréchet space with the following family of seminorms:

$$\|f\|_n := \sup_{t\in[a,a+n]} \|f(t)\|_{\mathcal{U}}, \quad n \in \mathbb{Z}^+.$$

The space of $\mathcal{U}$-valued functions with $n \in \mathbb{Z}^+$ continuous derivatives on I is denoted by $C^n(I;\mathcal{U})$.

(ii) By $L^2(I;\mathcal{U})$ we denote the space of all $\mathcal{U}$-valued Lebesgue-measurable functions f defined on I, such that

$$\|f\|_{L^2(I;\mathcal{U})} := \left(\int_I \|f(t)\|^2_{\mathcal{U}} \,\mathrm{d}t\right)^{1/2} < \infty,$$

where $\|\cdot\|_{\mathcal{U}}$ denotes some arbitrary given admissible norm on $\mathcal{U}$.

(iii) The space $L^2_{loc}([a,\infty);\mathcal{U})$ consists of all functions f that lie locally in L^2: $\rho_{[a,b]}f \in L^2([a, b];\mathcal{W})$ for all $b > a$. This is a Fréchet space when equipped with the following family of seminorms:

$$\|f\|_n := \|\rho_{[a,a+n]}f\|_{L^2([a,a+n];\mathcal{U})}, \quad n \in \mathbb{Z}^+.$$

(iv) The space of functions $f \in L^2(I;\mathcal{U})$ that have a distribution derivative g in $L^2(I;\mathcal{U})$ is denoted by $H^1(I;\mathcal{U})$. By this we mean that $f \in L^2(I;\mathcal{U})$ lies in $H^1(I;\mathcal{U})$ if and only if there exists $g \in L^2(I;\mathcal{U})$ such that

$$\forall t \ge a : \quad f(t) = \int_a^t g(s)\,\mathrm{d}s. \tag{A.9}$$

If (A.9) holds for $f, g \in L^2_{loc}(I;\mathcal{U})$, then we write $f \in H^1_{loc}(I;\mathcal{U})$.

If $\mathcal{W}$ is a Kreĭn space and I is a subinterval of $\mathbb{R}$, then $L^2(I;\mathcal{W})$ is a Kreĭn space with the inner product $[w^1, w^2] := \int_I [w^1(t), w^2(t)]_{\mathcal{W}}\,\mathrm{d}t$, because every fundamental decomposition $\mathcal{W} = (\mathcal{W}_+, \mathcal{W}_-)$ induces the fundamental decomposition

$$L^2(\mathbb{R}^+;\mathcal{W}) = \big(L^2(\mathbb{R}^+;\mathcal{W}_+), L^2(\mathbb{R}^+;\mathcal{W}_-)\big).$$

The space $L^2_{loc}(I;\mathcal{U})$ is the same as $L^2(I;\mathcal{U})$ for all *finite* intervals $I \subset \mathbb{R}$.

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